2012

Delay-periodic solutions and their stability using averaging in delay-differential equations, with applications

Thomas W. Carr
Southern Methodist University, tcarr@smu.edu

Richard Haberman
Southern Methodist University, rhaberma@smu.edu

Thomas Erneux
Universite Libre de Bruxelles, TERNEUX@ULB.AC.BE

Delay-periodic solutions and their stability using averaging in delay-differential equations, with applications

T.W. Carr, R. Haberman
Department of Mathematics,
Southern Methodist University,
Dallas, TX 75275-0156, USA
Tel: 214-768-3460, Fax: 214-768-2355,
tcarr@smu.edu, rhaberma@smu.edu
T. Erneux
Optique Nonlinéaire Théorique
Université Libre de Bruxelles
Campus Plaine, Code Postal 231, 1050 Bruxelles, Belgium
terneux@ulb.ac.be

Abstract

Using the method of averaging we analyze periodic solutions to delay-differential equations, where the period is near to the value of the delay time (or a fraction thereof). The difference between the period and the delay time defines the small parameter used in the perturbation method. This allows us to consider problems with arbitrarily size delay times or of the delay term itself. We present a general theory and then apply the method to a specific model that has application in disease dynamics and lasers.

---

1. Introduction

Perturbation methods applied to dynamical systems rely on identifying a small parameter, e.g., ϵ, such that the reduced problem for $\epsilon = 0$ is easier to solve. Corrections to the solution of the reduced problem are constructed via the machinery of the perturbation method. For delay-differential equations (DDEs), the approach has generally been one of three alternatives: averaging, multiple scales, or matched asymptotics (see [1, 2, 3] and included references). In this work we use averaging to determine the existence and stability of nearly $\tau-$period solutions, where τ is the delay time. Generally, perturbation methods applied to DDEs require some separation of time scales (small delay or large delay) or for the delay term itself to be small. We show that solutions with arbitrary delay time and with $O(1)$-sized delay terms can be described, with the trade-off being that the period of solutions must be near that of the delay (or an integer fraction of the delay).

We consider perturbed DDEs of the form

$$\begin{aligned} \dot{x} &= f_1(x, x_\tau, y, y_\tau) + \epsilon f_2(x, x_\tau, y, y_\tau), \\ \dot{y} &= g_1(x, x_\tau, y, y_\tau + \epsilon g_2(x, x_\tau, y, y_\tau), \end{aligned} \tag{1}$$

where $\epsilon \ll 1$, $x = x(t)$, $x_\tau = x(t-\tau)$ (similarly for y), and the over dot indicates derivative with respect to t. We assume that ϵ measures the size of the dissipation such that when $\epsilon = 0$ and there is no delay ($\tau = 0$), the reduced problem has periodic solutions. More specifically, we assume that

$$\begin{aligned} \dot{X} &= f_1(X, X, Y, Y), \\ \dot{Y} &= g_1(X, X, Y, Y), \end{aligned} \tag{2}$$

has a first integral $E = F(X, Y)$ that describes periodic orbits.

In general, analysis of Eq. (1) is easiest if the delay is absent from the leading-order problem, that is, neither f_1 nor g_1 depend on x_τ or y_τ, such that the leading order problem is an ordinary differential equation (ODE). In the perturbation method the delay will be part of the correction problem such that f_2 and g_2 are evaluated using the solutions of the leading order problem. Thus, in the correction problem the delay terms are known functions and become simply inhomogeneous terms.

If the delay is small, i.e., $\tau = O(\epsilon)$, then the delay terms in the leading order problem can be expanded via a Taylor series, e.g., $x(t - \epsilon\tau) = x(t) - \epsilon\tau\dot{x}(t) + O(\epsilon^2)$. This idea is used to apply the method of multiple time scales to problems with delay [3], whereupon introducing a slow time $T = \epsilon t$, the delay term becomes $x(t-\tau) = x(t-\tau, T-\epsilon\tau) = x(t-\tau, t) - \epsilon\tau\partial_T x(t-\tau, T) + O(\epsilon^2)$. Thus, the $O(\epsilon)$ term of the expansion contributes a slow-time derivative to the correction problem. Of course, the delay in the fast time must still be dealt with by other means. In any case, we note that it has long been known that series expansions based on small delay can change the properties of the problem, e.g., stability of equilibrium may change [4]. Hence, results based on small-delay expansions should be carefully checked against numerical solutions to ensure that they maintain fidelity.

A third approach is to allow the delay to remain in the leading order problem but to take advantage of solution properties such as pulsations that allow for the construction of solutions other than by direct methods. For example, in [5, 6] the authors use ideas from matched asymptotic expansions to patch together a complete solution based on approximate solutions constructed over different intervals of time.

In this paper we use the method of averaging applied to DDEs [1], where application of the method is facilitated by looking for solutions whose period is near to that of the delay time, or integer fractions thereof. This is the key technical advance of this work and, to our knowledge, it has not previously been utilized. More specifically, we look for solutions where $x_\tau - x = O(\epsilon)$ (similarly for y) such that the solution is nearly τ-periodic. In this way, we allow for solutions of arbitrary-sized delay and the delay term may be part of the leading-order problem, as we have indicated in f_1 and g_1 in Eq. (1). We then use averaging to determine how the amplitude of the periodic solutions depends on the system parameters and the delay τ.

In the next section we present the general calculation ultimately deriving conditions for the existence and stability of τ-periodic solutions to Eq. (1). In Sec. 3 we demonstrate the method on a generic predator-prey type model and compare the results to those from numerical simulations. We conclude with a discussion in

Sec. 4.

2. General formulation

2.1. Evolution of the energy

As discussed in the introduction, we look for nearly τ-periodic solutions, where

$$x_\tau - x = \epsilon u(t), \quad y_\tau - y = \epsilon v(t). \tag{3}$$

Eq. (1) become

$$\begin{aligned}
\dot{x} &= f_1(x, x + \epsilon u, y, y + \epsilon v) + \epsilon f_2(x, x + \epsilon u, y, y + \epsilon v), \\
\dot{y} &= g_1(x, x_\tau y, y_\tau) + \epsilon g_2(x, x + \epsilon u, y, y + \epsilon v), \\
\epsilon \dot{u} &= f_1(x + \epsilon u, x_\tau + \epsilon u_\tau, y, y_\tau + \epsilon v_\tau) - f_1(x, x_\tau y, y_\tau) + O(\epsilon), \\
\epsilon \dot{v} &= g_1(x + \epsilon u, x_\tau + \epsilon u_\tau, y, y_\tau + \epsilon v_\tau) - g_1(x, x_\tau y, y_\tau) + O(\epsilon),
\end{aligned} \tag{4}$$

where the $O(\epsilon)$ terms in the u and v equations contain the deviations of f_2 and g_2. Expanding for $\epsilon \ll 1$, we have

$$\begin{aligned}
\dot{x} &= f_1(x, x, y, y) + \epsilon \left(u \partial_{x_\tau} f_1 + v \partial_{y_\tau} f_1 + f_2\right) + O(\epsilon^2), \\
\dot{y} &= g_1(x, x, y, y) + \epsilon \left(u \partial_{x_\tau} g_1 + v \partial_{y_\tau} g_1 + g_2\right) + O(\epsilon^2),
\end{aligned} \tag{5}$$

where the arguments of the f_j and g_j are (x, x, y, y) just as shown for the leading-order problem. In principle, we should continue to monitor the evolution of u and v by expanding their evolution equations in Eq. (4) for small epsilon. However, we see that we will be able to reuse Eq. (3) to re-express our results solely in terms of x and y and will not need to explicitly determine the evolution of u and v. In any case, looking for nearly τ-periodic solutions implies that u and v remain bounded.

To solve by averaging we define new variables (E, Φ) that represent the energy and phase. The variable definitions are motivated by considering the unperturbed ($\epsilon = 0$) problem, which is

$$\begin{aligned}
\dot{X} &= f_1(X, X, Y, Y), \\
\dot{Y} &= g_1(X, X, Y, Y).
\end{aligned} \tag{6}$$

We assume that there are P-periodic solutions characterized by a first integral $E = F(X, Y)$ such that

$$\dot{E} = \partial_X F f_1 + \partial_Y F g_1 = 0. \tag{7}$$

We will not need the phase Φ and so do not consider it further.

We define the energy variable based on the $\epsilon = 0$ problem as $E = F(x, y)$ and the time evolution of E is given by

$$\begin{aligned}
\dot{E} &= \partial_x F(x,y)\dot{x} + \partial_y F(x,y)\dot{y} && (8)\\
&= \partial_x F f_1 + \partial_y F g_1 && (9)\\
&\quad + \epsilon \partial_x F \left(u \partial_{x_\tau} f_1 + v \partial_{y_\tau} f_1 + f_2\right) + \epsilon \partial_y F \left(u \partial_{x_\tau} g_1 + v \partial_{y_\tau} g_1 + g_2\right),
\end{aligned}$$

where all functions are evaluated with arguments $(x(E,\Phi), x(E,\Phi), y(E,\Phi), y(E,\Phi))$.

2.2. Averaging and nearly τ-periodic solutions

We now average over one period P of the unperturbed ($\epsilon = 0$) system to obtain

$$\begin{aligned}
\dot{\hat{E}} &= \epsilon \frac{1}{P} \int_0^P \left[\partial_x F \left(u \partial_{x_\tau} f_1 + v \partial_{y_\tau} f_1 + f_2\right)\right.\\
&\qquad \left. + \, \partial_y F \left(u \partial_{x_\tau} g_1 + v \partial_{y_\tau} g_1 + g_2\right)\right] dt + O(\epsilon^2), \qquad (10)
\end{aligned}$$

where according to the usual application of averaging, the functions on the right-hand side of Eq. (10) are evaluated with arguments (X, X, Y, Y) from the unperturbed ($\epsilon = 0$) system Eq. (6). The fact that there is no $O(1)$ term on the right-hand side of Eq. (10) is a consequence of our definition of E and indicative of the fact that $\hat{E}$ is nearly constant, i.e., slowly varying.

Recall that u and v are defined by Eq. (3) such that to leading order

$$\epsilon u(t) = X_\tau - X, \quad \epsilon v(t) = Y_\tau - Y. \qquad (11)$$

We have assumed that X and Y are P-periodic and that the period is closely related to the delay time, that is

$$\tau = nP - \epsilon P_1. \qquad (12)$$

Then

$$\begin{aligned}
\epsilon u(t) &= \left[X(t - nP) + \dot{X}(t - nP)\epsilon P_1 + O(\epsilon^2)\right] - X(t),\\
&= \left[X(t) + \dot{X}(t)\epsilon P_1 + O(\epsilon^2)\right] - X(t)),\\
&= \epsilon P_1 f_1(X, X, Y, Y) + O(\epsilon^2), \qquad (13)
\end{aligned}$$

and similarly

$$\epsilon v(t) = \epsilon P_1 g_1(X, X, Y, Y) + O(\epsilon^2). \qquad (14)$$

Eq. (10) becomes

$$\begin{aligned}
\dot{\hat{E}} &= \epsilon \frac{1}{P} \int_0^P \left[\partial_x F \left(P_1 f_1 \partial_{x_\tau} f_1 + P_1 g_1 \partial_{y_\tau} f_1 + f_2\right)\right.\\
&\qquad \left. + \, \partial_y F \left(P_1 f_1 \partial_{x_\tau} g_1 + P_1 g_1 \partial_{y_\tau} g_1 + g_2\right)\right] dt + O(\epsilon^2), \qquad (15)
\end{aligned}$$

which we reorganize as

$$\begin{aligned}\dot{\hat{E}} &= \epsilon\frac{1}{P}\int_0^P\left(\partial_x F f_2 + \partial_y F g_2\right) dt \\ &+\epsilon P_1\frac{1}{P}\int_0^P\left[\partial_x F(f_1\partial_{x_\tau} f_1 + g_1\partial_{y_\tau} f_1) + \partial_y F(f_1\partial_{x_\tau} g_1 + g_1\partial_{y_\tau} g_1)\right]\, dt. \end{aligned} \tag{16}$$

(Henceforth, we will not indicate the $O(\epsilon^2)$ term). The functions in the integrals are all evaluated with arguments (X, X, Y, Y), and we have the standard result that the $O(\epsilon)$ term in the averaging result for $\hat{E}$ is independent of the phase. Further, the period will also be a function of the energy E such that

$$\epsilon P_1 = nP(\hat{E}) - \tau. \tag{17}$$

Thus, Eq. (16) has the form

$$\dot{\hat{E}} = \epsilon\mathcal{I}_1(\hat{E}) + \left[nP(\hat{E}) - \tau\right]\mathcal{I}_2(\hat{E}). \tag{18}$$

The first integral $\mathcal{I}_1$ characterizes the cumulative effect of the dissipation over one period; we expect that $\mathcal{I}_1$ will be negative. The second integral $\mathcal{I}_2$ characterizes the effective driving force of the delay on the system.

2.3. Existence and stability of periodic solutions

We can now use Eq. (18) to examine the existence and stability of periodic solutions to Eq. (1). Periodic solutions exist if the change in the average energy over one period is zero. Thus, periodic solutions satisfy $\hat{E} = E_0$ (a constant) and

$$0 = \epsilon\mathcal{I}_1(E_0) + \left[nP(E_0) - \tau\right]\mathcal{I}_2(E_0), \tag{19}$$

which is the bifurcation equation describing the amplitude of solutions E_0 in terms of the damping (ϵ), the delay (τ) and the other system parameters included in the integrals. We can analyze the stability of the periodic orbits by letting $\hat{E} = E_0 + z(t)$ in Eq. (18), expanding for z small and finding an ODE for z as

$$\dot{z} = \epsilon\left\{\mathcal{I}_1'(E_0) + \left[nP(E_0) - \tau\right]\mathcal{I}_2'(E_0) + nP'(E_0)\mathcal{I}_2(E_0)\right\}\, z, \tag{20}$$

where the prime indicates derivative with respect to argument. Thus, periodic orbits are stable (unstable) if the term in brackets is negative (positive).

In summary, we look for nearly τ-periodic solutions, which facilitates analytical progress at two points. First, using Eq. (3) we can expand for small ϵ leaving a leading-order problem that is integrable with periodic solutions. Second, we use τ-periodicity in Eq. (12) and (13) to simplify the integrals containing the delay terms. The end results are Eq. (19) and (20), which determine the existence and stability of solutions. In the next section we carry out this calculation on a specific example.

3. Example

As an example we consider (with $r = O(1)$ and $\epsilon \ll 1$)

$$\begin{aligned}\dot{x} &= -y - \epsilon x(a+by) + ry_\tau \\ \dot{y} &= x(1+y),\end{aligned} \tag{21}$$

which has applications in lasers [7, 8, 9, 10, 11], population epidemics [12, 6], and malaria infection [13, 14]. More generally, Eqs. (21) are a rescaled predator-prey system with the predator x regulated by a delayed version of the prey population y. In the case of lasers the delay term is feedback corresponding to a reinjection of the original signal after being reflected from some external surface. In the case of population epidemics, the delay term was feedback of temporarily immune individuals back into the susceptible class.

3.1. Evolution of the energy

We look for nearly τ-periodic solutions with $\epsilon v = y_\tau - y$ to obtain

$$\dot{x} = -Ry - \epsilon x(a+by) + \epsilon rv, \tag{22}$$

$$\dot{y} = x(1+y), \tag{23}$$

where $R = 1 - r$. If we take $\epsilon = 0$ we have

$$\dot{X} = -RY, \tag{24}$$

$$\dot{Y} = X(1+Y). \tag{25}$$

The (X, Y) subsystem has a first integral that motivates defining a new *energy* variable given by

$$E = \frac{1}{2}x^2 + Ry - R\ln(1+y). \tag{26}$$

For $\epsilon = 0$, $E' = 0$, while for $\epsilon \neq 0$ we have

$$\dot{E} = -\epsilon x^2(a+by) + \epsilon rxv. \tag{27}$$

3.2. Averaging and nearly τ-periodic solutions

We average both sides of the above equation over the period P to obtain

$$\dot{\hat{E}} \approx -\epsilon\frac{1}{P}\int_0^P X^2(a+bY)\,dt + r\frac{1}{P}\int_0^P X(Y_\tau - Y)\,dt, \tag{28}$$

where as indicated the integrals are evaluated using the $\epsilon = 0$ system and we made the substitution $\epsilon v = Y_\tau - Y$. For the $\epsilon = 0$ system we have $\dot{X} = -RY$ so that

$$\int_0^P X^2 Y\,dt = \int_0^P X^2(-\frac{1}{R}\dot{X})\,dt = 0, \tag{29}$$

where the final result is due to the periodicity of X. Thus, the energy evolves according to

$$\dot{\hat{E}} \approx -\epsilon a \frac{1}{P} \int_0^P X^2 \, dt + r \frac{1}{P} \int_0^P X(Y_\tau - Y) \, dt. \tag{30}$$

We look for nearly τ-periodic solutions such that using Eq. (12) and expanding the difference $(Y_\tau - Y)$ as we did in Eq. (13), the evolution equation for the averaged energy becomes

$$\dot{\hat{E}} \approx -\epsilon a \frac{1}{P} \int_0^P X^2 \, dt + \epsilon P_1 r \frac{1}{P} \int_0^P X^2(1+Y) \, dt. \tag{31}$$

Using Eq. (29) again, the equation for the average energy is then

$$\dot{\hat{E}} \approx (-\epsilon a + \epsilon P_1 r) \frac{1}{P} \int_0^P X^2 \, dt. \tag{32}$$

The integral on X^2 is a function of the energy, and so we define

$$\mathcal{I}(\hat{E}) = \frac{1}{P} \int_0^P X^2 \, dt. \tag{33}$$

In addition, the period also depends upon the energy, and we rewrite P_1 using Eq. (17). Thus, we have that

$$\dot{\hat{E}} \sim \left[-\epsilon a + r[nP(\hat{E}) - \tau] \right] \mathcal{I}(\hat{E}). \tag{34}$$

Note that both $\mathcal{I}$ and P are known functions. The former is defined above and the latter is given by

$$P(\hat{E}) = 2 \int_{Y_{min}(\hat{E})}^{Y_{max}(\hat{E})} \frac{dY}{X(1+Y)}, \tag{35}$$

where X is a function of $\hat{E}$ and Y via Eq. (26).

3.3. Existence and stability of periodic solutions

Periodic solutions Eq. (21) exist if the average energy $\hat{E}$ is a constant with respect to time, i.e., $d\hat{E}/dt = 0$. Thus, letting $\hat{E} = \hat{E}_0$ we have that

$$0 = \left[-\epsilon a + r[nP(\hat{E}_0) - \tau] \right] \mathcal{I}(\hat{E}_0). \tag{36}$$

The function $\mathcal{I}(\hat{E}_0)$ is zero only if the energy is zero. Thus, periodic solutions have an average energy determined by

$$0 = -\epsilon a + r[nP(\hat{E}_0) - \tau]. \tag{37}$$

More specifically, Eq. (37) provides an equation for the period:

$$P = \frac{1}{n} \left(\tau + \frac{\epsilon a}{r} \right). \tag{38}$$

The stability of the periodic solutions can be analyzed by letting $\hat{E} = \hat{E}_0 + z$ in Eq. (34) and linearizing for $z \ll 1$; we find that

$$\dot{z} = rnP'(\hat{E}_0)\mathcal{I}(\hat{E}_0)z. \tag{39}$$

The function $\mathcal{I}$ is positive by Eq. (33). The period monotonically increases with the amplitude [12] and so $P' > 0$. Thus, periodic orbits with period $P \approx \tau/n$ are stable (unstable) if $r < 0$ ($r > 0$).

3.4. Comparison with numerical simulations

Using Eq. (38) and given a relationship for $P = P(\hat{E}_0)$ or $P = P(X, Y)$, we can then invert to determine how the maximum amplitudes depend upon the the parameters. Such relationships can be determined from the results in [9] and are given by

$$x_{mx} = \frac{1}{4}RP + \sqrt{(\frac{1}{4}RP)^2 - R\ln(4)}, \text{ and } y_{mx} = \frac{1}{2R}x_{mx}^2, \tag{40}$$

which were derived in the limit of large-amplitude pulsating solutions. We compare our analytical results for the period, Eq. (38), the the maximum amplitudes, Eq. (40) and stability, Eq. (39) to the output of numerical simulation [15] and continuation [16].

We first show three example time series in Fig. 1 for the case when $\tau = 5\pi/2$. For small-values of r, the system will decay to the $(0, 0)$ steady state, and as r is increased, oscillatory solutions appear due to a Hopf bifurcation. Fig. 1a shows small-amplitude nearly harmonic solutions for a value of r just beyond the Hopf bifurcation point. As r is increased, the periodic solutions become pulsating as shown in Fig. 1b. Increasing r further leads to the chaotic solutions as shown in Fig. 1c, typically through a period-double sequence.

In Fig. 2-4 we compare the analytical results (blue curve) with those from numerical continuation [16] (red stars), for the case when $r < 0$. These results are generic for all values of delay that we have tested. In Fig. 2a $\tau = 5\pi/2$, there is a supercritical bifurcation, and the period asymptotes to $P \approx \tau$. There is excellent fit between the averaging result, Eq. (38) using $n = 1$. In Fig. 3a $\tau = 7\pi/2$ and the Hopf bifurcation is subcritical. After the limit point and as $|r|$ is increased, the period asymptotes to $P \approx \tau$ and there is excellent fit with the analytical result for $n = 1$. Finally, in Fig. 4a $\tau = 8\pi/2$, the Hopf bifurcation is supercritical, though the branch is less steep than in Fig. 2a. In this case the analysis fits using $n = 2$ in Eq. (38). In all three cases the branch of solutions considered originates from a Hopf bifurcation, which occurs when the period is approximately 2π; the value of n used in Eq. (38) is such that when the period asymptotes to $P\ \tau/n$ it connects to this primary branch. We have not attempted to numerically find disconnected branches of solutions that may use different values of n for the same value of the delay τ. Finally, whether the bifurcation is supercritical or subcritical can be predicted from the value of the delay and is described in [6].

In Figs. 2-4 (b and c) we show the maximum values of x and y compared to the results of Eq. (40). The results between the numerics and analysis track well. The quality of fit is constrained by the assumptions used to derive Eq. (40), which were for large-amplitude, pulsating solutions, with vanishing pulse width. Nevertheless, the analytical results track the numerical solutions reasonably well.

Eq. (39) predicts that the solutions shown in Fig. 2-4 are stable for $r < 0$. In both Fig. 2 ($\tau = 5\pi/2$) and Fig. 4 ($\tau = 8\pi/2$) the bifurcation is supercitical and initially stable. However, there is a period-doubling bifurcation for $|r|$ not too much larger than the value of Hopf bifurcation. In Fig. 3 ($\tau = 7\pi/2$) the branch of solutions is stable for $|r|$ greater than the value at the limit point, where the period is close to τ; however, it too also has a period-doubling bifurcation for not too much larger $|r|$. Thus, the linear stability result given by Eq. (39) is initially correct but does not predict the period-doubling bifurcation. Our hypothesis is that this is because the period-doubling bifurcation occurs when the period asymptotes to a constant and the amplitudes are still increasing, thus $P'(\hat{E}_0) \approx 0$. In this case the linear-stability result becomes inconclusive.

4. Discussion

We have shown that solutions with periods that are nearly τ-periodic can be constructed for arbitrary size solutions, delay time, and feedback strength (size of the delay term). Instead of assuming that one of the above is small, we look for solutions whose period is a small deviation from the delay time (or an integer fraction there of). In this respect, the idea is similar to that used when studying the development of chaos or resonances, where the system is perturbed from a known trajectory such as a homoclinic orbit or other periodic solution [17, 18].

The control scheme of Pyragus [19] is also related to the idea considered here. In Pyragus control a system without delay is assumed to have an unstable periodic orbit of period P. Delayed feedback of delay time $\tau \approx P$ is used to stabilize the desired orbit by forcing the difference $y(t-\tau) - y(t)$ to be small.

We have demonstrated the method on a simple model that has wide application. Eq. (21) and variations thereof have been analyzed considering a number of different limiting cases. Pieroux et al. in both [10] and [7] consider the case of a small-delay term ($r \ll 1$) and $r < 1$, which corresponded to weak, delayed, negative feedback in a laser. They use averaging similar to that performed here and obtain similar but not equivalent results. In particular, the definition of the energy function is slightly different because $r \ll 1$ in their work, and the evaluation of the delay terms in the integrals cannot take advantage of nearly τ-periodicity.

In [3] (Sec. 7.2.2) Erneux uses the Poincaré-Lindsedt method, while in Taylor et al. they use multiple-time scales to describe solutions with $r = O(1)$ but are restricted to small-amplitude solutions and small deviations from Hopf bifurcation points.

Taylor et al. [6], when examining temporary immunity in diseases, consider Eq. (21) with $r > 0$ corresponding to reinjection of recovered individuals back into the susceptible class. Their map approach allows for arbitrary size solutions, delay time and feedback strength but is restricted to $r > 0$, but requires that the delayed pulse and current pulse of infectious individuals do not overlap in time and so cannot describe the solutions considered here.

What we see is that the idea and method of analysis demonstrated here does not supersede or is not necessarily superior to these other approaches. Instead, it is a matter of what information is trying to be obtained and under what conditions. The advantage of the idea in this paper is that the solution size, delay time, and feedback size are unrestricted. The limitation is that it will capture only solutions whose period is related to the delay time.

References

[1] J.K. Hale and S.M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer, New York, 1993.

[2] A.C. Fowler. Asymptotic methods for delay equations. *J. Eng. Math.*, 53:271–290, 2005.

[3] T. Erneux. *Applied Delay Differential Equations*. Springer, 2009.

[4] R.D. Driver. *Ordinary and delay differential equations*. Springer-Verlag, New York, 1977.

[5] E.V. Grigorieva, S.A. Kahchenko, N.A. Loiko, and A.M. Samson. Nonlinear dynamics in a laser with negative delayed feedback. *Physica D*, 59:297–319, 1992.

[6] M.L. Taylor and T.W. Carr. An sir epidemic model with partial temporary immunity modeled with delay. *J. Math. Biol.*, 59:841–880, 2009.

[7] D. Pieroux and T. Erneux. Strongly pulsating lasers with delay. *Phys. Rev. A*, 53:2765–2771, 1996.

[8] T.W. Carr, I.B. Schwartz, M.Y. Kim, and R. Roy. Delayed-mutual coupling dynamics of lasers: scaling laws and resonances. *SIAM J. Dynamical Systems*, 5:699–725, 2006.

[9] T.W. Carr, L. Billings, I.B. Schwartz, and I. Triandaf. Bi-instability and the global role of unstable resonant orbits in a driven laser. *Physica D*, 147:59–82, 2000.

[10] D. Pieroux, T. Erneux, and K. Otsuka. Minimal model of a class-b laser with delayed feedback: Cascading branching of periodic solutions and period-doubling bifurcation. *Phys. Rev. E*, 50:1822–1829, 1994.

[11] T. Erneux and P. Mandel. Minimal equations for antiphase dynamics in multimode lasers. *Phys. Rev. A*, 52:4137–4144, 1995.

[12] I.B. Schwartz and H.L. Smith. Infinite subharmonic bifurcation in an SEIR epidemic model. *J. Math. Bio.*, 18:233–253, 1983.

[13] J.L. Mitchell and T.W. Carr. Oscillations in an intrahost model of *Plasmodium falciparum* malaria due to cross-reactive immune response. *Bull. Math. Biol.*, 72:590–610, 2010. DOI 10.1007/s11538-009-9462-2.

[14] J.L. Mitchell and T.W. Carr. Synchronous versus asynchronous oscillations for antigenically varying *Plasmodium falciparum* with host immune response. *J. Biol. Dynamics*, 0(0):1–25, 2010.

[15] L.F. Shampine and S. Thompson. Solving ddes in Matlab. *J. Appl. Numer. Math.*, 37:441–458, 2001. See also http://www.cs.runet.edu/ thompson/webddes/index.html.

[16] K. Engelborghs, T. Luzyanina, and G. Samaey. DDE-BIFTOOL v.2.00 user manual: a Matlab package for bifurcation analysis of delay differential equations. Technical Report TW-330, Dept. of Computer Science, K.U.Leuven, Leuven, Belgium, 2001.

[17] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer-Verlag, 1983.

[18] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems*. Springer-Verlag, New York, 1990.

[19] K. Pyragas. Continuous control of chaos by self-controlling feedback. *Phys. Lett. A*, 170:421–428, 1990.

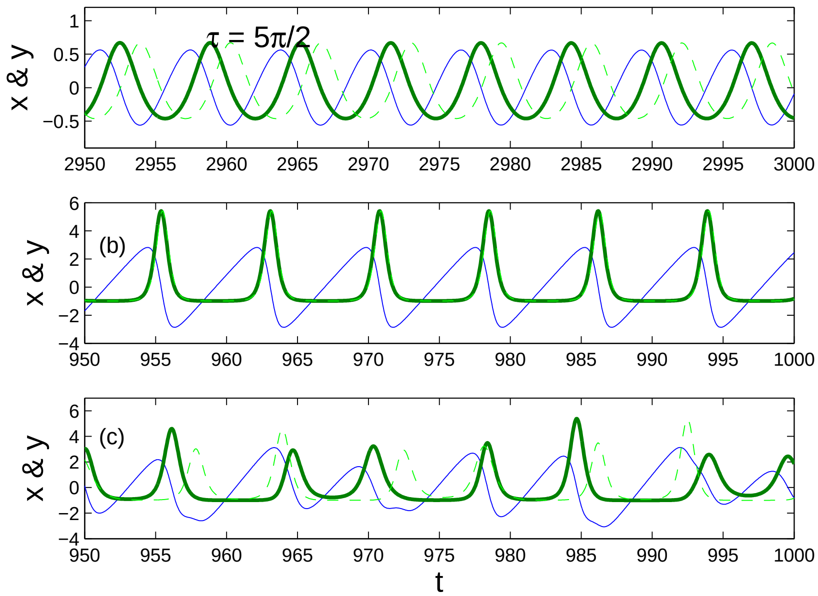

Figure 1: Time series [15] for x (thin solid), y (thick solid), and $y(t-\tau)$ (dashed) for $\tau = 5\pi/2$, $\epsilon = 0.014$, $a = 1.41$ and $b = 0.71$. (a) $r = -0.02$. (b) $r = -0.13$. (c) $r = -0.40$.

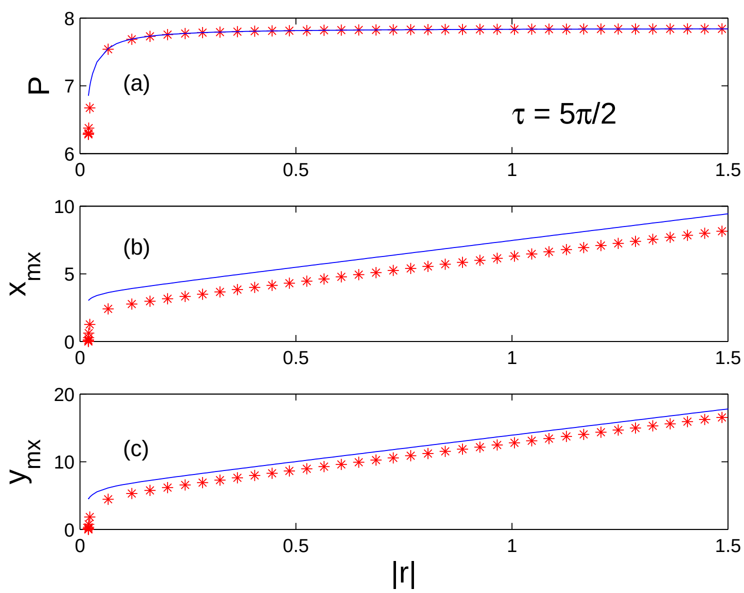

Figure 2: Bifurcation diagrams for $\tau = 5\pi/2 \approx 7.85$ ($n = 1$): analysis (solid line) vs. numerical continuation (stars). The Hopf bifurcation occurs when $|r| = 0.020$ and a period-doubling bifurcation occurs when $|r| = 0.145$. ($\epsilon = 0.014$, $a = 1.41$ and $b = 0.71$.)

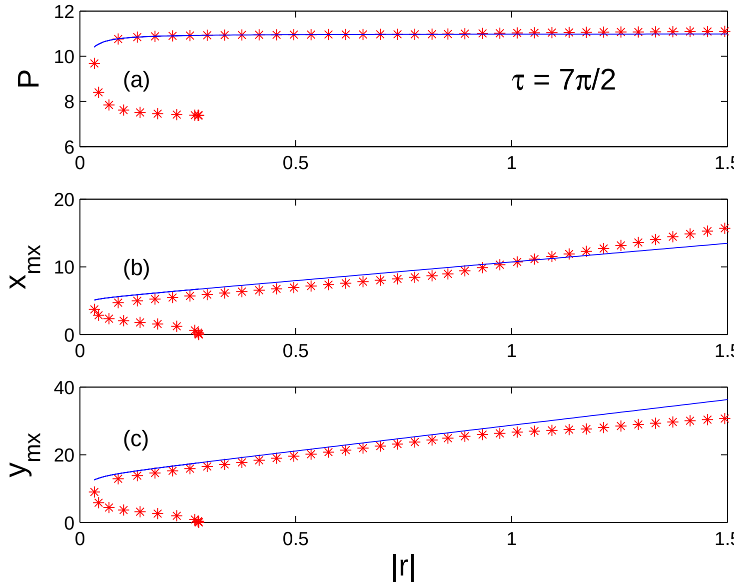

Figure 3: $\tau = 7\pi/2 \approx 11.00$ ($n = 1$): analysis (solid line) vs. numerical continuation (stars). The Hopf bifurcation occurs when $|r| = 0.275$, the limit-point at $|r| = 0.034$ and the period-doubling bifurcation at $|r| = 0.089$. ($\epsilon = 0.014$, $a = 1.41$ and $b = 0.71$.)

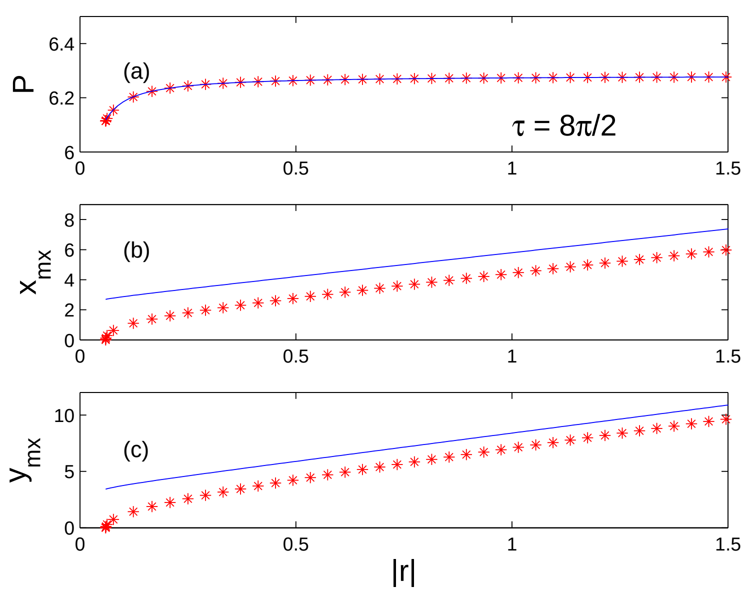

Figure 4: $\tau = 8\pi/2 \approx 12.57$ ($n = 2$): analysis (solid line) vs. numerical continuation (stars). The Hopf bifurcation occurs when $|r| = 0.060$ and the period-doubling bifurcation at $|r| = 0.192$. ($\epsilon = 0.014$, $a = 1.41$ and $b = 0.71$.)