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Multilevel Schur Complement Preconditioner for Multi-Physics Simulations

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Abstract

Advection-diffusion PDEs are prevalent in models of many physical applications in science and engineering. Within different parameter regimes (diffusion- vs. advection-dominated), there exist optimally scalable solvers for these PDEs, however no single solver yet applies well within all regimes of physical interest. Our goal is to find one method that is scalable in both regimes, through using a hybrid approach combining restrictive additive Schwarz (domain decomposition) and geometric multigrid.

Advection/Diffusion Equation

$$\partial_t u = \nabla \cdot (\beta u) + \mu \nabla \cdot (D \nabla u) + f$$

- After discretizing in both time and space, we obtain a linear algebraic system denoted by $Au = f$ to move from one time step to the next.
- We use a Krylov subspace method (GMRES) to solve this system.

Preconditioning

- Preconditioning can accelerate convergence of GMRES and enable scalability for increasing n (size of the domain).
- The right preconditioned system is $APP^{-1}u = f$.
- Letting $Pu = x$ we get $AP^{-1}x = f$.
- Ideally, we want $P^{-1} = A^{-1}$, but we cannot find A^{-1} for large systems, which leads to the use of preconditioners.

Motivation

- Multigrid [2] is optimally scalable for diffusion dominated problems.
  - Multigrid approximates a fine grid solution using a coarser grid, recursively.
  - For problems with smooth solutions, multigrid methods prove effective.
- Domain decomposition methods are optimal for advection dominated problems.
  - These approximate the solution by solving problems one subdomain at a time.
- We use a Schur complement domain decomposition preconditioner on the whole problem, with a multigrid solve for the interface only problem.

Schur Complement [1]

- Unknowns are broken up into two sets, I and Γ.
- I contains interior nodes inside each subdomain.
- Γ contains the subdomain interfaces.
- We then rewrite the system $Au = f$ as:

$$\begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} f_I \\ f_\Gamma \end{bmatrix}$$

- We use a Schur complement to solve this problem:

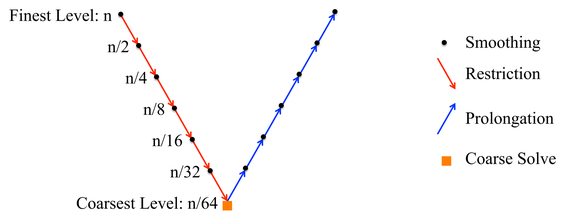
$$\begin{bmatrix} u_I \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} A_{II}^{-1}(f_I - A_{I\Gamma} S^{-1}(f_\Gamma - A_{\Gamma I} A_{II}^{-1} f_I)) \\ S^{-1}(f_\Gamma - A_{\Gamma I} A_{II}^{-1} f_I) \end{bmatrix}$$

where $S = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I\Gamma}$ is the Schur complement.

- $A_{II} b_I = f_I$ is directly solved using SuperLU.
- The focus of this work is with the interface solve: $Su_\Gamma = g_\Gamma$.

Interface Solve

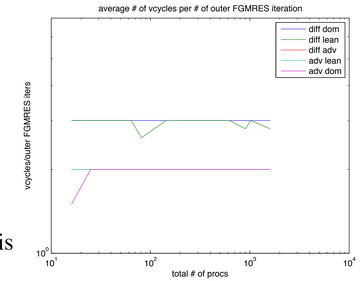
- We perform the interface solve $Su_\Gamma = g_\Gamma$ using GMRES preconditioned with a multigrid method.
- This preconditioner performs V-Cycles up and down the grid hierarchy, repeating until a set residual tolerance is met.
- At each level, we smooth the residual using a fixed number of FGMRES iterations.
- The coarsest level is solved to a set tolerance with FGMRES.

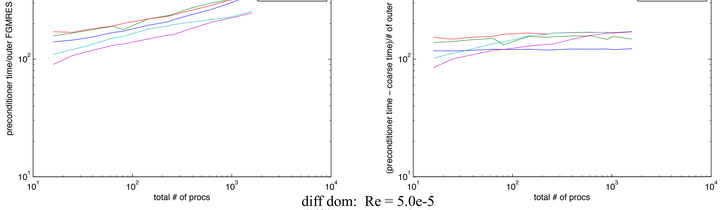


- We anticipate that the combination of a domain decomposition method with a multigrid method will allow our solver to scale well in both the advection- and diffusion-dominated regimes.

Results

- Right: The number of V-Cycles per outer FGMRES stays constant as the number of processors increases.
- Bottom Left: The total time in the preconditoner slightly increases as the number of processors increase.
- Bottom Right: Once the coarse solver time is removed, the total solution time is constant with an increase in processors.





diff dom: Re = 5.0e-5
diff lean: Re = 5.0e-3
diff adv: Re = 1
adv lean: Re = 1.99e2
adv dom: Re = 1.99e4

Conclusions

- While the approach promises scalability, adjustments must be made to the coarse level solve for optimal parallel scalability.
  - One could coarsen the interface further making it a much smaller problem to solve.
  - Alternatively, one could compute and store the coarsest interface matrix and solve the resulting system with a direct method.

References

[1] X. Tu and J. Li, *A Balancing Domain Decomposition Method by Constraints for Advection-Diffusion Problems*, Comm. In Appl. Math. And Comp. Sci. **3** (2008), 25-60.
[2] U. Trottenberg, C. Oosterlee, and A. Schüller, *Multigrid*, Academic Press, 2001.