Numerical Solution of Integral Equations in Solidification and Laser Melting

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Abstract
Many applications in solidification or melting are described by a two-phase Stefan problem with elliptical geometry. Here, we model a scanning laser as it heats and melts a metal surface and compute the evolution of the boundary between the solid and liquid material. Using Green’s representation theorem, the heat equation is reformulated as a system of nonlinear integro-differential equations in time. The unknown fluxes and radius of the solid-liquid interface are determined from this system and the Stefan condition. The integral equation is discretized with the Nyström method which leads to an efficient time stepping scheme to determine the position and velocity of the interface with given melting temperature and material properties. We first tested this method with the assumption of spherical symmetry on a problem with a known analytical solution, and are currently extending it to general geometries. Examples and numerical results are presented that demonstrate the effectiveness of the method.

Introduction
Laser heat sources are used in many methods of material production. A laser source can be easily controlled and has a concentrated heating area. Some applications include:
- Laser Engraving: used to develop metal plates for printing
- Laser Cutting: used for manufacturing
- Laser Ablation: converts solid material into vapor for removing excess material
- Laser Annealing: a method of strengthening a sheet of metal by scanning a laser evenly over the surface

Geometry of Problem
We consider a material which occupies all of $\mathbb{R}^n$, where the solid and liquid phase are separated by a smooth interface $\Gamma(t)$, whose size and location depend on time. We assume that the problem is governed by the heat equation where $u$ is the interface temperature:

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + q.$$  

The variables are defined as:
- $q$: Volume heat source,
- $u_0$: Initial condition,
- $\kappa$: Boundary condition,
- $\nu$: Velocity of the interface.

The constant $\alpha$ is the thermal diffusivity, $k$ is conductivity, and the subscripts ‘s’ and ‘l’ indicate the solid and liquid phase.

Integral Equation Formulation
Green’s representation formula for (1) can be derived and a solution of (1) satisfies

$$u(x,t) = \int G(x-y,t)q(y,t)dy,$$

where $V$ and $K$ are the single and double layer potentials defined by

$$G(x-y,t) = \int_0^t \frac{1}{4\pi(t-s)} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) ds,$$

$$K(x-y,t) = \int_0^t \frac{1}{4\pi(t-s)} \left(\frac{1}{t-s} - \frac{1}{t}\right) \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) ds.$$

The propagation of the interface is now

$$v = k g_{\nu} = k g_{\tau} + \frac{\partial u}{\partial \nu} \bigg|_{\nu_0}.$$ 

Results
The computed radius using $N = 100$ up to $N = 640$ time steps. Note that the curves for different time step sizes overlap.

Spherical Symmetry
In this simplified problem we assume the interface and source terms have spherical symmetry, and

$$q(0, t) = 0, u_0 = 0.$$ 

Because of the symmetry the heat equation can be written in spherical coordinates as

$$u_t = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = q.$$ 

The radius $r$ is spherical with $r$ as its radius. The surface integrals for the single and double layer operators are done analytically. We obtain an integral equation in time for the unknown radius. The time integrals are replaced by a singularly corrected quadrature rule, and we solve equation (2) in the solid and liquid phase for each time step. We discretize the operators with the Nyström Method, using a singularity corrected version of the quadrature rule.

$$K_i[u_i(t_i)] = \frac{1}{\varphi_i} \sum_{j \neq i} N_i j [f_j(t_j)] + \mu_k (u_{\nu_0} - u_{\nu_0}) (x_i, t_i).$$

The $\mu_j$ are determined to correct the singularities at $t = \tau$ and $t = \tau^2 = 0$. Some of these corrective weights can be computed in advance for efficiency since they do not depend on the radius of the interface.

Gibbs-Thompson Equation
The effects of surface tension are modeled through the Gibbs-Thompson equation, given by

$$u_l(t) = u_s(t) + \alpha K x.$$ 

We define $\alpha$ as

$$\alpha = \frac{\kappa}{\nu_0} = \frac{\kappa}{\nu_0} + \alpha u_s,$$

which satisfies the homogeneous heat equation. Then our system becomes

$$\frac{1}{\nu_0} \frac{\partial G}{\partial \nu} = \frac{\kappa}{\nu_0} \frac{\partial G}{\partial \nu},$$

where

$$\nu_0 = \frac{\partial u}{\partial \nu} \bigg|_{\nu_0}.$$ 

The propagation of the interface is now

$$v = k g_{\nu} = k g_{\tau} + \frac{\partial u}{\partial \nu} \bigg|_{\nu_0}.$$ 

Discretization of the 2D Integral Equation
We are currently looking at this problem in two dimensions, without the assumption of spherical symmetry. Before we can address the complications of a moving heat source, we must first look at a moving geometry. We developed a method to discretize an evolving surface of a general geometry over time and solve the heat equation for either $u_0$ or $u_l$.

The base of the surface is discretized with points

$$\{x_1, x_2, \ldots, x_j, \ldots, x_N, x_{N+1}, \ldots, x_N\}$$

at each time $t$, where $N$ is the spatial mesh size. To test the accuracy of the method we use a prescribed geometry and omit the initial conditions and source terms. Equation (2) becomes

$$\frac{1}{\varphi_i} \sum_{j \neq i} N_i j [f_j(t_j)] - \alpha K x_i (x_i, t_i) + \alpha K x_i (x_i, t_i)$$

and the double layer operator is similarly defined.

Results and Errors
The spatial discretization is illustrated above. At each time step $t$, each point $x$ is determined with information from all other points at $t$, and all previous times.

Current Work
We are currently extending this problem to solve for an unknown surface of general geometry in three dimensions with a moving laser heat source. We plan to use the Boundary Element Method on this 3D parabolic problem.

References