Blunt and Ballistic Impacts On Human Head Models: An Analytical And Numerical Study

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BLUNT AND BALLISTIC IMPACTS ON HUMAN HEAD MODELS:
AN ANALYTICAL AND NUMERICAL STUDY

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BLUNT AND BALLISTIC IMPACTS ON HUMAN HEAD MODELS:
AN ANALYTICAL AND NUMERICAL STUDY

A Dissertation Presented to the Graduate Faculty of
Bobby B. Lyle School of Engineering
Southern Methodist University
in
Partial Fulfillment of the Requirements
for the degree of
Doctor of Philosophy
with a
Major in Mechanical Engineering
by

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Head injuries, as a leading cause of death, have become a major health care issue for civilians and soldiers. There has been an urgent need to understand mechanisms of such injuries. The objective of this dissertation research is to study some head injuries and related mechanisms using analytical and computational models.

In Chapter 2, a new analytical (non-linear) model for the impact of a solid sphere on a fluid-filled spherical shell is developed by including the stress wave propagation effect in addition to the Hertzian contact deformations and the shell membrane and bending actions. A simplified (linearized) model incorporating the elastic energy loss due to the stress wave propagation is then formulated by using a linear force-deflection relation, which leads to a closed-form expression for the impact duration. By directly applying the newly obtained non-linear and linearized models, three representative problems simulating blunt head impacts are analyzed.

In Chapter 3, head injuries induced by golf ball impacts are studied through computational modeling. A full human body model and a three-piece golf ball model are integrated to construct a new finite element model, and LS-DYNA is employed to perform simulations. To assess head injury risks, the impact force, von Mises stress, pressure, and
first principal strain are computed in the current model and compared with existing experimental and simulation data.

In Chapter 4, a finite element model is developed for an Advanced Combat Helmet (ACH) and validated against the experimental data obtained at the Army Research Laboratory. Ballistic head impact simulations are then performed for an ACH placed on a ballistic dummy head form embedded with clay as specified in the current ACH testing standard by using the validated helmet model.

In Chapter 5, new constitutive models for hyperelastic materials are proposed using the upper triangular decomposition of the deformation gradient tensor, which are simpler than those based on the invariants of the right and left Cauchy-Green deformation tensors. Two examples are provided to illustrate applications of the new constitutive models, which can be adopted and further modified to simulate brain tissues.
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DEDICATION

This dissertation is dedicated to my father, my mother, my advisor Prof. Xin-Lin Gao.
Chapter

1. INTRODUCTION

Frequent collisions in contact sports and increasing use of improvised explosive devices (IDEs) in war zones have led to high risks of head injuries, such as traumatic brain injury (TBI). Such injuries can cause death and disability and have become a serious health care issue. Efforts have been made to study mechanisms of head injuries induced by blunt and ballistic impacts through using computational models (e.g., Hardy et al., 1994; Willinger et al., 1999, Zhang et al., 2001; Kleiven and Hardy, 2002; Horgan and Gilchrist, 2003; Takhounts et al., 2008).

However, most finite element based head models require extensive computation and tend to be time consuming. In contrast, analytical models can give a first estimate in a much shorter time. Some early models have been proposed by modeling the human head as a fluid-filled sphere (e.g., Engin, 1969; Kenner and Goldsmith; 1973). Among civilians, sports-related head injuries have not been well studied. In military conflicts, behind helmet blunt trauma (BHBT) induced by ballistic impacts is a serious injury type experienced by soldiers. Most of the constitutive models for brain tissues are based on the polar decomposition of the deformation gradient tensor, which tend to be tedious. Hence, simpler constitutive models are needed to simulate brain tissues. These motivated the current dissertation research.
In Chapter 2, a new analytical (non-linear) model and a simplified (linearized) model for blunt head impacts are developed. In Chapter 3, a computational model for head injuries induced by a golf ball strike is provided. In Chapter 4, a finite element-based model for ballistic impacts on a helmeted head form is presented. In Chapter 5, the constitutive models for hyperelastic materials based on the upper triangular decomposition of the deformation gradient tensor are proposed.
References


Chapter

2. ANALYTICAL MODELS FOR BLUNT IMPACTS

2.1 Introduction

Blunt impacts on human head can result in traumatic brain injuries (TBI). TBI is regarded as a signature injury in military field (e.g., O'Neil et al., 2013; Kulkarni et al., 2013; Young et al., 2015). Among civilians, TBI can be induced by sports and automobile accidents. TBI remains to be the most prevalent cause of death in adults aged less than 45 years and is also the primary reason for long-term disability. Among the people aged 65 and over, TBI is the second leading cause (after cancer) of death (e.g., Meaney et al., 2014).

There have been continuous efforts in developing models for impact-induced head injuries (e.g., Hardy et al., 1994; Willinger et al., 1999; Goldsmith, 2001; Goldsmith and Monson, 2005; Mao et al., 2013; Tse et al., 2014; Cotton et al., 2016; Li et al., 2016). In early models for head impacts, a fluid-filled spherical shell is used to represent a human head in order to simplify mathematical or experimental studies on complex dynamic responses of an impacted head (e.g., Engin and Liu, 1970; Kenner and Goldsmith, 1973; Khalil and Viano, 1982; Misra and Chakravarty, 1982; Charalambopoulos et al., 1998; El Baroudi et al., 2012). Most of recently proposed head impact models are based on the finite element method, which include anatomical details of a human head and can provide a wealth of information about the stress distribution, wave propagation and energy
dissipation in various parts of the head. However, such finite element models often require extensive computationas and tend to be time consuming. In contrast an analytical models can give a first estimate or a benchmark solution in a much shorter time. Hence, it is very desirable to have validated analytical head impact models.

The earliest analytical models for head impacts were proposed by Anzelius (1943) and Güttlinger (1950) by treating a human head as a rigid spherical shell filled with an inviscid fluid. These models were improved in Engin (1969) by regarding a head as a fluid-filled elastic spherical shell subjected to a radial impulsive load described by a Dirac delta function. The model of Engin (1969) was modified in Kenner and Goldsmith (1972) by extending it to loadings of finite duration. Young (2003) developed an analytical model for blunt head impacts through studying the impact of a solid sphere on a fluid-filled spherical shell based on the Hertz contact theory (1882) and the Reissner spherical shell theory (1946). Closed-form expressions for the impact duration and maximum acceleration in the shell (head) were obtained for the first time in Young (2003) based on some approximations. The model of Young (2003) was extended in Heydari and Jani (2010) by replacing the spherical shell with an ellipsoidal shell. In Mansoor-Baghaei and Sadegh (2011, 2015), closed-form solutions were derived for impacts of a spherical or an ellipsoidal shell on a fixed flat barrier (as an elastic half space) by using a linear approximation. However, the elastic energy loss due to the stress wave propagation (e.g., Hunter, 1957; Reed, 1985; Boettcher et al., 2017a) is not considered in these recent analytical models. This motivated the study presented here.

In this chapter, an analytical (non-linear) model for blunt head impacts is developed in Section 2.2 by including the stress wave propagation effect in addition to the Hertzian
contact deformations of the sphere-shell system and the membrane and bending actions in the shell. A simplified (linearized) model incorporating the elastic energy loss due to the stress wave propagation is then formulated by using a linear force-deflection relation to approximate the non-linear relation in the classical Hertz contact theory, which gives a closed-form expression for the impact duration. It is shown in Section 2.3 that the non-linear model reduces to the model of Young (2003) and the linearized one recovers the model of Mansoor-Baghaei and Sadegh (2011) when the stress wave propagation effect is not considered. In Section 2.4, three representative problems simulating blunt head impacts are analyzed by directly applying the two newly developed models. The numerical results predicted by the current models are plotted and compared with those given by the two existing models and with available finite element simulation results. This chapter concludes in Section 2.5 with a summary.

2.2 Formulation

Consider the impact of a solid sphere of radius $R_1$ traveling at a velocity of $v_1$ on a fluid-filled spherical shell with a thickness of $h$ and an outer radius $R_2$ moving at a velocity of $v_2$, as shown in Fig. 2.1. It is assumed that the impact is elastic and the deformation is localized in a small area near the collision site.
During the impact, three types of events are taking place: (a) the deformations of the two elastic bodies according to the Hertz contact theory; (b) the deflection of the fluid-filled spherical shell due to the membrane and bending actions; (c) the stress wave propagation in the shell. Methods for analyzing each of these three phenomena are briefly discussed next.

2.2.1 Hertz contact

For the problem of an elastic solid sphere impacting on an elastic spherical shell, the contact theory of Hertz (1882) (e.g., Johnson, 1982; Zhou et al., 2011; Gao and Zhou, 2013) gives the force-deflection relation as (e.g., Johnson, 1985; Maugis, 2000)

\[ F = k_H \delta_H^{3/2}, \quad (2.1) \]

where \( F \) is the applied (contact) force, \( \delta_H \) is the mutual approach of the centers of the two spherical bodies, and \( k_H \) is the Hertz contact stiffness defined by

\[ k_H = \frac{4}{3} R^{1/2} E, \quad (2.2) \]
where $R$ and $E$ are, respectively, the effective radius and Young’s modulus given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \frac{1}{E} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2},$$

(2.3)

in which $E_1$ and $\nu_1$ are, respectively, Young’s modulus and Poisson’s ratio of the solid sphere, and $E_2$ and $\nu_2$ are, respectively, Young’s modulus and Poisson’s ratio of the spherical shell.

Note that Eq. (2.1) is valid for smooth and frictionless contacts with small deformations and satisfying the conditions: $\delta_h/R \ll 1$ and $a/R \ll 1$, where $a$ is the radius of the maximum impact area (see Fig. 2.2).

2.2.2 Shell response

For a thin hollow spherical shell loaded by the applied force $F$, the force-deflection relation incorporating both the membrane and bending actions in the shell can be approximated by

$$F = k_{sh} \delta_{sh},$$

(2.4)

where $k_{sh}$ is the shell stiffness given by (e.g., Reissner, 1946; Young, 2003; Lazarus et al., 2012)

$$k_{sh} = \frac{8D}{L^2} = \frac{4E_s h^2}{R_2 \sqrt{3(1 - \nu_2^2)}},$$

(2.5)

in which $D$ is the bending stiffness of the shell expressed as

$$D = \frac{E_s h^3}{12(1 - \nu_2^2)},$$

(2.6)

and $L_b$ is a characteristic length scale defined by (Lazarus et al., 2012)
This approximation is accurate for spherical shells with \( h/R < 0.2, \ a/R \ll 1 \) and \( \delta_h/R \ll 1 \). It has been stated (Young, 2003) that Eqs. (2.4) and (2.5) are also valid for a spherical shell filled with a compressible or an incompressible fluid, since the bulk modulus of the fluid has been found to have no effect on the shell stiffness. Young and Morfey (1998) conducted an extensive study using a finite element model for the head as a fluid-filled spherical shell and found that the pressure response in the brain (fluid) is not sensitive to changes in the fluid bulk modulus over a wide range of values (from 218 MPa to 21.8 GPa). This study supports the above statement of Young (2003). Hence, Eqs. (2.4) and (2.5) are adopted to describe the membrane and bending effects in the fluid-filled spherical shell in the current study (see Fig. 2.1).

Note that \( \delta_h \) given by Eqs. (2.4) and (2.5) is the deflection of the thin shell arising from the membrane and bending actions induced by the impact force, which is only part of the total deformation (see Eqs. (2.11) and (2.15a)) in the radial direction and tends to be negligibly small compared to \( \delta_H \) (see Fig. 2.4(a)). Hence, the numerical error resulting from the assumption of Young (2003) that excludes the influence of the bulk modulus of the fluid should be insignificant. However, the effect of the fluid on the impact, which is incorporated in the current model through the linear momentum transfer (see Eq. (2.10)) and energy conversion (see Eq. (2.11)), can still be significant, depending on the fluid density \( \rho_f \) and shell inner radius \( R_f \) which are directly related to the mass of the fluid-filled shell \( m_z = \frac{4\pi}{3} \left( \rho_f R_f^3 + \rho_2 \left( R_z^3 - R_f^3 \right) \right) \).
2.2.3 Stress wave propagation

Three types of stress waves are generated during an elastic impact on an elastic half space, which include a compression (longitudinal) wave (P), a shear (transverse) wave (S) and a surface (Rayleigh) wave (e.g., Miller and Pursey, 1955). The propagation of these stress waves transports part of the impact energy away from the contact zone, which cannot be recovered in the impact duration. This is known as the elastic energy loss and has been extensively studied (e.g., Hunter, 1957; Reed, 1985; Weir and Tallon, 2005; Bao and Yu, 2015a, b; Farin et al., 2016; Boettcher et al., 2017a).

When a spherical shell is impacted, there are also P, S and Rayleigh waves traveling in the shell (e.g., Shah et al., 1969; Rossikhin et al., 2011). It has been found that the front of each stress wave propagating in a spherical shell is toroidal (e.g., Towfighi and Kundu, 2003; Yu et al., 2013), as schematically shown in Fig. 2.2.

Fig. 2.2 Stress wave propagation in a spherical shell during an impact.
Based on the model provided in Weir and Tallon (2005), which considers the impact of two identical spherical particles, the total elastic energy loss due to the stress wave propagation during the impact has been obtained as

\[ E_{\text{loss}} = \frac{1}{2} m v^2 \lambda, \quad (2.8) \]

where

\[ \lambda = 1.7244 \sqrt{2(1+\nu^2)} \left( \frac{\nu}{c_0} \right)^{3/5} \quad (2.9) \]

is the energy loss ratio (i.e., \( \lambda \equiv E_{\text{loss}}/(\frac{1}{2} m v^2) \)), \( m \) and \( v \) are, respectively, the reduced mass and initial relative velocity (see Eq. (2.13)), \( c_0 = \sqrt{E_2/\rho_2} \) is the longitudinal wave speed in an elastic bar made from the shell material, and \( \rho_2 \) is the density of the shell material. Experimental results are provided in Weir and Tallon (2005) to validate their model.

There are other models for estimating \( E_{\text{loss}} \), which are based on the work of Miller and Pursey (1955) for a finite circular disk vibrating normally on the surface of an isotropic elastic half-space (e.g., Hunter, 1957; Reed, 1985; Hayakawa and Kuninaka, 2004; Wu et al., 2005; Argatov, 2008; Farin et al., 2016; Boettcher et al., 2017a) or on the study of Zener (1941) for the impact by a sphere on an infinitely large thin plate (e.g., Tillet, 1954; Fisher, 1975; Mueller et al., 2015; Farin et al., 2016; Boettcher et al., 2017b).

For the more general case of a solid sphere impacting on a fluid-filled spherical shell considered in the current study, no exact energy loss formula has been published. This necessitates the use of approximate expressions. As a first approximation, Eqs. (2.8) and (2.9) are adopted, which are based on the model of Weir and Tallon (2005). Owing to the similarities in spherical shapes and finite sizes of two impacting bodies, these formulas are
believed to be more suitable for the current case than those based on the work of Miller and Pursey (1955) for impact on a semi-infinite elastic half-space or based on the study of Zener (1941) for impact on an infinitely large thin plate. Nevertheless, the power-law relation of $\lambda \propto (v / c_0)^{3/5}$ in Eq. (2.9) is the same as that identified in Hunter (1957), Wu et al. (2005), Argatov (2008) and Boettcher et al. (2017a) using different approaches, even though the coefficient differs in each case. In addition, the use of Eqs. (2.8) and (2.9) for fluid-filled spherical shells is supported by the finding of Young and Morfey (1998) and Young (2003) that the bulk modulus of the fluid has no effect on the stiffness of the shell.

2.2.4 New model considering the elastic energy loss due to the stress wave propagation

A new model for the impact of a solid sphere on a fluid-filled spherical shell shown in Fig. 2.1 is developed here by combining the effects of the Hertzian contact, shell membrane and bending actions, and stress wave propagation discussed in Sections 2.2.1–2.2.3. It is assumed that the contact area is obtained from the Hertz theory.

From the linear momentum conservation (e.g., Maugis, 2000; Young, 2003),

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_s,$$

where $m_1$ and $m_2$ are, respectively, the masses of the solid sphere and fluid-filled spherical shell, $v_1$ and $v_2$ are, respectively, the initial velocities of the mass centers of the solid sphere and fluid-filled spherical shell, and $v_s$ is the velocity of the mass center of the sphere-shell system after the contact. Note that $v_1$ and $v_2$ are collinear but in the opposite directions, as shown in Fig. 2.1. Also, the mass of the fluid $m_f = \frac{4\pi}{3} \rho_f R_s^3$ is part of $m_2$, where $\rho_f$ is the mass density of the fluid.
According to the principle of energy conservation,

\[
\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} (m_1 + m_2) v_s^2 + \frac{1}{2} k_{sh} \delta_{sh}^2 + \frac{2}{5} k_H \delta_H^{5/2} + \frac{1}{2} m \left( \frac{d\delta}{dt} \right)^2 + E'_{\text{loss}},
\]

(2.11)

which can be rewritten as

\[
\frac{1}{2} m v^2 = \frac{1}{2} k_{sh} \delta_{sh}^2 + \frac{2}{5} k_H \delta_H^{5/2} + \frac{1}{2} m \left( \frac{d\delta}{dt} \right)^2 + E'_{\text{loss}},
\]

(2.12)

where \(d\delta/dt\) is the mutual approach velocity, and \(E'_{\text{loss}}\) is the energy loss due to the stress wave propagation during the impact, and \(m\) and \(v\) are, respectively, the reduced (or effective) mass and initial relative velocity given by

\[
m = \frac{m_1 m_2}{m_1 + m_2}, \quad v = v_1 - v_2.
\]

(2.13)

Note that in reaching Eq. (2.12) use has been made of Eq. (2.10).

\(E'_{\text{loss}}\) in Eq. (2.12) is a function of time. The total elastic energy loss \(E_{\text{loss}}\), introduced in Eq. (2.8) is related to \(E'_{\text{loss}}\) through \(E_{\text{loss}} = E'_{\text{loss}}(T_c)\), where \(T_c\) is the impact duration. When the shell is sufficiently large such that the stress waves reflected or refracted at boundaries or interfaces cannot travel back to the impact site during the period of impact, \(E_{\text{loss}}\) is a constant depending only on the initial relative velocity and material properties. This is the case for all leading energy loss models including those of Zener (1941) and Hunter (1957), as summarized in Boettcher et al. (2017a, b) and reviewed earlier in Section 2.2.3.

The second last term on the right hand side of Eq. (2.11) represents the kinetic energy associated with the mutual approach of the two spherical bodies, whose centers of mass are moving towards each other with the velocity \(d\delta/dt\).

It should be mentioned that the last two terms on the right hand side of Eq. (2.12) are
not present in the analysis of Young (2003), since the stress wave propagation effect is not included and only the energy conservation at the maximum compression (i.e., $\delta = \delta_{\text{max}}$) is considered in Young (2003).

In addition, if the second and last terms on the right hand side of Eq. (2.11) are eliminated (i.e., without considering the membrane and bending actions in the shell and the energy loss), Eq. (2.11) will be reduced to the energy conservation equation for the Hertz contact (e.g., Maugis, 2000).

Combining Eqs. (2.1) and (2.4) gives

$$\delta_{sh} = \frac{k_H}{k_{sh}} \delta_H^{3/2}. \quad (2.14)$$

Then, it follows from Eq. (2.14) that the total mutual approach and its time derivative during the impact are

$$\delta = \delta_H + \delta_{sh} = \delta_H \left(1 + \frac{k_H}{k_{sh}} \delta_H^{1/2}\right), \quad (2.15a)$$

$$\frac{d\delta}{dt} = \frac{d\delta_H}{dt} + \frac{d\delta_{sh}}{dt} = \left(1 + \frac{3}{2} \frac{k_H}{k_{sh}} \delta_H^{1/2}\right) \frac{d\delta_H}{dt}. \quad (2.15b)$$

When $\delta$ reaches its maximum value, $d\delta/dt = 0$, $E_{\text{loss}}' = E_{\text{loss}}'(t_c) = E_{\text{loss}}'/2$, and Eq. (2.12) becomes, with the help of Eqs. (2.14) and (2.15a),

$$\frac{1}{2} m v^2 - \frac{1}{2} E_{\text{loss}} = \frac{1}{2} \frac{k_H^2}{k_{sh}} \delta_{\text{max}}^3 + \frac{2}{5} k_H \delta_{\text{Hmax}}^{5/2}, \quad (2.16)$$

where $\delta_{\text{Hmax}}$ is the maximum value of $\delta_H$. For a given expression of $E_{\text{loss}}$, Eq. (2.16), which is a non-linear algebraic equation, can be solved numerically to obtain $\delta_{\text{Hmax}}$. 

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For the elastic energy loss in Eq. (2.8) adopted in the current study, Eq. (2.16) changes to

\[
(1 - \frac{1}{2}\lambda)mv^2 = \frac{k_H}{k_{sh}}\delta_H^{3}\max + \frac{4}{5}k_H\delta_H^{5/2}\max,
\]

(2.17)

where \(\lambda\) is given in Eq. (2.9), \(m\) and \(v\) are defined in Eq. (2.13), and \(k_H\) and \(k_{sh}\) are, respectively, listed in Eqs. (2.2) and (2.5). By solving Eq. (2.17) numerically, \(\delta_H^{\max}\) will be determined.

To find the impact duration, Eq. (2.12) can be rewritten as

\[
\left(\frac{d\delta}{dt}\right)^2 + \frac{2E_{\text{loss}}'}{m} = v^2 - \frac{k_{sh}}{m}\delta_H^2 - \frac{4}{5}k_H\delta_H^{5/2}.
\]

(2.18a)

As a first approximation, consider \(E_{\text{loss}}' \approx \frac{E_{\text{loss}}}{T_c}\) and take \(\frac{2t}{T_c} \approx 1\) for a short-duration impact. Then, Eq. (2.18a) becomes, upon using these approximations and Eqs. (2.14) and (2.15b),

\[
dt = \frac{\left(1 + \frac{3}{2}\frac{k_H}{k_{sh}}\delta_H^{1/2}\right)d\delta_H}{\sqrt{v^2 - \frac{E_{\text{loss}}}{m} - \frac{1}{m}\left(\frac{k_H}{k_{sh}}\delta_H^3 + \frac{4}{5}k_H\delta_H^{5/2}\right)}}.
\]

(2.18b)

Integrating Eq. (2.18b) gives the half impact duration \(t_c\) as

\[
t_c = \int_{\delta_H^{\min}}^{\delta_H^{\max}} \frac{\left(1 + \frac{3}{2}\frac{k_H}{k_{sh}}\delta_H^{1/2}\right)d\delta_H}{\sqrt{v^2 - \frac{E_{\text{loss}}}{m} - \frac{1}{m}\left(\frac{k_H}{k_{sh}}\delta_H^3 + \frac{4}{5}k_H\delta_H^{5/2}\right)}}.
\]

(2.19)

The impact duration \(T_c\) can then be obtained from Eq. (2.19) as
where \( \delta_{Hmax} \) is defined in Eq. (2.16). When \( E_{loss} \) is known, the definite integral in Eq. (2.20) can be evaluated numerically to compute \( T_c \).

For the elastic energy loss identified in Eq. (2.8), Eq. (2.20) becomes

\[
T_c = 2\int_0^1 \frac{\delta_{Hmax} \left( 1 + \frac{3}{2} \frac{k_H}{k_{sh}} \delta_{Hmax}^{1/2} x^{1/2} \right)}{\sqrt{v^2 - \frac{E_{loss}}{m} - \frac{1}{m} \left( \frac{k_H^2}{k_{sh}} \delta_{Hmax}^3 x^3 + \frac{4}{5} \frac{k_H}{k_{Hmax}} \delta_{Hmax}^{5/2} x^{5/2} \right)}} \, dx, \tag{2.21}
\]

where \( \lambda \) is listed in Eq. (2.9).

For given material properties \((E_1, \nu_1, E_2, \nu_2)\), geometrical parameters \((R_1, R_2, h)\), initial velocities \((v_1, v_2)\), masses \((m_1, m_2)\), and energy loss \((E_{loss} or \lambda)\), \( \delta_{Hmax} \) and \( T_c \) can be obtained from Eqs. (2.16) and (2.20) or Eqs. (2.17) and (2.21), respectively. The maximum contact force \( F_{max} \) and the maximum acceleration \( a_{shmax} \) in the shell can then be readily determined from Eq. (2.1) and Newton’s second law as

\[
F_{max} = k_H \delta_{Hmax}^{3/2}, \tag{2.22a}
\]

\[
a_{shmax} = \frac{k_H \delta_{Hmax}^{3/2}}{m_2}. \tag{2.22b}
\]

Note that \( \delta_{Hmax} \) and \( T_c \) cannot be obtained in closed-form expressions from Eqs. (2.16) and (2.20) or Eqs. (2.17) and (2.21) owing to the non-linearity involved. However, closed-form formulas for \( \delta_{Hmax} \) and \( T_c \) can be derived by developing a simplified (linearized) model, as shown next.
2.2.5 Simplified model incorporating the stress wave propagation effect

The non-linearity of the new model presented in Section 2.2.4 arises from the Hertz contact theory, which provides a non-linear force-deflection relation (see Eq. (2.1)) that is used in the formulation (see Eq. (2.14)). However, this force-deflection relation can be linearized to simplify contact analyses (e.g., Hunt and Crossley, 1975; Yang and Sun, 1985; Flores and Lankarani, 2016). A method for linearizing the Hertz contact law has recently been used by Mansoor-Bagheri and Sadegh (2011, 2015) to derive analytical solutions for impacts of a spherical or ellipsoidal shell on a stationary flat barrier. This method is adopted in the current study.

Following Mansoor-Bagheri and Sadegh (2011, 2015), a linearized effective stiffness \( k_L \) for the Hertz contact is introduced such that the non-linear force-deflection relation in Eq. (2.1) can be approximately represented by

\[
F = k_L \delta_H,
\]

where \( k_L \) is determined by equating the strain energy based on the non-linear relation in Eq. (2.1) and that based on the linear relation in Eq. (2.23). That is,

\[
U_{\text{Nonlinear}} = U_{\text{Linear}}; \\
U_{\text{Nonlinear}} = \int_0^{\delta_{H_{\text{max}}}} k_H \delta_H^{3/2} d\delta_H = \frac{2}{5} k_H (\delta_{H_{\text{max}}})^{5/2}, \quad U_{\text{Linear}} = \int_0^{\delta_{H_{\text{max}}}} k_L \delta_H d\delta_H = \frac{1}{2} k_L (\delta_{H_{\text{max}}})^2,
\]

which gives

\[
k_L = \frac{4}{5} k_H (\delta_{H_{\text{max}}})^{1/2},
\]

where \( \delta_{H_{\text{max}}}^{L} \) is the maximum value of \( \delta_H \) based on the linearized Hertz contact relation, which is yet to be determined. Note that in Mansoor-Bagheri and Sadegh (2011, 2015)
\( \delta_{H_{\text{max}}} \) is taken to be the value given in Johnson (1972) based on the classical Hertz contact theory (without considering the membrane and bending actions in the shell and the elastic energy loss due to the stress wave propagation). The value provided in Johnson (1972), as an upper bound of \( \delta_{H_{\text{max}}} \), can be directly obtained from Eq. (2.12) by setting \( k_{sh} = 0 \) and \( E_{loss}^e = 0 \). That is, with \( d\delta dt = 0, k_{sh} = 0 \) and \( E_{loss}^e = 0 \), Eq. (2.12) immediately yields the maximum value of \( \delta_H \) based on the classical Hertz contact theory as

\[
\delta_{H_{\text{max}}}^C = \left( \frac{5}{4} \frac{mv^2}{k_H} \right)^{2/5}.
\]  

(2.26)

The use of this maximum value in Mansoor-Baghaei and Sadegh (2011, 2015) suggests that their models should provide an upper bound for the maximum contact force \( F_{\text{max}} \) (see Eq. (2.22a)) in each respective case.

From Eqs. (2.4) and (2.23),

\[
\delta_{sh} = \frac{k_L}{k_{sh}} \delta_H .
\]

(2.27)

Then, the total mutual approach and its time derivative during the impact become, upon using Eq. (2.27),

\[
\delta = \delta_H + \delta_{sh} = \delta_H \left( 1 + \frac{k_L}{k_{sh}} \right),
\]

(2.28a)

\[
\frac{d\delta}{dt} = \frac{d\delta_H}{dt} + \frac{d\delta_{sh}}{dt} = \left( 1 + \frac{k_L}{k_{sh}} \right) \frac{d\delta_H}{dt} .
\]

(2.28b)

Clearly, Eqs. (2.27) and (2.28a, b) are all linear in \( \delta_H \), which differs from that exhibited by Eqs. (2.14) and (2.15a, b).
When $\delta=\delta_{\text{max}}$, $d\delta/dt = 0$, $E_{\text{loss}}' = E_{\text{loss}}'(t_c) = E_{\text{loss}}' / 2$, and Eq. (2.12) gives, with the help of Eqs. (2.25), (2.27) and (2.28a),

$$\frac{1}{2} m v^2 = \frac{2}{5} \left( 1 + \frac{k_L}{k_{sh}} \right) k_H (\delta_{\text{Hmax}}^L)^{5/2} + \frac{1}{2} E_{\text{loss}}$$

which can be rearranged to obtain

$$\delta_{\text{Hmax}}^L = \left[ \frac{5 m v^2 - E_{\text{loss}}}{4 \left( 1 + \frac{k_L}{k_{sh}} \right) k_H} \right]^{2/5}$$

Note that Eq. (2.30) is an implicit expression for $\delta_{\text{Hmax}}^L$, since $k_L$ is a function of $\delta_{\text{Hmax}}^L$ (see Eq. (2.25)). However, as an identity from Eq. (2.29), Eq. (2.30) will be directly used as a closed-form expression to facilitate the derivation of the impact duration formula below.

To obtain the value of $\delta_{\text{Hmax}}^L$, using Eq. (2.25) in Eq. (2.29) yields

$$\frac{1}{2} m v^2 - \frac{1}{2} E_{\text{loss}} = \frac{2}{5} k_H (\delta_{\text{Hmax}}^L)^{5/2} + \frac{8}{25} k_H^2 (\delta_{\text{Hmax}}^L)^3,$$

which can be numerically solved to get $\delta_{\text{Hmax}}^L$ for given material, geometrical, kinetic energy, and elastic energy loss parameters.

For the elastic energy loss identified in Eq. (2.8), Eqs. (2.30) and (2.31) read

$$\delta_{\text{Hmax}}^L = \left[ \frac{5 m v^2 (1 - 0.5 \lambda)}{4 \left( 1 + \frac{k_L}{k_{sh}} \right) k_H} \right]^{2/5},$$

$$mv^2 (1 - \frac{1}{2} \lambda) = 4 \frac{5}{5} k_H (\delta_{\text{Hmax}}^L)^{5/2} + 16 \frac{k_H^2}{25 k_{sh}} (\delta_{\text{Hmax}}^L)^3.$$
where $\lambda$ is given in Eq. (2.9), $m$ and $v$ are defined in Eq. (2.13), $k_L$ is derived in Eq. (2.25), and $k_H$ and $k_{sh}$ are, respectively, listed in Eqs. (2.2) and (2.5).

Next, using Eqs. (2.27) and (2.28b) in Eq. (2.12) yields

$$dt = \frac{\left(1 + \frac{k_L}{k_{sh}}\right) d\delta_H}{\sqrt{v^2 \left(1 - \frac{2E'_t}{mv^2}\right) - \frac{1}{m} \left(\frac{4}{5} k_H \delta_H^{5/2} + \frac{k_L k_H}{k_{sh}} \delta_H^2\right)}}. \quad (2.34)$$

From Eq. (2.25), it follows that

$$k_L \delta_H^2 = \frac{4}{5} k_H (\delta_H^{\max})^{1/2} \delta_H^2 \approx \frac{4}{5} k_H \delta_H^{5/2}, \quad (2.35)$$

where use has been made of the approximation $\delta_H^{\max} \approx \delta_H$, which should be fairly accurate when $\delta_H$ is small. Substituting Eq. (2.35) into Eq. (2.34) and integrating the resulting equation will yield, with the approximations $E'_t \approx \frac{E_{loss}}{T_c}$ and $\frac{2t}{T_c} \approx 1$ mentioned earlier,

$$t_c^L = \int_0^{\delta_H^{\max}} \frac{\left(1 + \frac{k_L}{k_{sh}}\right)}{\sqrt{v^2 \left(1 - \frac{2E_{loss}}{mv^2}\right) - \frac{1}{m} \left(\frac{4}{5} k_H \delta_H^{5/2} + \frac{k_L k_H}{k_{sh}} \delta_H^2\right)}} d\delta_H \quad (2.36)$$

as the half impact duration based on the linearized Hertz contact relation.

The impact duration $T_c^L$ can then be obtained from Eq. (2.36) as

$$T_c^L = 2 \int_0^{\delta_H^{\max}} \frac{d\delta_H}{\sqrt{\frac{1}{v^2} \left(1 - \frac{E_{loss}}{mv^2}\right) - \frac{1}{5} \left(\frac{k_H}{k_{sh}}\right) \delta_H^{5/2}}} \equiv 2 \int_0^{\delta_H^{\max}} \frac{d\delta_H}{\sqrt{\frac{1}{v^2} \left(1 - \frac{E_{loss}}{mv^2}\right) - \frac{1}{5} \left(\frac{k_H}{k_{sh}}\right) \delta_H^{5/2}}} \quad (2.37)$$
where the definite integral $I$ can be analytically evaluated to be

$$I = \int_{\delta_{H_{\text{max}}}}^{\delta_{L_{\text{max}}}} \frac{d\delta_{H}}{\sqrt{1 - \frac{4}{5} \frac{k_{H}}{mv^2}[1 - \frac{E_{\text{loss}}}{mv^2}]\left(1 + \frac{k_{L}}{k_{sh}}\right)\delta_{H}^{5/2}}} = \delta_{L_{\text{max}}}^{L} \frac{dx}{\sqrt{1 - x^{5/2}}} = \frac{2\sqrt{\pi}}{5} \Gamma(2/5) \frac{\Gamma(9/10)}{\Gamma(1)} \delta_{H_{\text{max}}}^{L}, \quad (2.38)$$

in which $\Gamma(\cdot)$ is the gamma function. Note that use has been made of Eq. (2.30) in reaching the second equality in Eq. (2.38).

Using Eq. (2.38) in Eq. (2.37) finally gives, with the help of Eq. (2.29),

$$T_{c}^{L} = \frac{2.943}{\sqrt{v}} \frac{\left(1 + \frac{k_{L}}{k_{sh}}\right)\delta_{H_{\text{max}}}^{L}}{1 - \frac{E_{\text{loss}}}{mv^2}} = \frac{3.679mv}{\sqrt{1 - \frac{E_{\text{loss}}}{mv^2}}} \frac{\delta_{H_{\text{max}}}^{L}}{k_{H} (\delta_{H_{\text{max}}}^{L})^{3/2}}, \quad (2.39)$$

where $\delta_{H_{\text{max}}}^{L}$ is defined in Eq. (2.31).

For the elastic energy loss identified in Eq. (2.8), Eq. (2.39) has the form:

$$T_{c}^{L} = \frac{3.679mv\sqrt{1 - 0.5\lambda}}{k_{H} (\delta_{H_{\text{max}}}^{L})^{3/2}}, \quad (2.40)$$

where $\lambda$ is given in Eq. (2.9), $m$ and $v$ are defined in Eq. (2.13), $k_{H}$ is listed in Eq. (2.2), and $\delta_{H_{\text{max}}}^{L}$ is to be obtained from Eq. (2.33).

From Eqs. (2.28a) and (2.25), the maximum mutual approach based on the linearized Hertz contact relation is obtained as

$$\delta_{\text{max}}^{L} = \left[1 - \frac{4}{5} \frac{k_{H}}{k_{sh}} (\delta_{H_{\text{max}}}^{L})^{2}\right] \delta_{H_{\text{max}}}^{L}, \quad (2.41)$$

The maximum contact force $F_{\text{max}}$ and the maximum acceleration $a_{sh_{\text{max}}}$ in the fluid-filled shell can then be readily determined from Eqs. (2.23) and (2.25) and Newton’s second law as
\[ F_{\text{max}}^L = \frac{4}{5} k_H (\delta_{H\text{max}}^L)^{3/2}, \]  
(2.42a)

\[ a_{H\text{max}}^L = \frac{4k_H (\delta_{H\text{max}}^L)^{3/2}}{5m_2}. \]  
(2.42b)

2.3. Reduction of the New Models

It is shown in this section that the new models developed in Section 2.2 can be reduced to the models of Young (2003) and Mansoor-Baghaei and Sadegh (2011).

2.3.1 Impact without including the elastic energy loss

Using Eq. (2.1) in Eq. (2.16) gives that at the time of maximum deflection,

\[ mv^2 - E_{\text{loss}} = \frac{F_{\text{max}}^2}{k_{sh}} + \frac{4}{5} \frac{F_{\text{max}}^{5/3}}{k_H^{2/3}}, \]  
(2.43)

where \( F_{\text{max}} \) is the maximum contact force associated with the maximum deflection.

Substituting Eqs. (2.2), (2.3), (2.5) and (2.13) into Eq. (2.43) yields, after some algebra,

\[ \left( \frac{F_{\text{max}}}{E_{\text{sol}} R_{sh}^3} \right)^2 \left( \frac{R_{sh}}{h} \right)^2 \left( \frac{c_0}{v} \right)^2 \left( 1 + \frac{m_{sh}}{m_{sol}} \right) \sqrt{3(1 - \nu_{sh}^2)} + \frac{3}{5} \left( \frac{4^{1/3}}{3} \right) \left( \frac{F_{\text{max}}}{E_{\text{sh}} R_{sh}^3} \right) \left( \frac{c_0}{v} \right)^2 \left( 1 + \frac{m_{sh}}{m_{sol}} \right) \left( 1 + \frac{R_{sh}}{R_{sol}} \right)^{1/3} \]

\[ \times \left[ (1 - \nu_{sh}^2) + (1 - \nu_{sol}^2) \frac{E_{\text{sh}}}{E_{\text{sol}}} \right]^{2/3} + \frac{E_{\text{loss}}}{R_{sh}^3 \rho_{sh} v^2} \left( 1 + \frac{m_{sh}}{m_{sol}} \right) - \frac{m_{sh}}{R_{sh}^3 \rho_{sh}} = 0, \]  
(2.44)

where \( R_{\text{sol}} = R_1, E_{\text{sol}} = E_1, \nu_{\text{sol}} = \nu_1, m_{\text{sol}} = m_1, R_{sh} = R_2, E_{sh} = E_2, \nu_{sh} = \nu_2, m_{sh} = m_2, \rho_{sh} = \rho_2, \) and use has also been made of \( c_0 = \sqrt{E_2/\rho_2}. \) Clearly, Eq. (2.44) shows that the non-dimensional maximum contact force \( F_{\text{max}}/(E_{\text{sol}} R_{sh}^3) \) depends on the non-dimensional parameters \( m_{sh}/m_{sol}, R_{sh}/R_{sol}, E_{sh}/E_{sol}, h/R_{sh}, v/c_0, \nu_{sh}, \nu_{sol}, m_{sh}/(\rho_{sh} R_{sh}^3) \) and
where $E_{\text{loss}} = \frac{v^2 \rho_s R_h^3}{(v^2 \rho_s R_h^3)}$. Note that in Eq. (2.44) $m_h = \frac{4\pi}{3} \left[ \rho_f R_f^3 + \rho_s \left( R_h^3 - R_f^3 \right) \right]$, which includes the mass of the fluid filling the shell.

When the elastic energy loss due to the stress wave propagation is not considered, $E_{\text{loss}} = 0$ and Eq. (2.43) reduces to

$$mv^2 = \frac{F_{\text{max}}^2}{k_h} + \frac{4}{5} \frac{F_{\text{max}}^{5/3}}{k_h^{2/3}},$$

(2.45)

which is the same as that provided in Young (2003) without considering the elastic energy loss due to the stress wave propagation. In addition, with $E_{\text{loss}} = 0$, Eq. (2.44) will be reduced to the equation (2.42) in Young (2003) if $\left( \frac{4}{3} \right)^{1/3}$ is replaced by 1. These verify and support the current model.

Moreover, when $E_{\text{loss}} = 0$, Eq. (2.20) becomes

$$T_c = 2 \int_0^{\infty} \frac{\delta_{H_{\text{max}}} \left( 1 + \frac{3}{2} \frac{k_h}{k_{sh}} \delta_{H_{\text{max}}}^{1/2} x^{1/2} \right)}{\sqrt{v^2 - \frac{1}{m} \left( \frac{k_h^2}{k_{sh}} \delta_{H_{\text{max}}}^3 x^3 + \frac{4}{5} k_h \delta_{H_{\text{max}}}^{5/2} x^{3/2} \right)}} \, dx,$$

(2.46)

which can be evaluated numerically to obtain the impact duration $T_c$.

Note that in Young (2003) an approximate relation was used to obtain a closed-form expression for $T_c$ without considering the stress wave propagation effect, whereas in the current model $T_c$ is derived in an integral form (see Eq. (2.20) or (2.46)) directly from the energy conservation principle by incorporating the elastic energy loss due to the stress wave propagation.

For the case with $E_{\text{loss}} = 0$, Eqs. (2.31) and (2.39) simplifies to
\[ mv^2 = \frac{4}{5} k_H (\delta_{H_{\text{max}}}^L)^{5/2} + \frac{16}{25} \frac{\delta_{L_{\text{max}}}^L}{k_{sh}} \delta_{H_{\text{max}}}^L, \quad (2.47) \]

\[ T_c^L = \frac{3.679mv}{k_H (\delta_{H_{\text{max}}}^L)^{3/2}}, \quad (2.48) \]

which are the two formulas for computing the maximum value of \( \delta_H \) and the impact duration \( T_c \) given by the current simplified model without including the stress wave propagation effect.

2.3.2 Impact of a spherical shell on a flat barrier without considering the elastic energy loss

When the sphere becomes a stationary flat elastic barrier (with \( v_1 = 0, R_1 \to \infty \) and \( m_1 \to \infty \)) and the elastic energy loss due to the stress wave propagation is neglected, Eqs. (2.30) and (2.39) degenerate to

\[ \delta_{H_{\text{max}}}^L = \left[ \frac{5}{4} \frac{m_2 v_2^2}{(1 + \frac{k_L}{k_{sh}})k_H} \right]^{2/5}, \quad (2.49) \]

\[ T_c^L = \frac{2.943 \left(1 + \frac{k_L}{k_{sh}}\right) \delta_{H_{\text{max}}}^L}{v_2}, \quad (2.50) \]

where use has also been made of Eq. (2.13). In Eqs. (2.49) and (2.50),

\[ k_H = \frac{4}{3} R_2^{1/2} E, \quad (2.51) \]

which follows directly from Eqs. (2.2) and (2.3), and \( k_{sh} \) and \( k_L \) are given in Eqs. (2.5) and (2.25), respectively.
A comparison shows that the formulas obtained here in Eqs. (2.49) and (2.50) are the same as those derived in Mansoor-Bagheri and Sadegh (2011) except that $k_L$ involved in their formulas is based on the upper bound of $\delta_{H\max}$ listed in Eq. (2.26), as stated near Eq. (2.25). Note that Newton’s second law is directly used by Mansoor-Bagheri and Sadegh (2011) to derive the governing equation, which is different from the current approach. This agreement validates the simplified model developed in Section 2.2.5.

For the case with $v_1 = 0$, $R_1 \to \infty$, $m_1 \to \infty$ and $E_{loss} = 0$, Eqs. (2.16) and (2.20) reduce to

\[
\frac{1}{2} m_2 v_2^2 = \frac{1}{2} \frac{k_H}{k_{sh}} \delta_{H\max}^3 + \frac{2}{5} k_H \delta_{H\max}^{5/2},
\]

\[
T_c = 2 \int_0^1 \frac{\delta_{H\max} \left(1 + \frac{3}{2} \frac{k_H}{k_{sh}} \delta_{H\max}^{1/2} x^{1/2}\right)}{\sqrt{v_2^2 - \left(\frac{k_H}{k_{sh}} \delta_{H\max}^3 + \frac{4}{5} k_H \delta_{H\max}^{5/2} x^{5/2}\right)}} dx,
\]

where $k_H$ and $k_{sh}$ are listed in Eqs. (2.51) and (2.5), respectively. Equations (2.52) and (2.53) are the formulas for computing $\delta_{H\max}$ and $T_c$ given by the new non-linear model presented in Section 2.2.4.

2.4. Applications to blunt head impacts

Three representative problems simulating blunt head impacts are analyzed in this section by directly applying the two new analytical models developed in Section 2.2.

2.4.1 Fluid-filled spherical shell impacting on a rigid half space

The new non-linear model is applied here to the problem of a fluid-filled spherical shell
impacting on a rigid half space, which can simulate the blunt impact of a human head on
the ground or on an automobile.

In this case, the solid sphere becomes a fixed, rigid, half space with \( v_{\text{sol}} = 0, E_{\text{sol}} \rightarrow \infty, m_{\text{sol}} \rightarrow \infty \) and \( R_{\text{sol}} \rightarrow \infty \), which gives \( R = R_{sh}, E = E_{sh}/(1 - v_{sh}^2) \) according to Eq. (2.3).

Also, \( v_{sh} = -v_0 \) (see Fig. 2.1). As a result, Eq. (2.44) reduces to

\[
\left( \frac{F_{\text{max}}}{E_{sh} R_{sh}^2} \right)^2 \left( \frac{R_{sh}}{h} \right)^2 \left( \frac{c_0}{v_0} \right)^2 \frac{3(1-v_{sh}^2)}{4} + \frac{3}{5} \left( \frac{F_{\text{max}}}{E_{sh} R_{sh}^2} \right)^{5/3} \left( \frac{c_0}{v_0} \right)^2 \left( 1-v_{sh}^2 \right)^{2/3} + \frac{E_{\text{loss}}}{R_{sh}^3 \rho_{sh} v_0^2} - \frac{m_{sh}}{R_{sh}^3 \rho_{sh}} = 0.
\]

(2.54)

When the fluid-filled spherical shell represents a human head, the material and geometrical parameters given in Table 2.1 (Engin, 1969) can be used for the shell.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \rho_{sh} ) (kg/m(^3))</th>
<th>( E_{sh} ) (GPa)</th>
<th>( v_{sh} )</th>
<th>( R_f ) (m)</th>
<th>( h ) (m)</th>
<th>( \rho_f ) (kg/m(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head shell</td>
<td>2136.89</td>
<td>13.79</td>
<td>0.25</td>
<td>0.074295</td>
<td>0.00381</td>
<td>1002.01</td>
</tr>
</tbody>
</table>

With \( v_{sh} = 0.25 \), the elastic energy loss is obtained from Eqs. (2.8) and (2.9) as

\[
E_{\text{loss}} = \frac{1}{2} m v^2 \left[ 2.7265 \left( \frac{v}{c_0} \right)^{3/5} \right],
\]

(2.55)

where \( c_0 = \sqrt{E_{sh}/\rho_{sh}} = 2.540.34 \) m/s.

The variation of the non-dimensional maximum force \( F_{\text{max}}/\left(E_{sh} R_{sh}^2\right) \) with the shell thickness ratio \( h/R_{sh} \) is shown in Figs. 2.3(a) and 2.3(b) for \( v_0/c_0 = 0.001 \) (or \( v_0 = 2.5403 \) m/s) and 0.004 (or \( v_0 = 10.1613 \) m/s), respectively. Fig. 2.3(c) displays how \( F_{\text{max}}/\left(E_{sh} R_{sh}^2\right) \)
changes with \( v_0/c_0 \) when \( h/R_{sh} = 0.0488 \). The numerical results plotted in Figs. 2.3(a)-(c) are obtained from Eq. (2.54) along with Eq. (2.55) and Table 2.1, with
\[
m_{sh} = \frac{4\pi}{3} \left[ \rho_f R^3_f + \rho_{sh} \left( R^3_{sh} - R^3_f \right) \right].
\]

It is observed from Figs. 2.3(a)-(c) that the maximum contact force \( F_{max} \) increases monotonically with the increase of the shell thickness \( h \) or the initial impact velocity \( v_0 \). Also, Figs. 2.3(a)-(c) quantitatively show that \( F_{max} \) is reduced when the elastic energy loss due to the stress wave propagation is considered, which is as expected. In addition, from Figs. 2.3(a)-(c), it is seen that when \( h/R_{sh} \) is sufficiently small (with \( h/R_{sh} < 0.06 \) for \( v_0/c_0 = 0.001 \) and \( h/R_{sh} < 0.04 \) for \( v_0/c_0 = 0.004 \) here) or \( v_0 \) is small (with \( v_0/c_0 < 0.002 \) or \( v_0 < 5.081 \) m/s here), the effect of the stress wave propagation on the maximum impact force \( F_{max} \) can be ignored. However, when the impact velocity gets large, this effect of the stress wave propagation becomes more significant and should be considered.

Fig. 2.3 \( F_{max}/(E_{sh} R^3_{sh}) \) changing with \( h/R_{sh} \) for (a) \( v_0/c_0 = 0.001 \) (or \( v_0 = 2.5403 \) m/s), (b) \( v_0/c_0 = 0.004 \) (or \( v_0 = 10.1613 \) m/s), and (c) with the velocity ratio \( v_0/c_0 \) for \( h/R_{sh} = 0.0488 \) for a human head impacting on a rigid half space.
2.4.2 Non-lethal projectile impacting on a fluid-filled spherical shell

In this section, the new non-linear model is employed to analyze the impact of a solid sphere on a fluid-filled spherical shell, which simulates the blunt impact of a non-lethal projectile (NLP) on a human head. NLPs are emerging as an alternative to metallic bullets to be used in riot control, peacekeeping operations, hostage rescue missions, and armed conflicts (e.g., Sahoo et al., 2016). The parameters of an NLP (a rubbery material) are listed in Table 2.2, which are taken from Sahoo et al. (2016).

Table 2.2 Material parameters for the non-lethal projectile (Sahoo et al., 2016)

<table>
<thead>
<tr>
<th>Material</th>
<th>$E_{sol}$ (MPa)</th>
<th>$\rho_{sol}$ (kg/m$^3$)</th>
<th>$R_{sol}$ (mm)</th>
<th>$V_{sol}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLP</td>
<td>6.0</td>
<td>650</td>
<td>22.0</td>
<td>0.499</td>
</tr>
</tbody>
</table>

In the present case, the NLP is represented by the solid sphere, and the human head, which is stationary, is simulated using the fluid-filled spherical shell. As a result, $v_1 = v_{sol} = v_0$, $v_2 = v_{sh} = 0$, and hence $v = v_0$ from Eq. (2.13).

The maximum mutual approach $\delta_{max}$ changing with the impact velocity $v_0$ of the NLP predicted by the current non-linear model is shown in Fig. 2.4(a) along with $\delta_{Hmax}$ and $\delta_{shmax}$. The numerical values of $\delta_{Hmax}$ plotted are obtained from Eqs. (2.17) and (2.9) and Tables 2.1 and 2.2, whereas those of $\delta_{shmax}$ and $\delta_{max}$ are computed using Eqs. (2.14) and (2.15a) after $\delta_{Hmax}$ has been determined.
It is seen from Fig. 2.4(a) that $\delta_{\text{shmax}}$ (i.e., the portion of $\delta_{\text{max}}$ due to the shell membrane and bending actions) is much smaller than $\delta_{\text{Hmax}}$ (i.e., the portion of $\delta_{\text{max}}$ due to the Hertz contact). Also, $\delta_{\text{Hmax}}$ and hence $\delta_{\text{max}}$ increase rapidly with the increase of $v_0$.

The variations of the maximum contact force $F_{\text{max}}$ with $h$ and $v_0$ are displayed in Figs. 2.4(b) and 2.4(c), respectively. The values of $F_{\text{max}}$ plotted are obtained from Eqs. (2.22a), (2.17) and (2.9). Clearly, Figs. 2.4(b) and 2.4(c) show that $F_{\text{max}}$ increases with the increase of $h$ or $v_0$.

In Fig. 2.4(c), the values of $F_{\text{max}}$ predicted by the current non-linear model are also compared with the finite element simulation results provided in Sahoo et al. (2016) for temporoparietal impacts, which were experimentally validated. It is seen that the predictions by the current model agree well with the simulations of Sahoo et al. (2016) for low-velocity impacts with $v_0 < 62.5$ m/s. For medium- and high-velocity impacts with $v_0 > 71.5$ m/s, the predicted values of $F_{\text{max}}$ are lower than those obtained in the finite element simulations of Sahoo et al. (2016). The reason for this is that the current analytical model
is based on the Hertz contact theory, which is accurate only for quasi-static and low-velocity impacts.

The variations of the impact duration $T_c$ with the shell thickness $h$ and the impact velocity $v_0$ are plotted in Figs. 2.5(a) and 2.5(b). The numerical results shown in Figs. 2.5(a) and 2.5(b) are obtained from Eqs. (2.21) and (2.17) (for the non-linear model) and Eqs. (2.40) and (2.33) (for the linearized model) along with Eqs. (2.8) and (2.9) and Tables 2.1 and 2.2.

From Fig. 2.5(a), it is seen that $T_c$ decreases with the increase of $h$ for the given impact velocity. Also, it is observed from Fig. 2.5(b) that $T_c$ decreases as $v_0$ increases, as expected. Clearly, Figs. 2.5(a) and 2.5(b) show that the predictions of $T_c$ by the non-linear and linearized models are quite close. Hence, the closed-form expression for $T_c$ given by the simplified model in Eq. (2.40) can be used to estimate the impact duration in the first place.

![Fig. 2.5 (a) $T_c$ varying with $h$ (with $v_0 = 10.1613$ m/s); (b) $T_c$ changing with $v_0$ (with $h = 0.00381$ m) for the impact of a non-lethal projectile on a human head.](image)
2.4.3 Fluid-filled spherical shell impacting on another fluid-filled spherical shell

Human head blunt impacts also include head-to-head impacts, such as two football players’ head collision. The current new non-linear model is applied here to characterize such impacts by simulating a head collision as the impact of one fluid-filled spherical shell on another one with the same parameters.

In this case, \( m_1 = m_2 = m_{sh} \), \( R_1 = R_2 = R_{sh} \), \( \nu_1 = \nu_2 = \nu_{sh} \), \( E_1 = E_2 = E_{sh} \) and \( \nu_1 = -\nu_2 = \nu_{sh} = \nu_0 \) and Eqs. (2.16) and (2.20) become

\[
m_{sh}v_0^2 - \frac{1}{2}E_{loss} = \frac{\sqrt{3}}{36} \frac{E_{sh}}{(1-\nu_{sh}^2)^{3/2}} \frac{R_{sh}^2}{h^2} \delta_{H_{\text{max}}}^3 + \frac{2\sqrt{2}}{15} \frac{E_{sh}}{(1-\nu_{sh}^2)} \sqrt{R_{sh} \delta_{H_{\text{max}}}^{5/2}}, \tag{2.56}
\]

\[
T_c = 2\int_0^1 \delta_{H_{\text{max}}} \left( 1 + \frac{3}{4} \frac{1}{\sqrt{6(1-\nu_{sh}^2)}} \frac{R_{sh}^{3/2}}{h^2} \delta_{H_{\text{max}}^{1/2}} \right) \frac{\sqrt{3}}{9} \frac{E_{sh}}{(1-\nu_{sh}^2)^{3/2}} \frac{R_{sh}^2}{h^2} \delta_{H_{\text{max}}}^3 x^3 + \frac{8\sqrt{2}}{15} \frac{E_{sh}}{1-\nu_{sh}^2} R_{sh}^{1/2} \delta_{H_{\text{max}}}^{5/2} x^{5/2} \right) dx, \tag{2.57}
\]

where use has also been made of Eqs. (2.2), (2.3), (2.5) and (2.13). Using the parameters for a human head given in Table 2.1 in Eqs. (2.56) and (2.57), \( \delta_{H_{\text{max}}} \) and \( T_c \) for given values of \( \nu_{sh} \) and \( E_{loss} \) can be numerically obtained. With \( \delta_{H_{\text{max}}} \) determined, the maximum contact force \( F_{\text{max}} \) and maximum acceleration \( a_{sh_{\text{max}}} \) can then be computed using Eqs. (2.22a, b).

Figure 2.6(a) shows the variation of the maximum contact force \( F_{\text{max}} \) with the shell thickness \( h \). It is seen that \( F_{\text{max}} \) increases monotonically with the increase of \( h \). Also, it is observed that the current model considering the elastic energy loss predicts lower values for \( F_{\text{max}} \) than the models of Young (2003) and Mansoor-Baghaei and Sadegh (2011), both of which do not consider the energy loss induced by the stress wave propagation. In addition, Fig. 2.6(b) reveals that \( F_{\text{max}} \) increases monotonically with the increase of the impact velocity \( v_0 \). Again, it is clearly shown that the predicted values of \( F_{\text{max}} \) by the current
model are lower than those given by the models Young (2003) and Mansoor-Bagheri and Sadegh (2011). Finally, it is seen from both Figs. 2.6(a) and 2.6(b) that the values of $F_{\text{max}}$ predicted by the model of Mansoor-Bagheri and Sadegh (2011) are the highest among the three models, which is consistent with the observation made in Section 2.2.5 that the model of Mansoor-Bagheri and Sadegh (2011) provides an upper bound.

![Graph](image)

Fig. 2.6 $F_{\text{max}}$ changing (a) with $h$ (with $v_0 = 2.5403$ m/s); and (b) with $v_0$ (with $h = 0.00381$ m) for the impact of two fluid-filled spherical shells.

The variations of the impact duration $T_c$ with the shell thickness $h$ and impact velocity $v_0$ are displayed in Figs. 2.7(a) and 2.7(b).
It is seen from Figs. 2.7(a) and 2.7(b) that $T_c$ decreases monotonically with the increase of $h$ or $v_0$. Also, Figs. 2.7(a) and 2.7(b) show that the values of $T_c$ predicted by the current model considering the energy loss due to the stress wave propagation are higher than those predicted by the models of Young (2003) and Mansoor-Baghaei and Sadegh (2011), which do not include the stress wave propagation effect.

2.5. Summary

A new analytical (non-linear) model for the impact of a solid sphere on a fluid-filled spherical shell is provided by considering the elastic energy loss due to the stress wave propagation along with the Hertz contact deformations and shell membrane and bending effects. Also, a simplified (linearized) model including the stress wave propagation effect is developed by using a linear force-deflection relation, which gives a closed-form expression for the impact duration. If the stress wave propagation effect is ignored, the new
non-linear and linearized models reduce to those of Young (2003) and Mansoor-Baghaei and Sadegh (2011), respectively. Three problems representing blunt head impacts are analyzed by directly applying the two newly developed models. Numerical results are obtained using the current models and compared to those predicted by the two existing analytical models and to available finite element simulation results, which shows a good agreement and thereby supports the two newly developed models.
References


Young, P. G., 2003, An analytical model to predict the response of fluid-filled shells to impact—a model for blunt head impacts, *Journal of Sound and Vibration* **267**, 1107-


3. COMPUTATIONAL MODELING OF BLUNT IMPACTS

3.1 Introduction

Sports-related head injuries, such as concussions arising from collision of football players, brain injuries induced by impacts of soccer or tennis balls, and skull fractures caused by golf ball strikes, have attracted increasing attention (e.g., Meaney and Smith, 2011; Chanda et al., 2016; Zemper et al., 2016; Ho et al., 2017). Golf is becoming a popular sport among all ages (e.g., Farrally et al., 2003; Lunn and Kelly, 2017). However, head injuries can be induced by flying golf balls (e.g., Rahimi et al., 2005; Nguyen and Kaplan, 2008; McGuinness et al., 2016; Walsh et al., 2017). When a golf ball strikes on a human head, the kinetic energy of the flying golf ball is imparted to the head over a small impact area and in a short impact duration (e.g., Roberts et al., 2001; Lee and Wang, 2010), which can cause skull fracture, contusion and intracranial trauma (e.g., Lindsay et al., 1980; Nicholas et al., 1998; Rahimi et al., 2005; Nguyen and Kaplan, 2008; Katagiri et al., 2012).

There have been continuous efforts to understand the mechanisms of head injuries induced by golf ball impacts, especially for children (e.g., Macgregor, 2002; Rahimi et al., 2005; Fountas et al., 2006). For example, an early survey by Nicholas et al. (1998) found that contusion is the major head injury related to golf ball impacts. Also, a traumatic basal
subarachnoid hemorrhage caused by a high-speed golf ball impact was investigated by Watanabe-Suzuki et al. (2003) through a case study of a 50-year-old male. In addition, a number of computational studies have been conducted to investigate such head injuries. A finite element model of a golf ball striking on a human head was proposed by Lee and Wang (2010), where the stress and energy flow patterns were evaluated for various striking velocities, ball falling angles, and impact locations. Katagiri et al. (2012) simulated the forehead impact by a golf ball using a finite element head model and a three-layer finite element model for the golf ball. They implemented a stress-based skull fracture criterion. More recently, Pearce and Young (2014) studied the impact of a golf ball on a human head with a short duration, where the intracranial pressure (ICP) in the brain was evaluated using a finite element head and neck model. In all of these computational studies, only a human head model or a human head and neck model was considered. Since the impact-induced motion of a human head depends on the support of not only the neck but also the other parts of the human body below the neck, the use of a full human body model can better represent the impact of a golf ball on a human head, even though employing a head model or a head-neck model simplifies simulations.

In this chapter, head injuries induced by a golf ball impact on a human head are evaluated using a full human body finite element model. In Section 3.2, a three-piece golf ball model and the full human body model are described. The latter is the 50th percentile detailed pedestrian occupant male human body model (GHBMC M50-P) developed by the Global Human Body Models Consortium (GHBMC). The constitutive relations for the golf ball and the human head are also discussed in this section. Validations of the golf ball model and the human head model are given separately in Section 3.3. In Section 3.4, the
baseline case of the frontal impact on a human head by a golf ball traveling at a velocity of
35 m/s is first studied, which is followed by an investigation on the effects of impact
location, velocity, and angle. In Section 3.5, the simulation results are discussed, and head
injury risks due to the golf ball impacts are explored. This chapter concludes in Section 3.6
with a summary.

3.2 Model Description

The finite element (FE) model of the golf ball is first introduced, which is followed by
a brief description of the GHBMC full human body FE model. All of the FE simulations
are performed using the code LS-DYNA (2017) in the current study.

3.2.1 Golf ball model

3.2.1.1 Three-piece golf ball finite element model

The FE mesh of a three-piece golf ball is shown in Fig. 3.1, which includes an outer
cover, a middle mantle, and a core. The cover is made from an ionomer resin, while both
the mantle and core are made from polybutadiene rubber. The dimensions of the golf ball
are taken from the measured values provided in Tanaka et al. (2006). The FE model
contains two layers of elements for the cover, another two layers for the mantle, and a
spherical domain of elements for the core, all being hexahedral elements. The relevant
information about the FE model for the three-piece golf ball is given in Table 3.1.
Table 3.1 Properties of the FE model for the golf ball

<table>
<thead>
<tr>
<th>Section</th>
<th>Outer diameter (mm)</th>
<th>No. of nodes</th>
<th>No. of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core</td>
<td>35.4</td>
<td>15625</td>
<td>13824</td>
</tr>
<tr>
<td>Mantle</td>
<td>38.8</td>
<td>10374</td>
<td>6912</td>
</tr>
<tr>
<td>Cover</td>
<td>42.8</td>
<td>10374</td>
<td>6912</td>
</tr>
<tr>
<td>Total</td>
<td>-</td>
<td>36373</td>
<td>27648</td>
</tr>
</tbody>
</table>

The constitutive behavior of the three-piece golf ball is described using hyperelasticity and viscoelasticity, as was done in Tanaka et al. (2013). For the ionomer resin cover, the Mooney-Rivlin hyperelasticity model is used, and for the polybutadiene rubber mantle and core, a combined hyperelasticity and linear viscoelasticity model is adopted, which is known as MAT_077_H in LS-DYNA (2017).

3.2.1.2 Material properties

The strain energy density function for an unconstrained hyperelastic material can be written as (e.g., Ogden, 1984)

\[ W(I_1, I_2, I_3) = \sum_{p,q=0}^{\infty} C_{pq} (I_1 - 3)^p (I_2 - 3)^q + g(I_3), \]  

(3.1)
where $p, q = 1, 2, 3, \ldots$, and $I_1, I_2, I_3$ are the three invariants of the right Cauchy-Green deformation tensor.

When $p = 1$ and $q = 1$, Eq. (3.1) reduces to the Mooney-Rivlin model for unconstrained materials (e.g., Holzapfel, 2000):

$$W(I_1, I_2, I_3) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + g(I_3), \quad (3.2)$$

where $C_{10}$ and $C_{01}$ are two material constants.

For rubbery materials that can be regarded as incompressible, Eq. (3.2) can be further simplified as (e.g., Feng et al., 2016)

$$W(I_1, I_2) = C_{10}(I_1 - 3) + C_{01}(I_2 - 3), \quad (3.3)$$

which is the Mooney-Rivlin strain energy density function for incompressible materials. This model is adopted in the current study, which is included in MAT_077_H in LS-DYNA (2017) as an option that requires inputting only $C_{10}$ and $C_{01}$. The values for these two parameters are listed in Table 3.2, which are taken from Tanaka et al. (2013).

For an incompressible material, the second Piola-Kirchhoff stress tensor $S^e_{ij}$ in terms of the invariants $I_i$ are given by (e.g., Holzapfel, 2000)

$$S^e_{ij} = 2 \left( \frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2} \right) \delta_{ij} - 2 \frac{\partial W}{\partial I_2} C_{ij} - p C^{-1}_{ij}, \quad (3.4)$$

where $C_{ij}$ are the components of the right Cauchy-Green deformation tensor, $\delta_{ij}$ is the Kronecker delta, $p$ is the hydrostatic pressure, and the superscript “$-$” denotes the inverse tensor.

To describe viscoelastic responses of rubber, the following constitutive relation can be used (e.g., Christensen, 1980, 1982)
\[
S_y^e = \int_{0}^{t} G_{ijkl}(t - \tau) \frac{\partial E_{kl}}{\partial \tau} d\tau,
\]  

where \( S_y^e \) is the viscous part of the second Piola-Kirchhoff stress, \( E_{kl} \) are the components of the Green-Lagrangian strain tensor, \( G_{ijkl} \) are the stress relaxation functions, and \( t \) is the current time.

The total second Piola-Kirchhoff stress in the rubber mantle or core can then be obtained as (e.g., Kulkarni et al., 2016)

\[
S_y = S_y^e + S_y^v.
\]  

For isotropic viscoelastic materials, the most general form of \( G_{ijkl} \) is (e.g., Christensen, 1982)

\[
G_{ijkl} = \frac{1}{3} \left[ g_2(t) - g_1(t) \right] \delta_y \delta_{kl} + \frac{1}{2} g_1(t)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]

where \( g_1(t) \) and \( g_2(t) \) are two independent relaxation functions. Each of these two relaxation functions can be represented as a Prony series given by (e.g., David et al., 2011)

\[
g_i(t) = \sum_{j=1}^{n} G_{ij} e^{-\beta_j t},
\]

where \( G_{ij} \) and \( \beta_j \) are constants. When only one term is considered, the simplest representations will be obtained for \( g_1(t) \) and \( g_2(t) \).

In the current study, the three-element viscoelasticity model shown in Fig. 3.2 is adopted, which uses one term to represent \( g_1(t) \) and is incorporated in MAT_077_H in LS-DYNA. This model includes a spring and a slider in series to represent frequency-independent frictional damping via the parameters \( G \) and \( SIGF \). The values for the four
parameters involved in the model (see Fig. 3.2) are given in Table 3.2, which were initially obtained by Tanaka et al. (2013) from fitting experimental data.

![Viscoelastic model diagram]

**Fig. 3.2 Viscoelastic model.**

<table>
<thead>
<tr>
<th>Material parameters for the golf ball model (Tanaka et al., 2013)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Section</strong></td>
</tr>
<tr>
<td>Core</td>
</tr>
<tr>
<td>Mantle</td>
</tr>
<tr>
<td>Cover</td>
</tr>
</tbody>
</table>

3.2.1.3 Mesh sensitivity

The mesh sensitivity has been studied by simulating the impact of a golf ball on a steel target (a square plate). Four types of mesh, i.e., coarse, medium, medium-fine, and fine, are considered, and the element number in each piece of the golf ball for every mesh type is listed in Table 3.3. The FE mesh for each of the four mesh types is shown in Fig. 3.3.

**Table 3.3 No. of elements for four mesh types**

<table>
<thead>
<tr>
<th>Section</th>
<th>Core</th>
<th>Mantle</th>
<th>Cover</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse</td>
<td>1728</td>
<td>1728</td>
<td>1728</td>
<td>5184</td>
</tr>
<tr>
<td>Medium</td>
<td>13824</td>
<td>6912</td>
<td>6912</td>
<td>27648</td>
</tr>
<tr>
<td>Medium-fine</td>
<td>46656</td>
<td>23328</td>
<td>23328</td>
<td>93312</td>
</tr>
<tr>
<td>Fine</td>
<td>110592</td>
<td>55296</td>
<td>69120</td>
<td>235008</td>
</tr>
</tbody>
</table>
The dimensions and boundary support (clamped on all four edges) of the square plate are shown in Fig. 3.4(a). The steel plate is discretized using hexahedral solid elements (totaling 19,845). The material properties of the steel plate used in the simulations here are given in Table 3.4. The FE simulation results for the impact of the three-piece golf ball on the steel plate at an impact velocity of 35 m/s are displayed in Fig. 3.4(b). It is seen that the impact force-time history curves for the four types of mesh are very close to each other and they all agree well with the experimental curve of Tanaka et al. (2013). Hence, the medium mesh type (size) shown in Fig. 3.3(b) is adopted for the golf ball in the rest of the current study.

Fig. 3.3 Four FE mesh types: (a) coarse, (b) medium, (c) medium-fine, and (d) fine.
### Table 3.4 Material properties of the steel plate

<table>
<thead>
<tr>
<th>Material</th>
<th>Density $\rho$ (kg/m$^3$)</th>
<th>Young modulus $E$ (GPa)</th>
<th>Poisson’s ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7800</td>
<td>210</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Fig. 3.4 (a) FE simulation of a golf ball impact on a clamped square steel plate (with $a = b = 250$ mm, $h = 20$ mm); (b) the impact force-time history at an impact velocity of 35 m/s for the four types of mesh.

3.2.2 Full human body model

#### 3.2.2.1 FE model of a human head

The 50th percentile detailed pedestrian occupant male human body model developed by the Global Human Body Models Consortium (GHBMC M50-P) (e.g., Untaroiu et al., 2015; Schwartz et al., 2015) is adopted in the current study. The GHBMC M50-P model includes the head, neck, thorax, abdomen, pelvis, and lower extremities.

The head model embedded in the GHBMC M50-P model was developed at Wayne State University (WSU) (e.g., Mao et al., 2013), which is shown in Fig. 3.5.
The neck model was provided by University of Waterloo in Canada (e.g., Fice et al., 2011), and the thorax model was constructed at University of Virginia (e.g., Li et al., 2010). In addition, the abdomen model was implemented by Virginia Tech (e.g., Gayzik et al., 2011), and the pelvis-lower extremity model was contributed by University of Virginia and University of Alabama-Birmingham (e.g., Kim et al., 2014; Yue and Untaroiu, 2014). Detailed descriptions for these finite element models can be found in the references cited above.

Fig. 3.5 The GHBMC M50-P/WSU head model: (a) isometric view of the head model with brain exposed; (b) medium sagittal view of the head model; (c) skull and facial bones; (d) brain; (e) falx and tentorium; (f), (g), (h) brain sectional views in horizontal, sagittal, and coronal directions (Mao et al., 2013).

3.2.2.2 Material properties

The WSU human head model embedded in the GHBMC M50-P model (see Fig. 3.5) includes the flesh, skin, facial bones, skull, brain, cerebrospinal fluid, falx, tentorium, pia,
arachnoid, and dura (e.g., Zhang et al., 2001; Mao et al., 2013). To capture large
deformations of and rate effects on the brain tissue, a second-order Ogden hyperelasticity
model and a linear viscoelasticity model represented by a six-term Prony series (Kleiven,
2007), which differs from the linear viscoelasticity model used in the WSU head model
(e.g., Zhang et al., 2001), are employed in the current study. The three-layer skull bone
(including the outer and inner compact/cortical bone and the middle cancellous bone) is
modeled as an elastic-plastic material, the flesh is treated as a viscoelastic material, and the
skin and membranes are regarded as linear elastic materials. The WSU head model has
been validated in terms of the brain pressure, relative skull-brain motion, skull deformation,
and facial response. More details about the mesh, material properties, and validations of
the WSU head model can be found in Mao et al. (2013).

3.3 Model Validation

3.3.1 Validation of the FE model for the golf ball

To validate the FE model for the golf ball described in Section 3.2.1, the impact by a
golf ball on a steel square plate is simulated. The plate is clamped on its four edges. The
dimensions and support of the plate are indicated in Fig. 3.4(a), and the material properties
of the plate are given in Table 3.4. The finite element mesh for the steel plate here is the
same as that used in Section 3.2.1.3, with the total number of hexahedral elements being
19,845. The CONTACT_AUTOMATIC_SURFACE_TO_SURFACE module in LS-
DYNA (2017) is adopted to simulate the interaction between the golf ball and the steel
target with a friction coefficient of 0.3.
The simulation results for the time history of the impact force and the time history of the golf ball diametric deformation in the impact direction are shown in Fig. 3.6, where they are also compared with the experimental data of Tanaka et al. (2013). An impact velocity of 35 m/s (i.e., the golf ball velocity just before the impact) is considered. It is seen from Figs. 3.6(a) and 3.6(b) that both the impact force- and deformation-time history curves predicted by the current FE model agree well with the experimental curves of Tanaka et al. (2013).

Figure 3.7 shows a comparison of the golf ball shape change predicted by the new model and that observed in experiments of Pincott and Blicblau (2014) for a golf ball impact on a TiN coated titanium plate at an impact velocity of 46 m/s. It is seen that the shape change of the golf ball predicted by the current model agrees fairly well with that obtained experimentally.
Fig. 3.7 Comparison of the golf ball shape change at an impact velocity of 46 m/s: (a) the prediction by the current FE model; (b) the experimental observation of Pincott and Blicblau (2014).

3.3.2 Validation of the FE model for the impact of a golf ball on a human head

In order to validate the FE model described in Section 3.2, the simulation results from the model are compared against those of Pearce and Young (2014). The constitutive relations and material parameters for all parts of the head except for the scalp at the impact site are adopted from Pearce and Young (2014) for the simulation included herein, while those for the other parts of the human body and for the three-piece golf ball remain the same as those described in Section 3.2. The FE model for the full human body impacted by a golf ball is illustrated in Fig. 3.8(a), with a local magnification of the human head and golf ball.

The simulation results for the pressure (defined as the mean normal stress) at the coup and contrecoup sites in the brain are respectively shown in Figs. 3.8(b) and 3.8(c), where they are also compared to those provided in Pearce and Young (2014). It is seen that the current simulation results agree fairly well with those of Pearce and Young (2014). The
time sequence showing the pressure distribution in the brain is displayed in Fig. 3.9, where each time step is associated with a peak value of the curves shown in Figs. 3.8(b) and 3.8(c).

Fig. 3.8 (a) A golf ball impact on a human head from the back using the GHBMC M50-P full body model; (b) the pressure-time history at the coup site; (c) the pressure-time history at the contrecoup site.

Fig. 3.9 Time sequence showing the evolution of the intracranial pressure (ICP).
3.4 Results

3.4.1 Baseline case

The impact by a golf ball on a human head from the front at an impact velocity of 35 m/s is considered as the baseline case in the current study, which uses the FE models and material properties for the three-piece golf ball and the GHBMC M50-P full human body described in Section 3.2.

The simulation results for the time history of the impact force between the golf ball and human head are shown in Fig. 3.10(a) with a peak value of 8.45 kN and a duration of 0.62 ms, which are consistent with the experiments of Roberts et al. (2001). The von Mises stress distribution in the outer compact bone is displayed in Fig. 3.10(b), and the time-history curves of the maximum von Mises stress in the outer compact bone and cancellous bone are plotted in Fig. 3.10(c). It is found that the maximum von Mises stress in the outer compact and cancellous bones has a peak value of 58.62 MPa and 5.26 MPa at $t = 0.34$ ms and $t = 0.62$ ms, respectively. The pressure and the first principal strain at the five positions of the brain shown in Fig. 3.10(d), which are similar to those used in Freitas et al. (2014) and Li et al. (2016), are plotted in Figs. 3.10(e) and 3.10(f), respectively. Figure 3.10(e) shows that the maximum positive and negative values of the pressure at the impact site (P1) is 490.7 kPa and −383.6 kPa, respectively. Figure 3.10(f) reveals that the peak value of the first principal strain at P1 is 0.0278.
Fig. 3.10 Simulation results from the baseline model of the frontal impact at a velocity of 35 m/s: (a) the impact force-time history; (b) the von Mises stress distribution in the outer compact bone at the peak impact force; (c) the time-history of the maximum von Mises stress in the outer compact and cancellous bones; (d) five positions in the brain; (e) the pressure-time history at the five positions; (f) the first principal strain-time history at the five positions.

Figure 3.11 shows two time sequences in the brain. The upper row displays the evolution of the pressure in the brain during the golf ball impact. It is found that the maximum positive and negative values (in magnitude) of the pressure at the coup site in the brain (i.e., coup pressure) are 553.1 kPa at $t = 0.74$ ms and $-451.8$ kPa at $t = 0.48$ ms, respectively. The lower row depicts the time sequence for the first principal strain at the coup site in the brain, which reaches a peak value of 0.1131 at $t = 0.74$ ms.
3.4.2 Effects of the impact location

In addition to the frontal impact, two other locations, i.e., lateral and crown impacts, are investigated. The von Mises stress distributions in the outer compact bone at the time of the peak impact force for each of the three impact locations are shown in Fig. 3.12(a). It is found that the maximum von Mises stress is located near the impact site. The time history of the impact force is plotted in Fig. 3.12(b), which reveals that the frontal impact generates the highest impact force (peak value), which is followed by the crown and lateral (right) impacts. The maximum von Mises stress in the outer compact bone at the coup site for the three impact locations are plotted in Fig. 3.12(c). It is seen that the lateral impact leads to the largest von Mises stress, while the frontal and crown impacts result in smaller values of the von Mises stress that are close to each other.
The pressure and first principal strain at the five locations in the brain shown in Fig. 3.10(d) are displayed in Fig. 3.13. The pressure has a peak value of 0.96 MPa at $t = 0.20$ ms and $-1.07$ MPa at $t = 0.58$ ms at P2 for the lateral (right) impact (see Fig. 3.13(a)), and the pressure reaches its maximum positive and negative values (in magnitude) of 0.63 MPa at $t = 0.82$ ms and $-0.68$ MPa at $t = 0.54$ ms respectively at P5 for the crown impact (see Fig. 3.13(b)). The maximum value of the first principal strain is 0.073 at $t = 1.34$ ms at P2 for the lateral (right) impact (see Fig. 3.13(c)) and 0.074 at $t = 0.70$ ms at P5 for the crown impact (see Fig. 3.13(d)).

Fig. 3.12 Effects of the impact location: (a) the von Mises stress distribution in the outer compact bone at the peak impact force in each case – frontal (left), lateral (middle) and crown (right); (b) the time history of the impact force; (c) the time history of the maximum von Mises stress.
Fig. 3.13 Simulation results at the five points in the brain for the lateral (right) and crown impacts: (a) the pressure-time history for the lateral impact; (b) the pressure-time history for the crown impact; (c) the time history of the first principal strain for the lateral impact; (d) the time history of the first principal strain for the crown impact.

3.4.3 Effects of the impact velocity

Considering the possibilities of a golf ball striking on a human head at various velocities and from different angles, the effects of the impact velocity and impact angle are discussed in this and the next sections. The golf ball impact velocity varies from 15 m/s to 76 m/s.

The time history of the impact force in the frontal impact of a golf ball on a human head is shown in Fig. 3.14(a) for different impact velocities. It is seen that the impact force
increases with the increase of the impact velocity. The analytical model developed in Li et al. (2017) is used here to validate the FE simulation results for the impact force. In applying this model, the parameter values for the human head are taken from Table 3.1 of Li et al. (2017) and those for the golf ball are adopted from Pearce and Young (2014).

Figure 3.14(b) displays the impact force-velocity curve obtained from the current FE model, where it is also compared to that predicted by the analytical model (nonlinear) of Li et al. (2017). It is seen that the simulation results agree fairly well with predictions by the analytical model at low impact velocities, at which the analytical model applies.

The maximum von Mises stress in the outer compact bone is plotted in Fig. 3.14(c) for different impact velocities. It is observed that the maximum von Mises stress increases as the impact velocity increases, which is the same trend as that exhibited by the impact force (see Fig. 3.14(a)). The maximum von Mises stress reaches its largest value of 98.87 MPa at the impact velocity of 76 m/s and takes its smallest value of 17.14 MPa at the velocity of 15 m/s.

![Image](image_url)

Fig. 3.14 Frontal impact of the golf ball at different impact velocities: (a) the time history of the impact force; (b) the maximum impact force-velocity curve and its comparison with the predictions by the analytical model of Li et al. (2017); (c) the time history of the maximum von Mises stress.
Figures 3.15(a) and 3.15(b) show the pressure at the coup and contrecoup sites in the brain obtained from the simulations of the frontal impact at different impact velocities. It is observed that the pressure has the same variation trend as that of the impact force and of the maximum von Mises stress when the impact velocity changes. The pressure at both the coup and contrecoup sites has the largest value at the velocity of 76 m/s, which is followed by that at the impact velocity of 65 m/s, 55 m/s, 45 m/s, 35 m/s, 25 m/s, or 15 m/s respectively in a descending order.

3.4.4 Effects of the impact angle

Figure 3.16(a) shows the golf ball impact angle varying from 45 degrees to 90 degrees (normal to the head) in frontal impacts. The time history of the impact force is shown in Fig. 3.16(b). It is seen that the impact force decreases as the golf ball impact angle decreases (from 90 degrees). But the impact duration is close for all cases. The maximum von Mises stress-time history curves at different impact angles are displayed in Fig.
3.16(c). It is observed that the maximum von Mises stress decreases with the decrease of the impact angle, which is the same variation trend as that of the impact force. For the pressure in the brain, the effect of the impact angle is shown in Figs. 3.17(a) and 3.17(b) at the coup and contrecoup sites, respectively. It is observed that the pressure decreases when the impact angle decreases, like what is exhibited by the impact force and maximum von Mises stress.

Fig. 3.16 (a) The 90-deg frontal impact and an oblique frontal impact with a 45-deg impact angle; (b) the time history of the impact force at different impact angles; (c) the time history of the maximum von Mises stress at different impact angles.

Fig. 3.17 The time history of the pressure in the brain for frontal impacts with different impact angles: (a) at the coup site; (b) at the contrecoup site.
3.5 Discussion

3.5.1 Predictions based on the baseline model

For a golf ball impact on a human head, the main head injury comes from the skull fracture (e.g., Rahimi et al., 2005; Fountas et al., 2006; McGuinness et al., 2016). An experimental study was conducted on thirty-one unembalmed human cadaver heads using plate impact tests at an impact velocity of 4.3 m/s by Allsop et al. (1991). Their results showed that the impact force threshold for the skull fracture has an average value of 12.39 kN. Another experimental investigation of impacts on unembalmed intact human cadaver heads at an impact velocity in the range of 7.1~8.0 m/s was performed by Yoganandan et al. (1995) employing an electrohydraulic piston. They found a failure load 11.9 kN (± 0.9 kN) for dynamic loading. A more recently study by Delye et al. (2007) using human cadaver heads without embalming but including scalp, soft tissue and intracranial contents revealed that the skull fracture has a threshold value of 10.239 kN (± 2.562 kN) at impact velocities ranging from 6.91~6.99 m/s. A comparison shows that the maximum impact force of 8.45 kN predicted by the baseline model in the current study (see Fig. 3.10(a)) is less than the mean threshold value for dynamic skull fracture reported in the above-mentioned experimental studies, which indicates that the skull fracture should not happen when the golf ball velocity is 35 m/s and the impact is frontal.

Note that the experimental results used above for the comparison were obtained at lower impact velocities than what is encountered in a typical golf ball strike. The reason for this is that there has been a lack of testing values for the skull fracture force at high-velocity impacts in the existing literature. According to the experimental study of Yoganandan et al. (1995), the critical value for the skull fracture force increases with the
increase of the impact velocity. As a result, the conclusion drawn above based on the comparison with the existing experimental values obtained at lower impact velocities would still hold if a higher critical value of the skull fracture force measured at a higher impact velocity were to become available and to be used.

The predicted values of the maximum von Mises stress in the outer compact and cancellous bones are respectively 58.62 MPa and 5.26 MPa, as shown in Fig. 3.10(c). The experimental study conducted by Melvin et al. (1970) using tension tests for the compact bone and compression tests for the cancellous bone found that in the strain rate range of 0.01 to 100 s\(^{-1}\), the critical stress value for skull fracture varies respectively from 68.95 MPa to 96.53 MPa for the compact bone and from 8.96 MPa to 219.94 MPa for the cancellous bone. A more recently study on the zygomatic bone fracture by Schaller et al. (2012) reported a threshold value of 153 MPa. A comparison with these experimentally determined values reveals that the von Mises stress in the compact and cancellous bones predicted by the baseline model in the current study should not result in skull fracture, which is the same conclusion as that based on the impact force mentioned above.

The peak positive and negative values of the coup pressure are found to be 490.7 kPa and −383.6 kPa respectively in the baseline model of the current study (see Fig. 3.10(e)). Ward et al. (1980) proposed a critical ICP value of 235 kPa for a serious brain injury and 173 kPa for a minor or no brain injury in a short duration (between 1 and 10 ms). Zhang et al. (2004) reported that the critical values of the ICP for the brain injury in a duration of 10 to 20 ms are about 53~130 kPa at the coup site, and −128 ~ −48 kPa at the contrecoup site. Another experimental study by Zhang et al. (2007) found that the ICP ranges from 644.6 to −92.8 kPa by using spherical head models with gelatin and Sylgard simulants impacted
by two commonly-used handgun projectiles. A more recent study by Freitas et al. (2014) on a helmet-protected head surrogate under ballistic impacts identified an ICP value of 255 kPa at an impact velocity of 428–438 m/s for a short duration, which can lead to moderate cranial injuries. Clearly, the critical values obtained in these studies differ from each other due to the differences in the head models and impact loading conditions. The peak pressure values obtained in the current study are within the range of the critical ICP values reviewed above, which indicates that there may exist mild brain injuries without skull fracture at the impact velocity of 35 m/s.

The maximum value of the first principal strain is 0.0278 according to the baseline model in the current study (see Fig. 3.10(f)). Bain and Meaney (2000) experimentally found that the conservative, optimal, and liberal strain threshold values are 0.13, 0.18, 0.28, respectively, for mild traumatic brain injury (mTBI). Zhang et al. (2004) reported that the critical strain values for 25%, 50% and 80% probability of mTBI are 0.14, 0.19 and 0.24, respectively. Compared with these strain thresholds, the critical strain value obtained in the current study is much smaller, which indicates that there should be no mTBI induced by the golf ball impacts considered here if the critical strain is used as an injury criterion.

Note that due to the lack of threshold values of strain for brain injuries at high-velocity impacts (with a strain rate of 250 – 350/s in the current simulations), the experimental threshold values of strain found in Bain and Meaney (2000) are used in the comparison above. These threshold values were obtained from the testing data for guinea pigs at a strain rate of 30 – 60/s, and their use for human brain injuries is based on the notion that strain-based injury criteria are similar across species (Ommaya et al., 1967; Bain and Meaney, 2000). In fact, in their study on blunt impact-induced TBI of a human head, Zhang et al.
(2004) used the strain threshold values that are very close to those of Bain and Meaney (2000), as indicated above.

3.5.2 Effects of the impact location, velocity and angle

As indicated by the results of the parametric study on the impact force, the maximum von Mises stress in the skull, and the pressure and first principal strain in the brain, which are displayed in Figs. 3.12–3.17, the impact location, velocity, and angle have significant effects on head injury risks. The impact force values for all the three impact locations, as shown in Figs. 3.12(b), are lower than the threshold values for skull fracture experimentally obtained in Allsop et al. (1991), Yoganandan et al. (1995) and Delye et al. (2007), as mentioned above in Section 3.5.1. However, the maximum von Mises stress in the lateral (right) impact, as displayed in Fig. 3.12(c), has a peak value of 71.27 MPa, which exceeds the skull fracture threshold values found in the existing experimental studies of Melvin et al. (1970) reviewed earlier. Thus, the lateral (right) impact by a flying golf ball on the human head has a higher risk of skull fracture than the frontal and crown impacts. For the pressure in the brain, its maximum value always happens near the impact site, as shown in Fig. 3.10(e) (at P1), Fig. 3.13(a) (at P2) and Fig. 3.13(b) (at P5). Compared with the other two impact locations, the lateral (right) impact leads to a larger pressure value (see Figs. 3.10(e), 3.13(a) and 3.13(b)) and a higher first principal strain value (see Figs. 3.10(f), 3.13(c) and 3.13(d)) at the impact site.

The pressure values for the frontal, lateral and crown impacts, as shown in Figs. 3.10(c), 3.13(a) and 3.13(b), have all exceeded the mean threshold values for mTBI reported in
Ward et al. (1980), Zhang et al. (2004, 2007) and Freitas et al. (2014), which may result in brain injuries such as intracranial hemorrhage.

The effect of impact velocity of the golf ball on the human head has been described by evaluating the impact force, von Mises stress in the skull and pressure in the brain, as shown in Figs. 3.14 and 3.15. The skull has a higher risk of fracture at a larger impact velocity. When the impact velocity is 45 m/s, the impact force and von Mises stress have a peak value of 10.69 kN and 75.95 MPa respectively for the frontal impact (see Figs. 3.14(a) and 3.14(c)). Both of these two values exceed the mean threshold values for skull fracture reviewed in Section 3.5.1, which indicates that when the golf ball impact velocity reaches 45 m/s in the frontal impact, there may be skull fracture. For the brain pressure, Fig. 3.15 shows that when the impact velocity reaches 35 m/s, the pressure has a maximum positive value of 490.7 kPa and a maximum negative value (in magnitude) of −383.6 kPa, which exceed the brain injury threshold values reviewed above and may result in mTBI.

The effect of impact angle has been illustrated in Figs. 3.16 and 3.17, which indicate that the normal (90-deg) impact has the highest risk of head injury among all the impacts with an oblique impact angle ranging from 45 to 90 degrees.

The parameter values at different impact velocities and impact angles are summarized in Table 3.5, where they are also compared to those predicted by the baseline model. It is seen from Table 3.5 that the impact velocity has a significant effect on the head response, as shown in columns 3 and 4. Also, the values of the impact force and brain pressure at the impact velocity \( v_0 = 76 \text{ m/s} \) are more than doubled those predicted by the baseline model (with \( v_0 = 35 \text{ m/s} \)). In addition, Table 3.5 shows that when the impact angle changes from 90 degrees to 45 degrees, the values of all the parameters (see column 5, with \( v_0 = 35 \text{ m/s} \))
become significantly lower than those given by the baseline model (with a 90-deg impact angle) (see column 2).

Table 3.5 Effects of the impact velocity and angle on the head response

<table>
<thead>
<tr>
<th></th>
<th>Baseline model</th>
<th>Model with (v_0 = 76) m/s</th>
<th>Model with (v_0 = 15) m/s</th>
<th>Model with (\alpha = 45^\circ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max impact force (kN)</td>
<td>8.45</td>
<td>18.51</td>
<td>3.24</td>
<td>5.74</td>
</tr>
<tr>
<td>Max first principal strain in the brain</td>
<td>0.0278</td>
<td>0.16</td>
<td>0.0057</td>
<td>0.014</td>
</tr>
<tr>
<td>Max pressure in the brain (MPa)</td>
<td>0.49</td>
<td>2.03</td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td>Max von Mises stress in the compact bone (MPa)</td>
<td>58.62</td>
<td>98.87</td>
<td>17.14</td>
<td>13.41</td>
</tr>
<tr>
<td>Max von Mises stress in the cancellous bone (MPa)</td>
<td>5.26</td>
<td>13.90</td>
<td>4.08</td>
<td>4.29</td>
</tr>
</tbody>
</table>

3.6 Summary

Head injuries caused by golf ball impacts are evaluated using a newly constructed finite element model that integrates the GHBMC M50-P full human body model and a three-piece golf ball model. Both the golf ball and full body models are validated against existing experimental data or simulation results. The frontal impact at an impact velocity of 35 m/s is first simulated as the baseline case, which is followed by a study on the effects of impact location, velocity and angle. The numerical results show that the golf ball impacts at all the three locations (frontal, lateral and crown) can result in mild TBIs, while the lateral (right) impact leads to higher risks of skull fracture. In addition, the simulation results reveal that the impact force, maximum von Mises stress, pressure and first principal strain all increase as the impact velocity or the impact angle increases.
References


Ogden, R.W., 1984, *Non-linear elastic deformations*, Ellis Harwood Ltd., Chichester,
England.


4. MODELING OF BALLISTIC IMPACTS ON A HELMETED HEAD FORM

4.1 Introduction

Combat helmets have been used for centuries to protect soldiers in battlefields. Such helmets have evolved from the first generation made of steel to the current generation made of composites (e.g., Walsh et al., 2005; Kulkarni et al., 2013). The ACH, the currently serving helmet in the U.S. Army, is made from Kevlar® K129 fibers bonded with a phenolic resin matrix (e.g., Walsh et al., 2005; Kulkarni et al., 2013). Continuing efforts are being made to further reduce the helmet weight. This has led to the development of the Enhanced Combat Helmet, which has been under development since 2007 for the U.S. Marine Corps and the U.S. Army (e.g., Kulkarni et al., 2013).

Combat helmets made from advanced composites provide enhanced protection against penetrating head injuries from ballistic and shrapnel threats and have saved lives of many soldiers. However, the reduced weight of such a composite helmet tends to result in a larger BFD, which can lead to head injuries known as BHBT. BHBT has emerged as a serious injury type experienced by soldiers in battlefields (e.g., Carroll and Soderstrom, 1978; Sarron et al., 2000; Cannon, 2001; Hisley et al., 2011; Prat et al., 2012) and has recently
received increased attention (e.g., Freitas et al., 2014b; Rafaels et al., 2015; Young et al., 2015).

The helmet BFD is defined as the maximum depth recorded by a piece of clay embedded in a ballistic dummy head form (e.g., Committee, 2014). Digital image correlation (DIC) has emerged as a new technique for measuring surface deformations of materials. This technique has been used to measure dynamic deformations of composite laminates and combat helmets under ballistic impact (e.g., Hisley et al., 2011; Vargas-Gonzalez et al., 2011; Chocron et al., 2013; Freitas et al., 2014a). Unlike the conventional method that is only capable of recording the maximum BFD, DIC can provide the time history of the helmet BFD additionally. Both the maximum value and the time history of the helmet BFD are important in understanding ballistic impact-induced head injuries (e.g., Freitas et al., 2014b). Therefore, helmet models that can accurately predict not only the maximum value but also the time history of the helmet BFD are needed.

A number of studies have been undertaken to simulate deformations of composite helmets under ballistic impact. Khalil et al. (1974) used an axisymmetric head-helmet model to study the dynamic response of the helmet in a short-duration impact. A series of studies on the Personnel Armor System for Ground Troops (PASGT) helmet under ballistic impact were performed by van Hoof (1999), van Hoof and Worswick (2001) and van Hoof et al. (2001) using both experimental and numerical methods. Their studies showed that ballistic impact has localized effects (which are restricted to the impacted area) and the global motion of the helmet is negligible. Their results also revealed that an impact by the helmet interior on the skull could occur when the BFD exceeds the stand-off distance. In a numerical study, Baumgartner and Willinger (2005) investigated two types of impacts,
namely, a high velocity ballistic projectile toward a helmet protected head and a direct impact of a flash ball toward a nonprotected head. The prediction from their model showed a risk of skull fracture due to the helmet BFD. The influences of shell stiffness and impact direction on head injury were investigated by Aare and Kleiven (2007). They found that helmet shell deflections should not exceed the initial distance between the shell and the head in order to protect the head from the most injurious threat levels. Lee and Gong (2010) assessed the effects of different interior cushioning systems on head injury and concluded that the helmet together with its interior strap offers a good protection against small fragments but fares poorly against larger projectile rounds. A more recent study by Tan et al. (2012) used both finite element (FE) simulations and impact tests to evaluate the performance of the ACH as well as the effectiveness of its interior cushioning systems. Their study showed that softer foams with low hardness are more effective as shock absorbing materials against ballistic impacts.

These existing studies provide valuable information about composite helmet modeling. However, a major limitation is that the loading conditions considered in many of the existing models are not representative of actual ballistic impact events (with smaller BFD values predicted by a model than measured ones). For example, in the recent study of Jazi et al. (2014), a linear elastic material response is assumed for the helmet shell without considering any failure mode. This leads to the prediction of a small helmet BFD (of less than 12 mm). In another recent study by Tse et al. (2014), the maximum helmet deflection was found to be around 10.9 mm. These predicted values are much smaller than experimentally obtained helmet deflection values, which are normally larger than 25 mm (e.g., Hisley et al., 2011; Committee, 2014). On the other hand, the models that do simulate
realistic ballistic loading conditions are validated only against the maximum value of the helmet BFD measured in experiments. The dynamic BFD has not been considered in existing FE models. This motivated the current work.

In this chapter, a computational model for the ACH under ballistic impact is developed and validated against the experimental data on both the maximum value and time history of the helmet BFD obtained by Hisley et al. (2011) at the Army Research Laboratory. By using the validated helmet model, the ballistic impact on an ACH placed on a ballistic dummy head form with a clay insert is then simulated, which gives the helmet BFD as recorded by the clay, as specified in the current ACH testing protocol.

4.2 Model Description

4.2.1 Helmet Shell and Foam Pads.

The FE mesh of an ACH is shown in Fig. 4.1, which is for a large-size helmet as specified in the ACH operator’s manual (2010). The FE model is constructed using hexahedral elements (totaling 91,296 for this large-size helmet shell) with a one point integration procedure and a viscous hourglass control scheme. Twelve layers of elements are used through the thickness of the helmet shell based on a convergence study of FE models with different number of layers. The helmet shell is divided into four parts in order to properly model the delamination at the interface between two adjacent laminas according to the recommendation provided in the user manual of LS-DYNA (2015). The mesh is refined at the impact sites. In the study of helmet BFD using a ballistic dummy head form embedded with clay as a fixture (see Sec. 4.3.6), suspension foam pads (with
45,160 hexahedral elements in total for the large-size helmet; see Fig. 4.1(b)) are attached to the helmet to get proper positioning on the head form fixture.

![Image](image.png)

Fig. 4.1 (a) FE mesh of a large-size ACH shell and (b) FE mesh of foam pads. Here, “1” and “2” represent the two in-plane directions and “3” stands for the thickness direction.

The helmet shell is treated as an orthotropic elastic material which is represented using nine elastic constants including three Young’s moduli $E_{11}$, $E_{22}$, and $E_{33}$, three Poisson’s ratios $\nu_{12}$, $\nu_{13}$, and $\nu_{23}$, and three shear moduli $G_{12}$, $G_{23}$, and $G_{31}$ (e.g., Jones, 1999; Gao, 2001). A progressive damage model elaborated in Xiao et al. (2007) and Gama and Gillespie (2011) is used to describe the complex composite damage modes under high strain rate and high pressure loading conditions, which include fiber tension-shear failure, fiber compressive failure, fiber crush, through-thickness matrix failure, and delamination (e.g., Refs. Xiao et al., 2007; Gama and Gillespie, 2011; Carrillo et al., 2012; Jordan et al., 2014). This damage model has been implemented in LSDYNA (2015) as MAT 162 (MAT_COMPOSITE_DMG_MSC), which is used in the current simulations.

Although a number of studies have been conducted to characterize the mechanical properties of Kevlar fibers (e.g., Wang and Xia, 1998; Lim et al., 2011) and Kevlar fabrics (e.g., Bilisik and Turhan, 2009; Zhu et al., 2011), there is a lack of experimental studies on
composite laminas made from Kevlar fibers embedded in a thermosetting resin. The majority of the existing numerical studies on composite combat helmets (e.g., Aare and Kleiven, 2007; Lee and Gong, 2010; Zhang et al., 2013; Jazi et al., 2014) adopted the parameter values from van Hoof et al. (2001). Following these studies, the parameter values provided in van Hoof et al. (2001) are taken as the baseline values in the current study, which are updated using the more recent data on Kevlar 129 composite panels reported in Gower et al. (2008). For those parameter values that are still not available but needed for the progressive damage model, they are estimated based on relevant studies, as noted in Table 4.1. The values of the properties for the Kevlar 129 fiber/phenolic resin composite as the ACH shell material are listed in Table 4.1.

### Table 4.1 Material properties and parameters for the helmet shell

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>$\rho = 1230 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_{11} = E_{22} = 22 \text{ GPa}, E_{33} = 9 \text{ GPa}$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu_{12} = 0.25, \nu_{13} = \nu_{23} = 0.33$</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$G_{12} = 0.77 \text{ GPa}, G_{23} = G_{31} = 2.715 \text{ GPa}$</td>
</tr>
<tr>
<td>Tensile strength</td>
<td>$S_{1T} = S_{2T} = 800 \text{ MPa}$</td>
</tr>
<tr>
<td>Compression strength</td>
<td>$S_{1C} = S_{2C} = 60 \text{ MPa}^{(a)}$</td>
</tr>
<tr>
<td>Normal strength</td>
<td>$S_N = 34.5 \text{ MPa}$</td>
</tr>
<tr>
<td>Fiber Crush strength</td>
<td>$S_{FC} = 1200 \text{ MPa}$</td>
</tr>
<tr>
<td>Fiber shear strength</td>
<td>$S_{FS} = 1086 \text{ MPa}$</td>
</tr>
<tr>
<td>Matrix shear strength</td>
<td>$S_{12} = 77 \text{ MPa}, S_{23} = S_{31} = 898 \text{ MPa}$</td>
</tr>
<tr>
<td>Delamination coefficient</td>
<td>$S = 1.2^{(b)}$</td>
</tr>
<tr>
<td>Coulomb friction angle</td>
<td>$\Phi = 10 \text{ degrees}^{(b)}$</td>
</tr>
<tr>
<td>Coefficient for strain rate dependent strength</td>
<td>$C_{rate1} = 0.0257^{(c)}$</td>
</tr>
<tr>
<td>Coefficient for strain rate dependent axial, shear, or transverse modulus</td>
<td>$C_{rate2,3,4} = 0.0246^{(c)}$</td>
</tr>
<tr>
<td>Scale factor for residual compressive strength</td>
<td>$S_{RFC} = 0.3^{(b)}$</td>
</tr>
<tr>
<td>Element eroding axial strain</td>
<td>$E_{LIMIT} = 4.5^{(d)}$</td>
</tr>
<tr>
<td>Limit damage parameter for elastic modulus reduction</td>
<td>$\omega = 0.9975^{(d)}$</td>
</tr>
<tr>
<td>Limit compressive relative volume for element eroding</td>
<td>$ECRS = 0.001^{(b)}$</td>
</tr>
<tr>
<td>Limit expansive relative volume for element eroding</td>
<td>$EEXP = 5.0^{(b)}$</td>
</tr>
<tr>
<td>Coefficient for strain rate softening property for fiber and matrix damage</td>
<td>$AM_1 = AM_2 = 0.5, AM_3 = 0.1, AM_4 = 20^{(d)}$</td>
</tr>
</tbody>
</table>
Notes:
(a) The value of 60 MPa for $S_{1C} = S_{2C}$ is estimated. This estimate is supported by the experimental study of Zhu et al. (1992), where a failure stress of 60 MPa was obtained for laminated Kevlar composites.
(b) Assumed value which has been reported for common composite laminates in Gama and Gillespie (2011).
(c) Fitted from the experimental data of Wang and Xia (1998).
(d) Parameter values reported in Gama and Gillespie (2011) are chosen as a starting point, and the values are tuned to match the deformations experimentally observed by Hisley et al. (2011).

The current ACH suspension foam pads are made from the Zorbium Action Pad (ZAP™) manufactured by Team Wendy (Cleveland, OH), which is a polyurethane-based foam material consisting of one hard layer and one soft layer (e.g., Moss and King, 2011; Fitek and Meyer, 2013; Zhang et al., 2013). The material model MAT_LOW_DENSITY_FOAM available in the LS-DYNA material library, which is suitable for describing responses of foam materials under large deformations, is employed to simulate both the hard (density: 63 kg/m³) and soft (density: 61 kg/m³) layers of the ZAP™ foam. This material model requires a nominal stress–strain curve controlling the foam response under compressive loading. The experimental data on the Team Wendy foam reported in Moss and King (2011) are adopted and extrapolated to higher strain rates to account for the rate-dependent material behavior under ballistic impacts. Other parameters needed in the simulation of the ACH foam pads, including a hysteretic factor and a shape factor, are obtained from Fitek and Meyer (2013).

4.2.2 Full Metal Jacket (FMJ) Bullet.

The dimensions and FE mesh of a 9mm FMJ bullet are shown in Fig. 4.2. The dimensions shown are specified according to the NIJ Standard (1981). The bullet, weighting about 8 g, contains a brass jacket and a lead core. The basic properties of the
brass jacket and lead core at the reference state are listed in Table 4.2. Hexahedral elements (totaling 1610, with 426 elements for the brass jacket and 1184 elements for the lead core) are used in discretizing the FMJ bullet. The brass jacket is modeled using the Johnson–Cook constitutive model (e.g., Johnson and Cook, 1983; Li et al., 2002), which is capable of describing material behavior under large strains, high strain rates and high temperatures, like in a ballistic impact.

The Johnson–Cook model can be written as (e.g., Johnson and Cook, 1983; Li et al., 2002)

$$
\bar{\sigma} = \left( A + B \bar{\varepsilon}^m \right) \left( 1 + C \ln \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0} \right) \left[ 1 - \left( \frac{T^*}{T_r} \right)^n \right],
\tag{4.1}
$$

where $A$, $B$, $C$, $m$ and $n$ are material constants, $\bar{\sigma}$ is the von Mises equivalent (flow) stress, $\bar{\varepsilon}$ is the equivalent plastic strain, $\dot{\varepsilon}$ is the plastic strain rate, $\dot{\varepsilon}_0$ is a reference strain rate, and $T^*$ is the homologous temperature defined by $T^* = (T - T_r) / (T_m - T_r)$, with $T_r$, $T_m$ and $T$ being the room temperature, melting temperature, and workpiece temperature, respectively. The material constants for the brass jacket are listed in Table 4.3, which are
taken from Johnson and Cook (1983), with $T_r$ and $T_m$ being the same as those given in Børvik et al. (2009).

The shear modulus $G$ and the von Mises equivalent stress $\bar{\sigma}$ for the lead core under a high temperature and a high strain rate are obtained using the Steinberg–Guinan constitutive model given by (Steinberg et al., 1980)

\[
G = G_0 \left[ 1 + \left( \frac{G'_p}{G_0} \right) \frac{p}{\eta^{1/3}} + \left( \frac{G'_T}{G_0} \right) (T - 300) \right],
\]

\[
\bar{\sigma} = \bar{\sigma}_0 \left[ 1 + \beta (\bar{\varepsilon} + \bar{\varepsilon}_i)^k \right] \left[ 1 + \left( \frac{\bar{\sigma}'_p}{\bar{\sigma}_0} \right) \frac{p}{\eta^{1/3}} + \left( \frac{G'_T}{G_0} \right) (T - 300) \right],
\]

where $p$, $\bar{\varepsilon}$, $T$ are, respectively, the pressure, equivalent plastic strain and temperature, $G_0$ and $\bar{\sigma}_0$ are, respectively, the initial values of $G$ and $\bar{\sigma}$ (at the reference state with $T = 300$ K, $p = 0$, $\bar{\varepsilon} = 0$), $\eta$ is the relative volume defined as the initial specific volume $v_0$ divided by the specific volume $v$), $\beta$ and $k$ are work-hardening parameters, $\bar{\varepsilon}_i$ is the initial equivalent plastic strain (normally zero), $G'_p$ and $G'_T$ are, respectively, the derivatives of $G$ with respect to the pressure and temperature at the reference state, and $\bar{\sigma}'_p$ is the derivative of $\bar{\sigma}$ with respect to the pressure at the reference state. The properties and parameters involved in Eqs. (4.2) and (4.3) are listed in Tables 4.2 and 4.4.

The responses of these two materials under high-pressure compression can be described using the Mie–Grüneisen equation of state given by (e.g., Steinberg et al., 1991; LS-DYNA, 2015)
where \( p \) is the pressure, \( \rho_0 \) is the initial density, \( M \) is the bulk sound speed, \( U \) is the energy per initial volume, \( \gamma_0 \) is the initial value of Grüneisen’s gamma, \( a \) is a non-dimensional first order volume correction to \( \gamma_0 \), \( S_1 \), \( S_2 \), and \( S_3 \) are non-dimensional coefficients of the Hugoniot slope, and \( \mu = \frac{\rho}{\rho_0} - 1 \) (with \( \rho \) being the density). The properties and parameters involved in Eqs. (4.4a) and (4.4b) are provided in Tables 4.2 and 4.5.

Table 4.2 Basic properties at the reference state (Steinberg et al., 1991; Tse et al., 2014)

<table>
<thead>
<tr>
<th>Material</th>
<th>( E_0 ) (GPa)</th>
<th>( G_0 ) (GPa)</th>
<th>( \nu_0 )</th>
<th>( \rho_0 ) (kg/m(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass jacket</td>
<td>110</td>
<td>40</td>
<td>0.375</td>
<td>8520</td>
</tr>
<tr>
<td>Lead core</td>
<td>-</td>
<td>8.6</td>
<td>-</td>
<td>11340</td>
</tr>
</tbody>
</table>

Table 4.3 Constants for the Johnson-Cook model applied to the brass jacket (Johnson and Cook, 1983)

<table>
<thead>
<tr>
<th>( A ) (MPa)</th>
<th>( B ) (MPa)</th>
<th>( C )</th>
<th>( m )</th>
<th>( n )</th>
<th>( T_m ) (K)</th>
<th>( T_r ) (K)</th>
<th>( \dot{\varepsilon}_0 ) (s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>505</td>
<td>0.009</td>
<td>1.68</td>
<td>0.42</td>
<td>1189</td>
<td>293</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4.4 Properties and parameters for the Steinberg-Guinan constitutive model applied to the lead core (Steinberg et al., 1980)

<table>
<thead>
<tr>
<th>( \sigma_0 ) (MPa)</th>
<th>( \beta )</th>
<th>( k )</th>
<th>( \varepsilon_i )</th>
<th>( G'_r/G_0 ) (TPa(^{-1}))</th>
<th>( G'_r/G_0 ) (kK(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>110</td>
<td>0.52</td>
<td>0</td>
<td>116</td>
<td>-1.16</td>
</tr>
</tbody>
</table>
Table 4.5 Parameters for the Mie-Grüneisen equation of state (Steinberg, 1991)

<table>
<thead>
<tr>
<th>Material</th>
<th>$M$ (m/s)</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$\gamma_0$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brass jacket*</td>
<td>3667</td>
<td>1.507</td>
<td>0.000</td>
<td>0.000</td>
<td>2.086</td>
<td>0.485</td>
</tr>
<tr>
<td>Lead core</td>
<td>2006</td>
<td>1.429</td>
<td>0.8506</td>
<td>-1.640</td>
<td>2.740</td>
<td>0.54</td>
</tr>
</tbody>
</table>

* The numerical values for the brass jacket listed here are obtained from those for copper (70%) and zinc (30%) provided in Steinberg (1991).

4.2.3 Energy Imparted to the Head and the Blunt Criterion.

The helmet will be in contact with the head and impart a force to the head when the helmet BFD is larger than the stand-off distance. To compare with the experimental data of Hisley et al. (2011), an approach similar to that used in Hisley et al. (2011) is adopted to calculate the energy and blunt criterion (BC) values. The energy imparted to the head from the moment when the helmet gets in contact with the head until when the maximum BFD is reached is estimated to be (Hisley et al., 2011)

$$K = \frac{1}{2} \rho(t) A_e(t) v^2(t) \left[ \text{stand-off distance} \right]_{\text{max BFD}},$$

(4.5)

where $v$ is the velocity of the helmet BFD region, $\rho$ is the areal density of the remaining plies of the helmet, and $A_e$ is the effective area of the helmet shell (i.e., the area of the helmet shell where the deformation is larger than the stand-off distance at the moment when the maximum BFD is reached).

The BC proposed by Sturdivan et al. (2004) and used in Hisley et al. (2011) for head injuries arising from the helmet BFD is defined by

$$\text{BC} = \ln \left( \frac{K}{TD} \right),$$

(4.6)

where $K$ is the impact kinetic energy (in Joules), $D$ is the diameter of the effective area on the helmet (in centimeters) (with a circular shape), and $T$ is the thickness of the skull (in...
millimeters). Note that $T = 6.8$ mm was used in Hisley et al. (2011) and is also adopted in the current study.

4.2.4 BFD Recorded by Clay Embedded in a Ballistic Dummy Head Form.

The ballistic dummy head form shown in Fig. 4.3 is constructed using the dimensions specified in the NIJ Standard (1981). An ACH (including the foam pad suspension system) is then fitted to the dummy head form (Fig. 4.3, right). The head form material is taken to be aluminum 6061-T6 (Committee, 2001), and the clay embedded in the head form to be Roma Plastilina No. 1 oil-based modeling clay whose properties are provided in Roberts et al. (2007) in a study of behind armor blunt trauma in accordance with the NIJ standard 0101.04. The relevant properties of the aluminum and clay used in the head form are listed in Table 4.6.

The material model MAT_SOIL_AND_FOAM in the LS-DYNA material library is adopted for the clay. An extra simulation is performed to ensure that the parameters used for the clay give an indentation depth within 1962 mm for a 63.5 mm diameter steel ball (104.3 g) dropped from 2 m onto a clay block, as specified in the NIJ Standard (1981).

The FE mesh for the head form embedded with clay is shown in Fig. 4.3, where 66,994 tetrahedral elements are used for the aluminum head form and 46,020 tetrahedral elements for the clay insert.
Fig. 4.3 Ballistic dummy head form with a clay insert. From left to right, the geometry, FE mesh of the dummy head, FE model of the dummy head form with clay embedded, and the final assembly of the helmet on the dummy head form. The geometry is adopted from Committee (2014).

Table 4.6 Material properties of the aluminum and clay (Roberts et al, 2007; Bae et al., 2008)

<table>
<thead>
<tr>
<th>Material</th>
<th>Density (kg/m³)</th>
<th>Young’s modulus (GPa)</th>
<th>Poisson’s ratio</th>
<th>Shear modulus (MPa)</th>
<th>Bulk modulus (GPa)</th>
<th>Yield strength (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>2710</td>
<td>68.9</td>
<td>0.33</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Clay</td>
<td>1750</td>
<td>-</td>
<td>-</td>
<td>50</td>
<td>40</td>
<td>0.3</td>
</tr>
</tbody>
</table>

4.3 Results and Discussion

4.3.1 Impact and Validation.

The work of Hisley et al. (2010; 2011) is the most comprehensive experimental study on the dynamic BFD of the ACH. The experimental data of Hisley et al. (2010; 2011) are chosen to validate the current simulation results. In order to compare with the BFD time history curve for an extra-large-size ACH provided in Hisley et al. (2011), the helmet size used in obtaining the simulation results presented in Sections 4.3.1–4.3.4 is also that of an extra-large-size ACH as specified in the ACH operator’s manual (Manual, 2010). The FE mesh for this extra-large helmet shell has 105,252 hexahedral elements.

Ballistic impact to the helmet from the right lateral side of the helmet is first simulated with a bullet velocity of 370 m/s. To compare with the experiments performed in Hisley
et al. (2010, 2011), the helmet shell is fixed at its left and right sides in the simulations. The simulation results for the dynamic sequence of the impact events reveal that the velocity of the helmet shell reaches a maximum at around 0.05 ms, and the BFD gets its maximum at 0.64 ms (see Fig. 4.4(a)). The deformation of the helmet when the BFD reaches its maximum is displayed in Fig. 4.4(b). Similar to what was observed in the experimental study of Hisley et al. (2010), the simulation results show that the impact area has a circular domain of approximately 110mm in diameter and the maximum BFD is 31.05 mm. The deformed bullet displays a classical mushroom shape with a diameter of 25.6mm in the final, permanently deformed state, which is close to the experimental finding of 26.9mm in diameter by Hisley et al. (2010).

The simulation results are further illustrated in Fig. 4.5 by plotting the time history of the BFD and the velocity profile of a point in the impact area with the maximum BFD. The deformation pattern in the impact area viewed from inside the helmet shell is displayed in Fig. 4.5(a). The time history of the BFD obtained in the simulations agrees well with the
The energy imparted to the head from the helmet and the computed values of the BC defined in Eq. (4.6) are listed in Table 4.7 for two cases with the stand-off distance being 12.7mm (case 1) and 19.1mm (case 2), respectively. A comparison shows that the energy obtained in the simulations is close to that measured experimentally by Hisley et al. (2010). It is seen from Table 4.7 that the larger stand-off distance in case 2 leads to significantly reduced energy imparted to the head and results in a much lower risk of head injuries according to the BC. The effective area obtained in the simulations is found to be larger than the experimental value.

Fig. 4.5 (a) The BFD viewed from inside, (b) the time history of the BFD, and (c) the velocity profile. The experimental data plotted for comparison are obtained from Hisley et al. (2011).
Table 4.7 The energy imparted to the head through the helmet BFD induced by a right-side ballistic impact, the effective area diameter, the blunt criterion values, and their comparisons with the experimental data of Hisley et al. (2011).

<table>
<thead>
<tr>
<th></th>
<th>Energy (J)</th>
<th>Effective area diameter (mm)</th>
<th>Blunt criterion (BC)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case 1</td>
<td>Case 2</td>
<td>Case 1</td>
</tr>
<tr>
<td>Simulation</td>
<td>193.5</td>
<td>68.0</td>
<td>61.6</td>
</tr>
<tr>
<td>Experimental</td>
<td>195.2</td>
<td>61.2</td>
<td>49.0</td>
</tr>
</tbody>
</table>

4.3.2 Energy absorption

The system has an initial kinetic energy of 547.6 J corresponding to a bullet of 8 g with a velocity of 370 m/s, as calculated from $K = \frac{1}{2}mv^2$. As shown in Fig. 4.6(a), the simulation results reveal that the total energy remains constant as expected, while the kinetic energy decreases rapidly during the impact and is converted into the internal energy of the bullet and helmet shell. The distribution of the internal energy is displayed in Fig. 4.6(b), which indicates that the helmet shell absorbs most of the internal energy (in the amount of 360.91 J at $t = 0.64$ ms) but the bullet dissipates a significant amount energy (in the amount of 120.61 J at $t = 0.64$ ms, accounting for about 25.0% of the total internal energy) through plastic deformations. Hence, it is very important to use an appropriate material model to describe the deformation behavior of the bullet.
4.3.3 Helmet Shell Failure Modes

The simulation results show that the composite laminate of the helmet shell exhibits complex damage modes under ballistic impact, as illustrated in Fig. 4.7. These include fiber damage (with the fiber direction being indicated by the arrows in Fig. 4.7(a) for warp yarns and in Fig. 4.7(b) for fill (weft) yarns; see, e.g., Gao and Mall (2000) for these and other relevant terms used for woven fabric composites), matrix damage, and delamination. It is seen from Fig. 4.7 that the fiber crush damage mode in Fig. 4.7(c) is not as dominant as the other four damage modes. The damage modes shown in Fig. 4.7 agree with those exhibited in the general damage process of a compliant composite laminate when it is penetrated by a small arm, in which fiber failure, fiber crush, matrix failure, and delamination are typically observed (e.g., Cheeseman and Bogetti, 2003; Hisley et al., 2010). This indicates that the orthotropic elasticity and progressive damage models used for the helmet shell in the current study successfully capture the complex damage process of the helmet shell under ballistic impact.
Fig. 4.7 Different damage modes of the helmet shell under ballistic impact when the BFD reaches its maximum. (a) Fiber damage in the warp direction; (b) fiber damage in the fill direction; (c) fiber crush damage; (d) perpendicular matrix (in-plane shear) damage; (e) parallel matrix (delamination) damage. Here, f1 through f5 represent the damage functions for the respective damage modes defined in Xiao et al. (2007) and adopted in MAT 162 of LS-DYNA (2015).

4.3.4 Effects of Impact Location and Direction

The simulation results for ballistic impacts at different locations and in different directions are shown in Fig. 4.8. It is observed that the helmet deformation patterns for the frontal, crown and lateral (right side) impacts are all circular (see Fig. 4.8, left), but different values of the BFD are obtained at different locations due to the difference in the helmet curvature. The BFD for the frontal impact has the largest value (of 40.8mm at $t = 0.87$ ms), which is followed by that for the crown impact (34.2mm at $t = 0.65$ ms) and then by that for the lateral impact (31.05mm at $t = 0.64$ ms) (see Fig. 4.8, left).

The impact angle of the bullet also has a significant effect on the BFD. For a right-side oblique impact with an impact angle of 60 deg, the maximum BFD is found to be 29.61mm compared to 31.05mm for the perpendicular impact (with an impact angle of 90 deg), and
the maximum value of the BFD is further reduced to 25.75mm for a 45-deg oblique impact, as illustrated in Fig. 4.8 (right).

![Figure 4.8](image)

Fig. 4.8 Effects of impact locations and directions on the helmet BFD. The time history of the BFD for the frontal, crown and lateral (right-side) impacts (from upper to lower) is shown on the left, and the time history of the BFD for the right-side oblique impact with an impact angle of 90 degrees, 60 degrees and 45 degrees (from upper to lower) are displayed on the right.

The simulation results shown in Fig. 4.8 reveal that for oblique impacts a decrease in the impact angle leads to a reduced BFD, thereby lowering the risk of skull fracture. But due to a larger rotational effect in an oblique impact, the risk for injury of the brain tissue, which is more vulnerable to a rotational acceleration, is expected to be higher, as shown in an earlier study by Aare and Kleiven (2007) using a head model.

### 4.3.5 Effect of Helmet Size

In order to study the effect of helmet size on the ballistic performance of an ACH, lateral (right side) impacts (with an impact angle of 90 deg) on helmets of four different sizes, namely, extra large, large, medium and small, as specified in the ACH operator’s manual (Manual, 2010), are simulated. The major dimensions for these four helmets as
specified in Manual (2010) and used in the current simulations are listed in Table 4.8. The bullet velocity in each case is 370 m/s.

<table>
<thead>
<tr>
<th>Helmet Shell Size</th>
<th>Length (mm)</th>
<th>Width (mm)</th>
<th>Height (mm)</th>
<th>Thickness (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>246.38</td>
<td>243.08</td>
<td>231.14</td>
<td>7.91</td>
</tr>
<tr>
<td>Medium</td>
<td>261.62</td>
<td>255.52</td>
<td>236.22</td>
<td>7.86</td>
</tr>
<tr>
<td>Large</td>
<td>266.70</td>
<td>264.16</td>
<td>241.30</td>
<td>7.81</td>
</tr>
<tr>
<td>X-large</td>
<td>279.40</td>
<td>271.78</td>
<td>256.54</td>
<td>7.80</td>
</tr>
</tbody>
</table>

* Maximum values for the ACH specified in Manual (2010).

The deformed shapes of the four helmets corresponding to their respective maximum BFD values are shown in Fig. 4.9(a), and the time history of the BFD for each of the four helmets obtained in the simulations is plotted in Fig. 4.9(b). The experimental curve provided in Hisley et al. (2011) for an extra-large-size ACH is also displayed for comparison, which is first shown in Fig. 4.5(b).

It is found that at the same bullet impact velocity of 370 m/s the maximum BFD values for the small-, medium-, large-, and extralarge- size helmets are, respectively, 33.84 mm, 32.00 mm, 31.87 mm, and 31.05 mm. From the time history curves shown in Fig. 4.9(b), it is seen that this order of helmet deformation is generally true for the entire impact duration. These observations indicate that the extra-large-size helmet can potentially provide the best protection to the head, which is followed by the large-, and then by the medium- and finally by the small-size helmets, as measured by the BFD.
Fig. 4.9 Effect of helmet size on the BFD. The experimental curve shown is obtained from Hisley et al. (2011).

4.3.6 BFD Measured by a Fixture with a Dummy/Clay Head Form

The current testing standard for the ACH includes mainly two types of testing—ballistic impact testing and blunt impact testing (e.g., Committee, 2014). The resistance to penetration and BFD are two measures of ballistic performance of a helmet. During a ballistic impact test, the helmet being tested is fixed to a head form packed with clay which records the maximum BFD.

The stand-off distance used in the current simulations of a large-size helmet on the dummy head form is, respectively, 20.3mm for the frontal impact, 25.6mm for the lateral (right side) impact, and 22.8mm for the crown impact (see Fig. 4.10), which are similar to the reported values of 22.5 mm, 25.6 mm, and 23.0mm in the three respective cases for a large-size helmet given in Committee (2014) based on experimental testing. This indicates good positioning of the helmet on the dummy head form, based on which the ballistic impact simulations are performed.
Fig. 4.10 Stand-off distance for the head form/clay at the right-side and crown impact locations (left) and at the frontal impact location (right), as marked by each short rectangular bar. The foam pads between the helmet shell and the dummy head/clay are not shown.

Figure 4.11 displays the simulation results of the BFD as recorded by the clay when the helmet is impacted at three different locations. The BFD values obtained are 16.56 mm, 11.39 mm, and 5.05 mm for the frontal, crown, and right-side lateral impacts, respectively. The descending order of these BFD values is the same as that, and the BFD values are similar to those, determined from ballistic testing of ACH as recorded by the clay (Committee, 2014; p. 35).

Fig. 4.11 BFD as recorded by the clay using the dummy head/clay as a fixture for (a) the frontal impact, (b) the crown impact, and (c) the right-side impact.

Note that owing to the specific configurations of the head form and foam pads attached to the helmet, the lateral (right side) impact simulated here is off-pad, with the bullet
striking point on the helmet shell located in the unsupported gap between two neighboring side foam pads (see Fig. 4.1(b)). As a result, there are two BFD areas recorded by the clay, with one on each side of the striking point, as shown in Fig. 4.11(c). This also explains why the BFD obtained here is smaller than the average value for right-side ballistic impacts reported in Committee (2014), which may have included more on-pad impacts. The differences between the on-pad and off-pad impacts on a helmet-head assembly are discussed in detail in Li et al. (2016).

4.4 Summary

An FE model for simulating the ballistic performance of the ACH is developed using orthotropic elasticity and a progressive damage model, which successfully capture the complex damage process of the helmet shell (made from the Kevlar 129 fiber/phenolic resin composite laminates) under ballistic impact. Both the maximum value and time history of the helmet BFD are considered in the current study, unlike existing works focusing on the maximum BFD only. It is found that the deformation pattern of the helmet shell predicted by the current model is close to that observed experimentally. In addition, both the maximum value and time history curve of the helmet BFD obtained in the simulations agree well with the experimental data. Furthermore, different BFD values are obtained for ballistic impacts at different locations and in different directions. The frontal impact is found to have the largest BFD, which is followed by a crown impact and then by a lateral impact. The simulation results for oblique ballistic impacts in different directions show that the helmet BFD decreases with the decrease of the impact angle. Also, the effect of helmet size on the ballistic performance of the ACH is studied by simulating helmets of
four different sizes—extra large, large, medium, and small—at the same bullet impact velocity. It is observed that the small-size helmet has the largest BFD, which is followed by the medium-, and then by the large- and finally by the extra large-size helmets, as measured by the BFD. Moreover, the ballistic impact on an ACH placed on a ballistic dummy head form embedded with clay as specified in the current ACH helmet testing protocol is analyzed. The simulation results for the helmet BFD as recorded by the clay in the head form match the experimental data well. The findings of the current study provide new insights into designing more effective combat helmets.
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5. CONSTITUTIVE MODELING

5.1 Introduction

Soft tissues, which can sustain large deformations, have been extensively studied. Such tissues can be characterized as nonlinear, strain rate-dependent, hyperelastic, and anisotropic materials (e.g., Humphrey, 2003; Chagnon et al., 2015). Soft tissues, such as brain (e.g., Ning et al., 2006; Velardi et al., 2006), skin (e.g., Tong and Fung 1976; Lanir, 1983), arteries (e.g., Holzapfel et al., 2000; Holzapfel and Ogden, 2010), tendon (e.g., Pioletti et al., 1998) or ligaments (e.g., Pioletti et al., 1998; Limbert et al., 2003) can be described using hyperelasticity (e.g., Chagnon et al., 2015).

Anisotropic hyperelastic materials can be modeled by employing strain energy density functions. A generic strain energy density function based on the Green strain tensor was proposed by Tong and Fung (1976), which was subsequently modified to have different forms (e.g., Fung et al., 1979; Humphrey, 2002; Chagnon et al., 2015). In these models, the strain components contribute to the strain energy density function with different weights and a large number of material parameters. Another form of the strain energy density function is based on the strain invariants (e.g., Chagnon et al., 2015).
Brain tissues are found to be transversely isotropic and can be treated as hyperelastic or visco-hyperelastic materials. For transversely isotropic materials, the strain energy density function can be divided into the sum of an isotropic part and an anisotropic part. The isotropic part is related to the first two invariants of the right Cauchy-Green deformation tensor $\mathbf{C}$, and it is often represented by a classical model such as the neo-Hookean model or Mooney-Rivlin model. The anisotropic part, which is related to the invariants $I_4$ and $I_5$ of $\mathbf{C}$, can be expressed in terms of polynomial strain invariants (e.g., Merodio and Ogden, 2005; Murphy, 2013). It can also be written in terms of the invariants $I_1$-$I_5$ in a power-law or exponential form (e.g., Weiss et al., 1996; Balzani et al., 2006; Schröder and Neff, 2003; Horgan and Saccomandi, 2005; Kulkarni et al., 2016). Strain energy functions in terms of physically motivated invariants have also been proposed (e.g., Criscione et al., 2001; Lu and Zhang, 2005; Shariff, 2017).

However, all these existing hyperelastic models are based on the polar decomposition of the deformation gradient $\mathbf{F}$. The polar decomposition of the deformation gradient $\mathbf{F}$ plays an important role in continuum mechanics (Truesdell and Noll, 1965). According to the polar decomposition theorem, $\mathbf{F}$, as a second-order tensor with a positive determinant (i.e., $\det \mathbf{F} > 0$), can be uniquely decomposed as (Truesdell and Noll, 1965; Gurtin, 1981)

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR},$$

(5.1)

where $\mathbf{U}$ and $\mathbf{V}$, given by $\mathbf{U} = (\mathbf{F}^T \mathbf{F})^{1/2}$ and $\mathbf{V} = (\mathbf{FF}^T)^{1/2}$ are, respectively, the right and left stretch tensors which are symmetric and positive definite, and $\mathbf{R}$ is a rotation tensor (with $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\det \mathbf{R} = 1$) that is obtainable from the decomposition as $\mathbf{R} = \mathbf{FU}^{-1}$ or $\mathbf{R} = \mathbf{V}^{-1}\mathbf{F}$. 

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The unique polar decomposition of $\mathbf{F}$ listed in Eq. (5.1) has been extended to the decomposition of the form (Boulanger and Hayes, 2001; Jaric et al, 2006):

$$\mathbf{F} = \mathbf{PG} = \mathbf{HP}, \quad (5.2)$$

where $\mathbf{P}$ is a rotation tensor (different from $\mathbf{R}$), and $\mathbf{G}$ and $\mathbf{H}$ are two non-symmetric second-order tensors each having three positive eigenvalues and three independent eigenvectors. This extended polar decomposition of $\mathbf{F}$ is non-unique.

There are other multiplicative decompositions of $\mathbf{F}$. Among them, an important one is the multiplicative decomposition of the elasto-plastic deformation gradient into its elastic and plastic parts (Lee, 1969; Lubarda, 2004), which is also known to be non-unique (Clifton, 1972; Casey and Naghdi, 1980).

Another important multiplicative decomposition is the lower triangular decomposition of $\mathbf{F}$ presented by Souchet (1993), where it was compared to the polar decomposition and applied to solve two important problems involving material time derivatives and compatibility conditions. This decomposition can be identified to be an extended polar decomposition of the form $\mathbf{F} = \mathbf{PG}$ listed in Eq. (5.2), with $\mathbf{G}$ being a lower triangular matrix satisfying $\mathbf{G}^T \mathbf{G} = \mathbf{C}$ and having three positive eigenvalues, where $\mathbf{C}$ is the right Cauchy-Green deformation tensor.

Twenty years after the lower triangular decomposition by Souchet (1993), an upper triangular decomposition of the deformation gradient $\mathbf{F}$ was proposed by Srinivasa (2012), which is a multiplicative decomposition based on the QR factorization that decomposes $\mathbf{F}$ into a product of an orthogonal matrix and an upper triangular matrix (Strang, 2006). That is,

$$\mathbf{F} = \mathbf{Q}\mathbf{F}, \quad (5.3)$$
where $Q$ is a $3 \times 3$ rotation matrix (different from $R$ and $P$ mentioned above), and $\tilde{F}$ is an upper triangular $3 \times 3$ matrix (with up to six non-zero components) which can be obtained from the right Cauchy-Green deformation tensor $C$ through a Cholesky factorization (Strang, 2006). This decomposition is unique, since $F$ is non-singular with $\det F > 0$.

A comparison of Eq. (5.3) with Eqs. (5.1) and (5.2) shows that the upper triangular decomposition can also be viewed as an extended polar decomposition. Compared to the polar decomposition $F = RU$ listed in Eq. (5.1), the upper triangular decomposition $F = Q\tilde{F}$ given in Eq. (5.3) has the following advantages: the six components of $\tilde{F}$ can be directly related to pure extensions and simple shear deformations; there is no need to compute the square root of $C = F^T F$ in order to determine $U$ and $R$; the upper triangular tensor $\tilde{F}$ is closed under both addition and multiplication; the components of the Cauchy stress can be directly expressed as derivatives of the strain energy density function with respect to the components of $\tilde{F}$, leading to simpler and more explicit expressions than those based on the invariants of $C$ (Zheng, 1994; Steigmann, 2002; Fu and Zhang, 2006; Kulkarni et al., 2016).

However, only one possibility of relating $\tilde{F}$ to pure stretch and simple shear deformations was considered in Srinivasa (2012). In the current paper, all of the possibilities of decomposing $\tilde{F}$ into a product of stretching and simple shear deformation matrices are studied. It is shown that

- There are totally 6 possibilities of decomposing $\tilde{F}$ into a product of matrices for one tri-axial stretch and two simple shear deformations. Only one of these possibilities was considered in Srinivasa (2012).
• There are totally 24 possibilities of decomposing $\hat{F}$ into a product of matrices for one tri-axial stretch and three simple shear deformations. None of these was examined in Srinivasa (2012).

The rest of this chapter is organized as follows. In Section 5.2, the upper triangular decomposition of the deformation gradient $F$ is recounted for completeness and tutorial purposes, which follows Srinivasa (2012) but starts from the first principles, is self-contained and has incorporated modifications to the original formulation of Srinivasa (2012). In Section 5.3, two types of decompositions of the distortion tensor $\hat{F}$ into a product of matrices for one tri-axial stretch and two or three simple shear deformations are considered, and all possible such decompositions are analyzed for the first time. In Section 5.4, the distortion tensor is shown to be frame-invariant, and the constitutive relations for both unconstrained and incompressible hyperelastic materials are then derived in terms of the distortion tensor. Two examples are provided in Section 5.5 to illustrate applications of the general constitutive relations obtained in Section 5.4. This chapter concludes in Section 5.6 with a summary.

5.2 Upper-triangular Decomposition of $F$

Consider a body $\mathcal{B}$ in the three-dimensional space $\mathcal{R}$, which occupies a region $\Omega$, known as the current (Eulerian) configuration, and is enclosed by the boundary $\mathcal{S}$ at the current time $t$. At the initial time $t_0$, $\mathcal{B}$ occupied the region $\Omega_0$, called the reference (Lagrangian) configuration. The initial position of a particle $\mathcal{P}$ in $\Omega_0$ at $t_0$ was $X = X(\mathcal{P}, t_0)$, which, after some motion $x = x(X, t)$, takes the current position $x = x(\mathcal{P}, t)$ in $\Omega$ at $t$. The
deformation gradient is then given by \( \mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X} \). It is known that \( \text{det} \mathbf{F} > 0 \) for any \( \mathbf{X} \) in \( \Omega_0 \) and \( t \geq t_0 \).

Note that \( \mathbf{F} \), mapping \( d\mathbf{X} \) in \( \Omega_0 \) to \( d\mathbf{x} \) in \( \Omega \), can be written as

\[
\mathbf{F} = (\mathbf{e}_i \cdot \mathbf{Fe}_j)\mathbf{e}_i \otimes \mathbf{e}_j \equiv \mathbf{f}_j \otimes \mathbf{e}_j, \quad (5.4)
\]

where \( \mathbf{e}_i \) \((i = 1, 2, 3)\) are three base vectors of a Cartesian coordinate system, and \( \mathbf{f}_j \equiv \mathbf{Fe}_i \) are the images of \( \mathbf{e}_i \) in the same vector space mapped by \( \mathbf{F} \). From \( \mathbf{f}_i \), an orthonormal basis \( \mathbf{e}'_i \) \((i = 1, 2, 3)\) can be readily constructed by using the Gram-Schmidt algorithm (Strang, 2006) as

\[
\mathbf{e}'_1 = \frac{\mathbf{f}_1}{|\mathbf{f}_1|}, \quad \mathbf{e}'_2 = \frac{\mathbf{f}_2 - (\mathbf{f}_2 \cdot \mathbf{e}'_1)\mathbf{e}'_1}{|\mathbf{f}_2 - (\mathbf{f}_2 \cdot \mathbf{e}'_1)\mathbf{e}'_1|}, \quad \mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2. \quad (5.5)
\]

The new base vectors \( \mathbf{e}'_i \) can be expanded in terms of the base vectors \( \mathbf{e}_i \) as

\[
\mathbf{e}'_i = (\mathbf{e}'_i \cdot \mathbf{e}_j)\mathbf{e}_j \equiv Q_{ij}\mathbf{e}_j, \quad (5.6)
\]

where \( Q_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j \) are the components of the orthogonal tensor:

\[
\mathbf{Q} = \mathbf{e}'_i \otimes \mathbf{e}_i, \quad (5.7)
\]

which represents the coordinate transformation from the system with the base vectors \( \mathbf{e}_i \) to the system with the base vectors \( \mathbf{e}'_i \), namely, \( \mathbf{e}'_i = \mathbf{Qe}_i \). Since \( \text{det} \mathbf{F} > 0 \) and \( \text{det} \tilde{\mathbf{F}} > 0 \), it follows that \( \text{det} \mathbf{Q} = +1 \), which says that \( \mathbf{Q} \) is a rotation (or a proper orthogonal tensor).

With \( \mathbf{F} \) given and \( \mathbf{Q} \) constructed, \( \tilde{\mathbf{F}} \) can then be determined via the QR factorization as, upon using Eqs. (5.3)–(5.5) and (5.7),

\[
\tilde{\mathbf{F}} = \mathbf{Q}^T \mathbf{F} = \sum_{i,j}^{i,j=1,2,3} \tilde{F}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad (5.8)
\]
where \( \tilde{F}_{ij} \) are given by

\[
\tilde{F}_{ij} = \begin{cases} 
\mathbf{e}_i \cdot \mathbf{f}_j, & i \leq j; \\
0, & i > j.
\end{cases}
\] (5.9)

In the matrix form, \( \tilde{F} \) can be written as

\[
\begin{bmatrix}
\tilde{F}_{11} & \tilde{F}_{12} & \tilde{F}_{13} \\
0 & \tilde{F}_{22} & \tilde{F}_{23} \\
0 & 0 & \tilde{F}_{33}
\end{bmatrix},
\] (5.10)

which is indeed an upper triangular matrix. This matrix can be readily inverted to obtain

\[
\begin{bmatrix}
\tilde{F}^{-1}_{ij}
\end{bmatrix} =
\begin{bmatrix}
\frac{1}{\tilde{F}_{11}} & -\frac{\tilde{F}_{12}}{\tilde{F}_{11} \tilde{F}_{22}} & \frac{\tilde{F}_{12} \tilde{F}_{23} - \tilde{F}_{13} \tilde{F}_{22}}{\tilde{F}_{11} \tilde{F}_{22} \tilde{F}_{33}} \\
0 & \frac{1}{\tilde{F}_{22}} & -\frac{\tilde{F}_{23}}{\tilde{F}_{22} \tilde{F}_{33}} \\
0 & 0 & \frac{1}{\tilde{F}_{33}}
\end{bmatrix},
\] (5.11)

as the matrix of \( \tilde{F}^{-1} \), which is also an upper triangular matrix with six non-zero components.

A typo in \( \begin{bmatrix} \tilde{F}^{-1}_{ij} \end{bmatrix} \) given by Srinivasa (2012) has been corrected in Eq. (5.11).

Note that the right Cauchy-Green deformation tensor \( \mathbf{C} \) can be written in terms of \( \tilde{F} \) as

\[
\mathbf{C} = \mathbf{F}^T \mathbf{F} = \tilde{F}^T \tilde{F},
\] (5.12)

where use has been made of Eq. (5.3). It then follows from Eqs. (5.10) and (5.12) and the symmetry of \( \mathbf{C} \) that the components of \( \tilde{F} \) can be obtained from the components of \( \mathbf{C} \) through a Cholesky factorization process as
After substituting $C_{ij} = F_{ki}F_{kj}$ (see Eq. (5.12)) into Eq. (5.13), the six components of $\bar{F}$ will then be fully determined from the components of $F$.

Finally, note that the rotation tensor $Q$ can also be computed from Eq. (5.3) as

$$Q = FF^{-1}.\quad (5.14)$$

Upon using Eqs. (5.11) and (5.13) and $C_{ij} = F_{ki}F_{kj}$ in Eq. (5.14), the nine components of $Q$ will be obtained from the components of $F$.

Clearly, Eqs. (5.11), (5.13) and (5.14) show that the determination of $\bar{F}$ and $Q$ in the upper triangular decomposition of $F$ does not involve the square root of $C$ or the inverse of $U$. The computation of the latter typically requires finding the eigenvalues and eigenvectors of $C$, which can be tedious. This indicates that the upper triangular decomposition of $F$ is computationally more advantageous than the polar decomposition of $F$. Nevertheless, with both being multiplicative, the polar and upper triangular decompositions are related, as schematically shown in Fig. 5.1. It is seen from Fig. 5.1 that the upper triangular decomposition can indeed be viewed as an extended polar decomposition, as stated earlier in Section 5.1.
It is shown in the next section that the upper triangular tensor $\hat{F}$, called the distortion tensor, can be directly related to stretch ratios (elongations) and amounts of simple shear, which have clear physical interpretations and can be experimentally measured.

5.3 Decomposition of the Distortion Tensor $\hat{F}$

The distortion tensor $\hat{F}$ can be multiplicatively decomposed into a product of matrices for one stretch and two simple shear deformations or for one stretch and three simple shear deformations. These two types of decompositions are considered separately below in this section.
5.3.1 Decomposition into one tri-axial stretch and two simple shear deformations

In this type of decomposition, \( \tilde{F} \) is decomposed into a product of matrices for one tri-axial stretch and two simple shear deformations. It has been found that there are totally six possibilities of decomposing \( \tilde{F} \) in this manner, which are examined individually next.

5.3.1.1 Case I.1: \( \tilde{F} = \Lambda \tilde{F}^{\beta \gamma} \tilde{F}^\alpha \)

This is the only case considered in Srinivasa (2012).

In this case, \( \tilde{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and two isochoric simple shear deformations \( \tilde{F}^\alpha \) and \( \tilde{F}^{\beta \gamma} \) in the following form:

\[
[\tilde{F}] = [\Lambda] [\tilde{F}^{\beta \gamma}] [\tilde{F}^\alpha] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & a & 0 \\ \beta & 0 & c \end{bmatrix} = \begin{bmatrix} a & a\alpha & a\beta \\ 0 & b & b\gamma \end{bmatrix},
\]

(5.15)

where

\[
a = \tilde{F}_{11}, \quad b = \tilde{F}_{22}, \quad c = \tilde{F}_{33}, \quad \alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{11}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}.
\]

(5.16)

As defined in Eq. (5.15), \( \tilde{F}^\alpha \) describes the simple shear of planes parallel to the \( X_1-X_3 \) coordinate plane in the \( e_1 \)-direction by an amount \( \alpha \), \( \tilde{F}^{\beta \gamma} \) accounts for the simple shear of planes parallel to the \( X_1-X_2 \) coordinate plane in the direction of \( \beta e_1 + \gamma e_2 \) by an amount \( \beta \) in the \( e_1 \)-direction and an amount \( \gamma \) in the \( e_2 \)-direction, and \( \Lambda \) represents the tri-axial stretch along the three Cartesian coordinate axes with ratios \( a, b \) and \( c \) (the three eigenvalues of \( \tilde{F} \)), respectively. Note that both of the two simple shear deformations involved here are isochoric, with \( \text{det} \tilde{F}^\alpha = 1 \) and \( \text{det} \tilde{F}^{\beta \gamma} = 1 \).

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This decomposition is schematically shown in Fig. 5.2, where the initial configuration is a unit cube, and the final configuration is a parallelepiped.

In each of the remaining cases to be discussed below, the initial and final configurations will be taken to be the same as those in this case so that the distortion tensor \( \mathbf{F} \) will stay the same, even though intermediate configurations will differ for each case. In addition, considering the large number of cases to be discussed, the same symbols \( a, b, c, \alpha, \beta \) and \( \gamma \) will be used in each case to denote the amounts of stretch and simple shear, although the values of \( \alpha, \beta \) and \( \gamma \) may be different in each case, where they are individually identified. Note that the values of \( a, b \) and \( c \) will remain unchanged for all cases, since they are the diagonal components of \( \mathbf{F} \) which is taken to be the same in each case.

### 5.3.1.2 Case I.2: \( \mathbf{F} = \mathbf{F}^{\alpha} \Lambda \mathbf{F}^{\beta \gamma} \)

In this case, \( \mathbf{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and two isochoric simple shear deformations \( \mathbf{F}^{\alpha} \) and \( \mathbf{F}^{\beta \gamma} \) in the following form:
\[
\begin{bmatrix}
F
\end{bmatrix} = \left[ \tilde{F}^{\beta\gamma} \right] [\Lambda] \tilde{F}^{\alpha} = \begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix} \begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
a & a\alpha & c\beta \\
0 & b & c\gamma \\
0 & 0 & c
\end{bmatrix},
\] (5.17)

where \( a, b \) and \( c \) are the diagonal components of \( \tilde{F} \) defined in Eq. (5.16), and

\[
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{33}}.
\] (5.18)

A comparison of Eqs. (5.17) and (5.18) with Eqs. (5.15) and (5.16) shows that \( \tilde{F} \) can be decomposed into a product of \( \Lambda \), \( \tilde{F}^{\alpha} \) and \( \tilde{F}^{\beta\gamma} \) in different orders. That is, the decomposition is not unique for a given \( \tilde{F} \) linking the same initial and final configurations.

This decomposition is schematically shown in Fig. 5.3, where the values of \( \beta \) and \( \gamma \) are different from those in Case I.1, as indicated in Eqs. (5.16) and (5.18).

![Fig. 5.3 Decomposition of \( \tilde{F} \): Case I.2](image)

### 5.3.1.3 Case I.3: \( \tilde{F} = \tilde{F}^{\alpha} \Lambda \tilde{F}^{\beta\gamma} \)

In this case, \( \tilde{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and two isochoric simple shear deformations \( \tilde{F}^{\alpha} \) and \( \tilde{F}^{\beta\gamma} \) in the following form:
where $a$, $b$ and $c$ are defined in Eq. (5.16), and

$$\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{22}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{11}} - \frac{\tilde{F}_{12} \tilde{F}_{23}}{\tilde{F}_{11} \tilde{F}_{22}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}. \quad (5.20)$$

Note that the values of $\alpha$, $\beta$, $\gamma$ in this case differ from those in Case I.2 given in Eqs. (5.17) and (5.18).

This decomposition is schematically shown in Fig. 5.4.

5.3.1.4 Case I.4: $\tilde{F} = \Lambda \tilde{F}^\alpha \tilde{F}^{\beta\gamma}$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and two isochoric simple shear deformations $\tilde{F}^\alpha$ and $\tilde{F}^{\beta\gamma}$ in the following form:

$$[\tilde{F}] = [\Lambda][\tilde{F}^\alpha][\tilde{F}^{\beta\gamma}] = \left[\begin{array}{cccccc} a & 0 & 0 & 1 & \alpha & 0 \\ 0 & b & 0 & 0 & 1 & \beta \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccccc} a & a\alpha & a\alpha\gamma + a\beta \\ 0 & b & b\gamma \\ 0 & 0 & c \end{array}\right], \quad (5.21)$$

where $a$, $b$ and $c$ are defined in Eq. (5.16), and
Note that the value of $\alpha$ in this case differs from that in Case I.3 given in Eqs. (5.19) and (5.20).

This decomposition is schematically shown in Fig. 5.5.

![Figure 5.5 Decomposition of $\tilde{F}$: Case I.4](image)

### 5.3.1.5 Case I.5: $\tilde{F} = \tilde{F}^{\beta\gamma} \tilde{F}^{\alpha} \Lambda$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and two isochoric simple shear deformations $\tilde{F}^{\alpha}$ and $\tilde{F}^{\beta\gamma}$ in the following form:

$$
[\tilde{F}] = [\tilde{F}^{\beta\gamma}] [\tilde{F}^{\alpha}] [\Lambda] =
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
a & b\alpha & c\beta \\
b & 0 & c\gamma \\
c & 0 & 0
\end{bmatrix}, \quad (5.23)
$$

where $a$, $b$ and $c$ are defined in Eq. (5.16), and

$$
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{22}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{33}}. \quad (5.24)
$$

Note that the values of $\alpha$, $\beta$, $\gamma$ in this case differ from those in Case I.4 given in Eqs. (5.21) and (5.22).
This decomposition is schematically shown in Fig. 5.6.

![Fig. 5.6 Decomposition of F: Case I.5](image)

5.3.1.6 Case I.6: $\tilde{F} = \tilde{F}^\alpha \tilde{F}^\beta \Lambda$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and two isochoric simple shear deformations $\tilde{F}^\alpha$ and $\tilde{F}^\beta$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & \gamma \\
0 & 0 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a & b \alpha & c \alpha \gamma + c \beta \\
b & 0 & c \gamma \\
c & 0 & 0 \\
\end{bmatrix},
$$

(5.25)

where $a$, $b$ and $c$ are defined in Eq. (5.16), and

$$
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{22}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}} - \frac{\tilde{F}_{12} \tilde{F}_{23}}{\tilde{F}_{22} \tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{33}}.
$$

(5.26)

Note that the value of $\beta$ in this case differs from that in Case I.5 given in Eqs. (5.23) and (5.24).

This decomposition is schematically shown in Fig. 5.7.
Fig. 5.7 Decomposition of \( \tilde{F} \) : Case I.6

This completes the discussion on the six possible decompositions of \( \tilde{F} \) into a product of matrices for one tri-axial stretch and two simple shear deformations. These decompositions are all independent, as indicated above.

5.3.2 Decomposition into one tri-axial stretch and three simple shear deformations

In this second type of decomposition, \( \tilde{F} \) is decomposed into a product of matrices for one tri-axial stretch and three simple shear deformations. There are totally 24 possibilities of mathematically and physically decomposing \( \tilde{F} \) in this manner. None of these was considered in Srinivasa (2012).

Out of the 24 decompositions of this type, six cases are entirely different from the six decompositions of the first type described in Section 5.3.1 and independent of each other. These independent cases are examined individually in this sub-section. The other 18 cases (non-independent) are discussed in Appendix.
5.3.2.1 Case II.1: $\tilde{F} = \tilde{F}^{\gamma} \Lambda \tilde{F}^{\beta} \tilde{F}^\alpha$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{F}^\alpha$, $\tilde{F}^\beta$ and $\tilde{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
  F \\
  \end{bmatrix} =
\begin{bmatrix}
  \tilde{F}^\gamma \\
  \Lambda \\
  \tilde{F}^\beta \\
  \tilde{F}^\alpha
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & \gamma \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  a & 0 & 0 \\
  0 & b & 0 \\
  0 & 0 & c
\end{bmatrix}
\begin{bmatrix}
  1 & 0 & \beta \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & \alpha & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
=
\begin{bmatrix}
  a & a\alpha & a\beta \\
  0 & b & c\gamma \\
  0 & 0 & c
\end{bmatrix},
$$

(5.27)

where $a$, $b$ and $c$ are the diagonal components of $\tilde{F}$ defined in Eq. (5.16), and

$$
\alpha = \frac{\tilde{F}_{12}^{11}}, \quad \beta = \frac{\tilde{F}_{13}^{11}}, \quad \gamma = \frac{\tilde{F}_{23}^{11}}{\tilde{F}_{33}^{11}}.
$$

(5.28)

Note that the value of $\gamma$ in this case differs from that in Case I.1 given in Eqs. (5.15) and (5.16).

As defined in Eq. (5.27), $\tilde{F}^\beta$ represents the simple shear of planes parallel to the $X_1$-$X_2$ coordinate plane in the $e_1$-direction by an amount $\beta$, $\tilde{F}^\gamma$ accounts for the simple shear of planes parallel to $X_1$-$X_2$ plane in the $e_2$-direction by an amount $\gamma$, and $\Lambda$ and $\tilde{F}^\alpha$ are defined near Eq. (5.16) in Section 5.3.1.

This decomposition is schematically shown in Fig. 5.8.
5.3.2.2 Case II.2: $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\beta \Lambda \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
\mathbf{F}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{F}^\alpha
\end{bmatrix}
\begin{bmatrix}
\Lambda
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}^\beta
\end{bmatrix}
\begin{bmatrix}
\mathbf{F}^\gamma
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 1 & 0 \\
c & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & \alpha \alpha & c \beta \\
b & 0 & b \gamma \\
c & 0 & c
\end{bmatrix}.
$$

(5.29)

where $a$, $b$ and $c$ are defined in Eq. (5.16), and

$$
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}.
$$

(5.30)

Note that the values of $\beta$ and $\gamma$ in this case differ from those in Case II.1 given in Eqs. (5.27) and (5.28).

This decomposition is schematically shown in Fig. 5.9.
5.3.2.3 Case II.3: \( \mathbf{F} = \mathbf{F}^\gamma \mathbf{F}^\alpha \Lambda \mathbf{F}^\beta \)

In this case, \( \mathbf{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and three isochoric simple shear deformations \( \mathbf{F}^\alpha \), \( \mathbf{F}^\beta \) and \( \mathbf{F}^\gamma \) in the following form:

\[
\left[ \mathbf{F} \right] = \left[ \mathbf{F}^\gamma \right] \left[ \mathbf{F}^\alpha \right] \left[ \Lambda \right] \left[ \mathbf{F}^\beta \right] = \begin{bmatrix}
1 & 0 & 0 & 1 & \alpha & 0 & a & 0 & 0 & 1 & 0 & \beta
0 & 1 & \gamma & 0 & 1 & 0 & 0 & b & 0 & 0 & 1 & 0
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & c & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
a & b \alpha & a \beta 
0 & b & c \gamma 
0 & 0 & c
\end{bmatrix},
\]

where \( a, b \) and \( c \) are defined in Eq. (5.16), and

\[
\alpha = \frac{\bar{F}_{12}}{\bar{F}_{22}}, \quad \beta = \frac{\bar{F}_{13}}{\bar{F}_{11}}, \quad \gamma = \frac{\bar{F}_{23}}{\bar{F}_{33}}.
\]

Note that the value of \( \alpha, \beta, \gamma \) in this case differ from those in Case II.2 given in Eqs. (5.29) and (5.30).

This decomposition is schematically shown in Fig. 5.10.
Fig. 5.10 Decomposition of \( \hat{\mathbf{F}} \): Case II.3

5.3.2.4 Case II.4: \( \hat{\mathbf{F}} = \hat{\mathbf{F}}^\alpha \hat{\mathbf{F}}^\beta \Lambda \hat{\mathbf{F}}^\gamma \)

In this case, \( \hat{\mathbf{F}} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and three isochoric simple shear deformations \( \hat{\mathbf{F}}^\alpha \), \( \hat{\mathbf{F}}^\beta \) and \( \hat{\mathbf{F}}^\gamma \) in the following form:

\[
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
a & b\alpha & a\beta + c\alpha \gamma \\
b & 0 & c\gamma \\
c & 0 & 0
\end{bmatrix},
\]

(5.33)

where \( a, b \) and \( c \) are defined in Eq. (5.16), and

\[
\alpha = \frac{\hat{F}_{12}}{\hat{F}_{22}}, \quad \beta = \frac{\hat{F}_{13}}{\hat{F}_{11}} \frac{\hat{F}_{23}}{\hat{F}_{12}}, \quad \gamma = \frac{\hat{F}_{23}}{\hat{F}_{33}}.
\]

(5.34)

Note that the value of \( \beta \) in this case differs from that in Case II.3 given in Eqs. (5.31) and (5.32)

This decomposition is schematically shown in Fig. 5.11.
5.3.2.5 Case II.5: $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\beta A \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\gamma$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $A$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
\tilde{\mathbf{F}}
\end{bmatrix} =
\begin{bmatrix}
\tilde{\mathbf{F}}^\beta
\end{bmatrix}
\begin{bmatrix}
A
\end{bmatrix}
\begin{bmatrix}
\tilde{\mathbf{F}}^\alpha
\end{bmatrix}
\begin{bmatrix}
\tilde{\mathbf{F}}^\gamma
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0
0 & 1 & 0
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0
0 & 1 & 0
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & a \alpha & a \alpha \gamma + c \beta
0 & b & b \gamma
0 & 0 & c
\end{bmatrix}
$$

where $a$, $b$ and $c$ are defined in Eq. (5.16), and

$$
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{11}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}.
$$

Note that the values of $\alpha$, $\beta$ and $\gamma$ in this case differ from those in Case II.4 given in Eqs. (5.33) and (5.34).

This decomposition is schematically shown in Fig. 5.12.
5.3.2.6 Case II.6: \( \tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha \Lambda \tilde{\mathbf{F}}^\gamma \)

In this case, \( \tilde{\mathbf{F}} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and three isochoric simple shear deformations \( \tilde{\mathbf{F}}^\alpha \), \( \tilde{\mathbf{F}}^\beta \) and \( \tilde{\mathbf{F}}^\gamma \) in the following form:

\[
\begin{bmatrix}
\mathbf{F}
\end{bmatrix} = \begin{bmatrix}
\mathbf{F}^\alpha
\end{bmatrix} \begin{bmatrix}
\Lambda
\end{bmatrix} \begin{bmatrix}
\mathbf{F}^\gamma
\end{bmatrix} = \begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 1
\end{bmatrix} = \begin{bmatrix}
a & b\alpha & b\alpha\gamma + c\beta \\
b & b\gamma & c
\end{bmatrix},
\]

where \( a, b \) and \( c \) are defined in Eq. (5.16), and

\[
\alpha = \frac{\tilde{F}_{12}}{\tilde{F}_{22}}, \quad \beta = \frac{\tilde{F}_{13}}{\tilde{F}_{33}} - \frac{\tilde{F}_{12} \tilde{F}_{23}}{\tilde{F}_{22} \tilde{F}_{33}}, \quad \gamma = \frac{\tilde{F}_{23}}{\tilde{F}_{22}}.
\]

Note that the value of \( \alpha \) in this case differs from that in Case II.5 given in Eqs. (5.35) and (5.36).

This decomposition is schematically shown in Fig. 5.13.
This completes the discussion on the six independent cases of the 24 mathematically and physically possible decompositions of $\tilde{F}$ into a product of matrices for one tri-axial stretch and three simple shear deformations. The other 18 cases of this second type of decomposition of $\tilde{F}$ are presented in Appendix.

It is seen from the analyses in this Section and in Appendix that the six components of $\tilde{F}$ can be directly related to stretch ratio in three orthogonal directions and amounts of simple shear in the three directions in 30 different ways. These stretch ratio and shear amounts are physical quantities that are measurable. Hence, the distortion tensor $\tilde{F}$ can be experimentally obtained and may be adopted as a basic kinematic variable in developing constitutive laws that incorporate measurable physical parameters, which is the case in the current study, as shown in the next Section.

5.4 Constitutive Relations in Terms of the Distortion Tensor $\tilde{F}$

For hyperelastic materials, the use of the principle of material frame indifference and the first and second laws of thermodynamics leads to (e.g., Holzapfel, 2000)

$$\dot{W} = J\sigma : D,$$

(5.39)
where $W$ is the strain energy density function per unit reference volume, $J$ is the volume ratio with $J = \text{det} \mathbf{F}$, $\sigma$ is the Cauchy stress, and $\mathbf{D}$ is the rate of deformation tensor defined by

$$
\mathbf{D} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T), \quad \mathbf{L} = \mathbf{F} \mathbf{F}^{-1}, \tag{5.40a, b}
$$

in which $\mathbf{L}$ is the velocity gradient tensor. Note that the overhead dot in Eqs. (5.39) and (5.40a, b) denotes the total time derivative of the corresponding quantity.

From Eq. (5.3), it follows that

$$
\dot{\mathbf{F}} = \dot{\mathbf{Q}} \mathbf{F} + \mathbf{Q} \dot{\mathbf{F}}, \quad \dot{\mathbf{F}}^{-1} = \dot{\mathbf{F}}^{-T} \mathbf{Q}^T. \tag{5.41a, b}
$$

Using Eqs. (5.41a, b) in Eq. (5.40b) gives

$$
\dot{\mathbf{L}} = \mathbf{\Omega} + \dot{\mathbf{Q}} \mathbf{L} \mathbf{Q}^T, \tag{5.42}
$$

where

$$
\mathbf{\Omega} = \dot{\mathbf{Q}} \mathbf{Q}^T = -\mathbf{\Omega}^T, \quad \dot{\mathbf{L}} = \dot{\mathbf{F}} \mathbf{F}^{-1} \mathbf{F}^{-1}. \tag{5.43a, b}
$$

Clearly, $\mathbf{\Omega}$, as defined in Eq. (5.43a), is a skew (or anti-symmetric) tensor.

Substituting Eqs. (5.40a) and (5.42) into Eq. (5.39) results in, with $\sigma = \sigma^T$ and $\mathbf{\Omega} = -\mathbf{\Omega}^T$ (from Eq. (5.43a)),

$$
\dot{W} = J\sigma : \dot{\mathbf{L}} \mathbf{Q}^T. \tag{5.44}
$$

Note that in reaching Eq. (5.44) use has been made of the fact that the Cauchy stress $\sigma$ is symmetric with $\sigma = \sigma^T$.

Equation (5.44) can be rewritten as

$$
\dot{W} = J\dot{\sigma} : \dot{\mathbf{L}}, \tag{5.45}
$$

where
\[ \tilde{\sigma} = Q^T \sigma Q \]  

(5.46)

is the Cauchy stress in the original coordinate system with the base vectors \( e_i \), which differs from what was stated in Srinivasa (2012). Clearly, Eq. (5.46) shows that \( \tilde{\sigma} \) is symmetric.

Note that the strain energy density function can be written as

\[ W = W(C) = \tilde{W}(\tilde{F}), \]  

(5.47)

where use has been made of Eq. (5.12).

Note that under a frame change represented by the second-order rotation tensor \( \Theta \) (with \( \Theta^{-1} = \Theta^T, \det \Theta = 1 \)), \( \tilde{F}^* = \Theta F \). It then follows from Eq. (5.3) that

\[ \tilde{F}^* = (Q^T F)^* = (Q^*)^T \Theta F = Q^T \Theta^T \Theta F = Q^T F = \tilde{F}, \]  

(5.48)

where the superscript "*" denotes the quantity in the new frame, and use has been made of the relation \( Q^* = \Theta Q \). This shows that the distortion tensor \( \tilde{F} \) is frame-invariant. Hence, \( \tilde{F} \) can be used as an independent variable to construct the strain energy function \( \tilde{W}(\tilde{F}) \), which plays a similar role to that of the frame-invariant \( C \) in the strain energy density function \( W(C) \).

It follows from Eqs. (5.47) and (5.43b) that

\[ \dot{W} = \frac{\partial \tilde{W}}{\partial \tilde{F}} : \dot{\tilde{F}} = \frac{\partial \tilde{W}}{\partial \tilde{F}} \tilde{F}^r : \tilde{L}. \]  

(5.49)

Combining Eqs. (5.45) and (5.49) then yields

\[ \tilde{\sigma} = \frac{1}{J} \frac{\partial \tilde{W}}{\partial \tilde{F}} \tilde{F}^r \quad \text{or} \quad \tilde{\sigma}_{ij} = \frac{1}{J} \frac{\partial \tilde{W}}{\partial \tilde{F}_{ik}} \tilde{F}_{jk}, \quad i \neq j. \]  

(5.50)

As indicated in Eq. (5.50), only the six components in the upper triangular part of the Cauchy stress matrix \( \tilde{\sigma} \) can be directly determined from Eqs. (5.45) and (5.49), since the
matrix $\mathbf{L}$ involved in the scalar products in the two equations is an upper triangular matrix (see Eqs. (5.43b), (5.10) and (5.11)). The other three components of $\bar{\sigma}$ (i.e., $\bar{\sigma}_{ij}$, $i > j$) can be readily obtained from the symmetry relations $\bar{\sigma}_{ji} = \bar{\sigma}_{ij}$. Note that the expression of $\bar{\sigma}$ derived in Eq. (5.50) differs from that provided in Srinivasa (2012) by a factor of $1/J$.

Equation (5.50) gives the constitutive relations for the six work-producing components of $\bar{\sigma}$ in terms of the distortion tensor $\bar{\mathbf{F}}$ for a hyperelastic material, which are based on the principle of material frame indifference, the first and second laws of thermodynamics, and the upper triangular decomposition $\mathbf{F} = \mathbf{Q} \bar{\mathbf{F}}$. Being expressed directly as derivatives of the strain energy density function with respect to the components of $\bar{\mathbf{F}}$, the Cauchy stress components $\bar{\sigma}_{ij}$ given in Eq. (5.50) have simpler expressions than those based on the invariants of $\mathbf{C}$ (e.g., Holzapfel, 2000; Fu and Zhang, 2006; Kulkarni et al., 2016).

Upon the full determination of $\bar{\sigma} = \mathbf{Q}^T \sigma \mathbf{Q} = \bar{\sigma}_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ (after using the symmetry additionally), the Cauchy stress in the transformed coordinate system with the base vectors $\mathbf{e}'_i$ (see Eq. (5.6)) can be obtained from Eqs. (5.46) and (5.50) as

$$\sigma = \mathbf{Q} \bar{\sigma} \mathbf{Q}^T = \bar{\sigma}_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j,$$

where $\bar{\sigma}_{ij}$ are defined in Eq. (5.50) and satisfy $\bar{\sigma}_{ji} = \bar{\sigma}_{ij}$.

For incompressible materials with $J = 1$, Eq. (5.51) becomes

$$\sigma = \bar{\sigma}_{ij} \mathbf{e}'_i \otimes \mathbf{e}'_j - p \mathbf{I},$$

where $p$ is a hydrostatic pressure (as a Lagrange multiplier). It follows from Eqs. (5.46), (5.50) and (5.52) that
\[ \bar{\sigma} = \frac{\partial W}{\partial F} F^T - p I. \]  

5.5 Examples

To illustrate the constitutive equations based on the strain energy density function \( W(\bar{F}) \) presented in Section 5.4, two examples are studied herein.

5.5.1 Tri-axial stretching

Consider the tri-axial stretch defined by

\[ x_1 = aX_1, \quad x_2 = bX_2, \quad x_3 = cX_3, \]

where \( a, b, c \) are constants, indicating that the deformation is homogeneous. The deformation gradient \( F \) of this motion is

\[
\begin{bmatrix}
F_{ij}
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix}.
\]

From Eqs. (5.12)–(5.14) and (5.55), the distortion \( \bar{F} \) and the rotation \( Q \) can be readily obtained as

\[
\begin{bmatrix}
\bar{F}_{ij}
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix}, \quad \begin{bmatrix}
Q_{ij}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Consider the strain energy density function provided in Ogden (1984) for an unconstrained isotropic hyperelastic material:

\[ W = \frac{\mu}{2} \left( \lambda_1^2 + \lambda_2^2 + 3 - 2 \ln J \right) + \frac{\mu_0}{2} (J - 1)^2, \]
and the corresponding one for an incompressible material:

\[ W = \mu \left( \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \right), \quad \lambda_1\lambda_2\lambda_3 = 1, \quad (5.58) \]

where \( \lambda_i \) are the three principal stretches, \( J = \det\mathbf{F} = \lambda_1\lambda_2\lambda_3 \), and \( \mu \) and \( \mu_0 \) are two material constants. Clearly, the strain energy density functions given in Eqs. (5.57) and (5.58) are symmetric in \( \lambda_i \). In particular, Eq. (5.58) is of the neo-Hookean type. These are principal stretch-based strain energy density functions that are similar to the ones elaborated in Valanis and Landel (1967), Ogden (1972), Rivlin (2006), and Horgan and Murphy (2007), which are more physical and simpler to use than those based on the three strain invariants for isotropic hyperelastic materials.

For the current tri-axial stretch deformation,

\[ \lambda_1 = a = \tilde{F}_{11}, \quad \lambda_2 = b = \tilde{F}_{22}, \quad \lambda_3 = c = \tilde{F}_{33}, \quad (5.59) \]

and Eqs. (5.57) and (5.58) for the isotropic hyperelastic materials can be respectively rewritten as

\[ \tilde{W} = \frac{\mu}{2} \left[ \tilde{F}_{11}^2 + \tilde{F}_{22}^2 + \tilde{F}_{33}^2 - 3 - 2\ln \left( \tilde{F}_{11} \tilde{F}_{22} \tilde{F}_{33} \right) \right] + \frac{\mu_0}{2} \left( \tilde{F}_{11} \tilde{F}_{22} \tilde{F}_{33} - 1 \right)^2, \quad (5.60) \]

\[ \tilde{W} = \frac{\mu}{2} \left( \tilde{F}_{11}^2 + \tilde{F}_{22}^2 + \tilde{F}_{33}^2 - 3 \right), \quad \tilde{F}_{11}\tilde{F}_{22}\tilde{F}_{33} = 1. \quad (5.61) \]

These strain energy density functions are now in terms of the components of \( \tilde{\mathbf{F}} \) and can therefore be directly used to obtain the Cauchy stress components through the constitutive relations derived in Section 5.4. Note that Eqs. (5.60) and (5.61) can be identically reduced to Eqs. (5.57) and (5.58) by directly replacing \( \tilde{F}_i \) with \( \lambda_i \) according to Eq. (5.59).

Using Eqs. (5.50), (5.56), (5.59) and (5.60) in Eq. (5.51) yields
\[
\left[ \sigma_{ij} \right] = \frac{1}{abc} \begin{bmatrix}
\mu(a^2 - 1) + \mu_0(abc - 1) & 0 & 0 \\
0 & \mu(b^2 - 1) + \mu_0(abc - 1) & 0 \\
0 & 0 & \mu(c^2 - 1) + \mu_0(abc - 1) \\
\end{bmatrix},
\]

(5.62)

and substituting Eqs. (5.50), (5.56), (5.59) and (5.61) into Eq. (5.52) gives

\[
\left[ \sigma_{ij} \right] = \begin{bmatrix}
\mu a^2 - p & 0 & 0 \\
0 & \mu b^2 - p & 0 \\
0 & 0 & \mu c^2 - p \\
\end{bmatrix}.
\]

(5.63)

It should be pointed out that the Cauchy stress components given in Eqs. (5.62) and (5.63) are relative to the original coordinate system with the base vectors \( e_i \), since \( Q = I \) (see Eq. (5.56)) in this case so that \( e'_i = e_i \) according to Eq. (5.6).

Note that the principal stress components obtained in Eq. (5.62) for the unconstrained isotropic material and in Eq. (5.63) for the incompressible isotropic material are the same as those given in Ogden (1984) for each respective case based on the polar decomposition.

5.5.2 Simple shear

Consider the simple shear of planes parallel to the \( X_1-X_3 \) coordinate plane in the \( e_1 \)-direction defined by (e.g., Peng et al., 2006)

\[
x_1 = X_1 + \alpha X_2, \quad x_2 = X_2, \quad x_3 = X_3,
\]

(5.64)

where \( \alpha \) is the amount of shear. The deformation gradient \( F \) of this motion is

\[
\left[ F_{ij} \right] = \begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

(5.65)
It then follows from Eqs. (5.12)–(5.14) and (5.65) that the distortion $\tilde{F}$ and the rotation $Q$ can be uniquely obtained as

$$
\begin{align*}
\begin{bmatrix}
\tilde{F}_{ij}
\end{bmatrix} &= \begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
Q_{ij}
\end{bmatrix} &= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\end{align*}
$$

(5.66a, b)

Equation (5.66a) can be rewritten as

$$
\begin{align*}
\begin{bmatrix}
\tilde{F}_{ij}
\end{bmatrix} &= \begin{bmatrix}
\tilde{F}_{11} & \tilde{F}_{12} & 0 \\
0 & \tilde{F}_{22} & 0 \\
0 & 0 & \tilde{F}_{33}
\end{bmatrix} = \begin{bmatrix}
a & d & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix}, \\
\end{align*}
$$

(5.67)

where $\tilde{F}_{11} = a = 1$, $\tilde{F}_{22} = b = 1$, $\tilde{F}_{33} = c = 1$, $\tilde{F}_{12} = d = \alpha$.

From Eqs. (5.12) and (5.67), it follows that

$$
\begin{align*}
\begin{bmatrix}
C_{ij}
\end{bmatrix} &= \begin{bmatrix}
a^2 & ad & 0 \\
ad & b^2 + d^2 & 0 \\
0 & 0 & c^2
\end{bmatrix}.
\end{align*}
$$

(5.68)

By solving the characteristic equation of the eigenvalue problem associated with $C$ listed in Eq. (5.68), i.e., $\text{det}[C_{ij} - \lambda^2\delta_{ij}] = 0$, the principal stretches $\lambda_i$ of the simple shear defined in Eq. (5.64) can be found to be

$$
\lambda_i^2 = \frac{a^2 + b^2 + d^2}{2} \pm \sqrt{\frac{(a + b)^2 + d^2}{2} \left[ (a - b)^2 + d^2 \right]},
$$

(5.69a)

$$
\lambda_2^2 = \frac{a^2 + b^2 + d^2}{2} - \sqrt{\frac{(a + b)^2 + d^2}{2} \left[ (a - b)^2 + d^2 \right]},
$$

(5.69b)

$$
\lambda_3^2 = c^2.
$$

(5.69c)

Using Eqs. (5.67) and (5.69a-c) in Eqs. (5.57) and (5.58), respectively, yields

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\begin{align}
\bar{W} &= \frac{1}{2} \mu \left[ \tilde{F}_{11}^2 + \tilde{F}_{22}^2 + \tilde{F}_{33}^2 + \tilde{F}_{12}^2 - 3 - 2 \ln \left( \tilde{F}_{11} \tilde{F}_{22} \tilde{F}_{33} \right) \right] + \frac{1}{2} \mu_0 \left( \tilde{F}_{11} \tilde{F}_{22} \tilde{F}_{33} - 1 \right)^2 
\end{align}

(5.70)

for the unconstrained isotropic hyperelastic material, and

\begin{align}
\bar{W} &= \frac{1}{2} \mu \left( \tilde{F}_{11}^2 + \tilde{F}_{22}^2 + \tilde{F}_{33}^2 + \tilde{F}_{12}^2 - 3 \right)
\end{align}

(5.71)

for the incompressible isotropic neo-Hookean material. These strain energy density functions are now in terms of the components of \( \tilde{F} \) and are ready to be used in the constitutive relations presented in Section 5.4.

Note that Eqs. (5.70) and (5.71) can be uniquely reduced to Eqs. (5.57) and (5.58) as follows. Using Eq. (5.67) in Eq. (5.70) gives

\begin{align}
\bar{W} &= \frac{1}{2} \mu \left[ a_1^2 + b_1^2 + c_1^2 + d_1^2 - 3 - 2 \ln (abc) \right] + \frac{1}{2} \mu_0 (abc - 1)^2.
\end{align}

(5.72)

From Eqs. (5.69a, b), it follows that

\begin{align}
\lambda_1^2 + \lambda_2^2 = a^2 + b^2 + d^2, \quad \lambda_1^2 \lambda_2^2 = a^2 b^2.
\end{align}

(3.73)

Substituting Eqs. (5.73) and (5.69c) in Eq. (5.72) immediately yields Eq. (5.57), noting that \( J = \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 = abc \). Similarly, using Eqs. (5.67), (5.69c) and (5.73) in Eq. (5.71) will readily give Eq. (5.58).

From Eqs. (5.50) and (5.67),

\begin{align}
\bar{\sigma}_{11} &= \frac{1}{J} \left( \frac{\partial \bar{W}}{\partial \tilde{F}_{11}} \tilde{F}_{11} + \frac{\partial \bar{W}}{\partial \tilde{F}_{12}} \tilde{F}_{12} \right), \quad \bar{\sigma}_{22} = \frac{1}{J} \frac{\partial \bar{W}}{\partial \tilde{F}_{22}} \tilde{F}_{22}, \quad \bar{\sigma}_{33} = \frac{1}{J} \frac{\partial \bar{W}}{\partial \tilde{F}_{33}} \tilde{F}_{33}, \\
\bar{\sigma}_{12} &= \frac{1}{J} \frac{\partial \bar{W}}{\partial \tilde{F}_{12}} \tilde{F}_{22}, \quad \bar{\sigma}_{13} = \frac{1}{J} \frac{\partial \bar{W}}{\partial \tilde{F}_{13}} \tilde{F}_{33}, \quad \bar{\sigma}_{23} = \frac{1}{J} \frac{\partial \bar{W}}{\partial \tilde{F}_{23}} \tilde{F}_{33}.
\end{align}

(5.74)

Substituting Eqs. (5.66a) and (5.70) into Eq. (5.74) gives

\begin{align}
\bar{\sigma}_{11} &= \mu \tilde{F}_{12}^2 = \mu \alpha^2, \quad \bar{\sigma}_{22} = 0, \quad \bar{\sigma}_{33} = 0, \quad \bar{\sigma}_{12} = \mu \tilde{F}_{12} \tilde{F}_{22} = \mu \alpha, \quad \bar{\sigma}_{13} = 0, \quad \bar{\sigma}_{23} = 0.
\end{align}

(5.75)
From Eqs. (5.51) and (5.75), it then follows that

\[
\begin{bmatrix}
\sigma_{ij}
\end{bmatrix} = \begin{bmatrix}
\mu \alpha^2 & \mu \alpha & 0 \\
\mu \alpha & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (5.76)

as the Cauchy stress tensor for the simple shear deformation of the unconstrained isotropic hyperelastic material satisfying Eq. (5.57). Note that the Cauchy stress components given in Eq. (5.76) are relative to the original coordinate system with the base vectors \( e_i \), since \( Q = I \) in this case (see Eq. (5.66b)) so that \( e'_i = e_i \) according to Eq. (5.6).

Similarly, using Eqs. (5.66a) and (5.71) in Eq. (5.74) yields

\[
\bar{\sigma}_{11} = \mu (\bar{F}_{11}^2 + \bar{F}_{12}^2) = \mu (1 + \alpha^2), \quad \bar{\sigma}_{22} = \mu \bar{F}_{23}^2 = \mu, \quad \bar{\sigma}_{33} = \mu \bar{F}_{33}^2 = \mu,
\]

\[
\bar{\sigma}_{12} = \mu \bar{F}_{12} \bar{F}_{22} = \mu \alpha, \quad \bar{\sigma}_{13} = 0, \quad \bar{\sigma}_{23} = 0.
\] (5.77)

From Eqs. (5.52) and (5.77), it follows that

\[
\begin{bmatrix}
\sigma_{ij}
\end{bmatrix} = \begin{bmatrix}
\mu(1+\alpha^2) - p & \mu \alpha & 0 \\
\mu \alpha & \mu - p & 0 \\
0 & 0 & \mu - p
\end{bmatrix}
\] (5.78)

as the Cauchy stress tensor for the simple shear deformation of the incompressible isotropic neo-Hookean material satisfying Eq. (5.58). Once again, the Cauchy stress components listed in Eq. (5.78) are relative to the original coordinate system with the base vectors \( e_i \) for the same reason stated above for the unconstrained material.

A comparison reveals that the stress components obtained in Eq. (5.76) for the unconstrained material and in Eq. (5.78) for the incompressible material are the same as those given in Ogden (1984) for each respective case based on the polar decomposition, but the procedure is more direct and simpler.
The two examples studied in this section clearly show that the upper triangular decomposition of the deformation gradient \( F = Q \tilde{F} \) and the general constitutive relations derived in terms of the distortion tensor \( \tilde{F} \) can be combined with a strain energy density function \( W(F) \), which is experimentally measurable, to provide an alternative method for determining stress components, which is simpler than the invariant-based approach in non-linear elasticity.

5.6 Summary

It is shown that the upper triangular decomposition of the deformation gradient into a product of an orthogonal tensor and an upper triangular distortion tensor can be viewed as an extended polar decomposition. The distortion tensor can be non-uniquely decomposed into a product of matrices for one tri-axial stretch and two simple shear deformations or for one tri-axial stretch and three simple shear deformations. It is found that there are 6 possible decompositions for the former, only one of which was studied earlier, and 24 possible decompositions for the latter, none of which was examined before. It is also shown that the distortion tensor is frame-invariant. The constitutive equations for both unconstrained and incompressible hyperelastic materials are derived in terms of the distortion tensor. The two problems of a tri-axial stretch and a simple shear deformation are studied as examples to illustrate the use of the general constitutive relations, which lead to the same results as those based on the polar composition and thereby validate the new approach based on the upper triangular decomposition of the deformation gradient. These newly proposed constitutive models can be further modified and applied to simulate brain tissues.
References


PUBLICATIONS

Journal papers:


Conference Publications:


APPENDIX

In this appendix, the remaining 18 cases of the second type of decomposition of \( \tilde{F} \) are analyzed, which are not independent of the six cases of the first type of decomposition presented in Section 5.3.1 and/or of the six cases of the second type of decomposition examined in Section 5.3.2.

**A.1 Case II.7: \( \tilde{F} = \tilde{F}^\gamma \Lambda \tilde{F}^\alpha \tilde{F}^\beta \)**

In this case, \( \tilde{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \Lambda \) and three isochoric simple shear deformations \( \tilde{F}^\alpha \), \( \tilde{F}^\beta \) and \( \tilde{F}^\gamma \) in the following form:

\[
\tilde{F} = \tilde{F}^\gamma \Lambda \tilde{F}^\alpha \tilde{F}^\beta
\]

\[
\begin{bmatrix}
    1 & 0 & 0 & a & 0 & 0 & 1 & \alpha & 0 & 1 & 0 & \beta \\
    0 & 1 & \gamma & 0 & b & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & b & c & \gamma \\
    0 & 0 & 1 & 0 & 0 & c & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & c
\end{bmatrix}
\]

Note that the values of \( a, b, c, \alpha, \beta \) and \( \gamma \) in Eq. (A1) for this case are the same as those given in Eqs. (5.27) and (5.28) for Case II.1. This confirms that \( \tilde{F}^\beta \tilde{F}^\alpha \) (in Case II.1) = \( \tilde{F}^\alpha \tilde{F}^\beta \) (in the current case).

This decomposition is schematically shown in Fig. A1.
A.2 Case II.8: $\tilde{\mathbf{F}} = \mathbf{\Lambda}\tilde{\mathbf{F}}^\beta\tilde{\mathbf{F}}^\gamma\tilde{\mathbf{F}}^\alpha$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{\Lambda}$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
\mathbf{F}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{\Lambda}
\end{bmatrix} \begin{bmatrix}
\tilde{\mathbf{F}}^\gamma
\end{bmatrix} \begin{bmatrix}
\tilde{\mathbf{F}}^\beta
\end{bmatrix} \begin{bmatrix}
\tilde{\mathbf{F}}^\alpha
\end{bmatrix} =
\begin{bmatrix}
a & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \alpha & 0
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 & 0 & 1 & 0 & 1 & \beta & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & b & 0 & 0 & 1 & \gamma & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix} =
\begin{bmatrix}
a & a\alpha & a\beta & 0 & b & b\gamma & 0 & 0 & c
\end{bmatrix}.
$$

(A2)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A2) for this case are the same as those given in Eqs. (5.15) and (5.16) for Case I.1. This confirms that $\tilde{\mathbf{F}}^{\beta\gamma}$ (in Case I.1) = $\tilde{\mathbf{F}}\tilde{\mathbf{F}}^\beta$ (in the current case).

This decomposition is schematically shown in Fig. A2.
A.3 Case II.9: \( \mathbf{F} = \mathbf{\Lambda} \mathbf{F}^{\alpha} \mathbf{F}^{\beta} \mathbf{F}^{\gamma} \)

In this case, \( \mathbf{F} \) is decomposed into a product of matrices for one tri-axial stretch \( \mathbf{\Lambda} \) and three isochoric simple shear deformations \( \mathbf{F}^{\alpha} \), \( \mathbf{F}^{\beta} \) and \( \mathbf{F}^{\gamma} \) in the following form:

\[
\begin{bmatrix}
\mathbf{F} \\
\end{bmatrix} = \begin{bmatrix}
\mathbf{\Lambda} \\
\end{bmatrix} \begin{bmatrix}
\mathbf{F}^{\alpha} \\
\mathbf{F}^{\beta} \\
\mathbf{F}^{\gamma} \\
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 & 1 & 0 & \beta & 1 & 0 & 0 & 1 & \alpha & 0 \\
0 & b & 0 & 0 & 1 & 0 & 0 & 1 & \gamma & 0 & 1 & 0 \\
0 & 0 & c & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
a & a\alpha & a\beta \\
0 & b & b\gamma \\
0 & 0 & c \\
\end{bmatrix}.
\]

(A3)

Note that the values of \( a, b, c, \alpha, \beta \) and \( \gamma \) in Eq. (A3) for this case are the same as those given in Eqs. (5.15) and (5.16) for Case I.1 and in Eqs. (A2) and (5.16) for Case II.8. This indicates that \( \mathbf{\tilde{F}}^{\alpha\beta} \) (in Case I.1) = \( \mathbf{\tilde{F}}^{\beta} \mathbf{\tilde{F}}^{\alpha} \) (in Case II.8) = \( \mathbf{\tilde{F}}^{\beta} \mathbf{\tilde{F}}^{\gamma} \) (in the current case).

This decomposition is schematically shown in Fig. A3.
A.4 Case II.10: $\mathbf{\tilde{F}} = \mathbf{\tilde{F}}^\gamma \mathbf{\tilde{F}}^\beta \mathbf{\Lambda F}^\alpha$

In this case, $\mathbf{\tilde{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{\Lambda}$ and three isochoric simple shear deformations $\mathbf{\tilde{F}}^\alpha$, $\mathbf{\tilde{F}}^\beta$ and $\mathbf{\tilde{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha & 0 & 0 \\
\beta & 0 & 0 \\
\gamma & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 0 & 1 \\
0 & 0 & c
\end{bmatrix}
\begin{bmatrix}
a & a\alpha & c\beta \\
b & 0 & c\gamma \\
c & 0 & c
\end{bmatrix}
$$

(A4)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A4) for this case are the same as those given in Eqs. (5.17) and (5.18) for Case I.2. This means that $\mathbf{\tilde{F}}_{\beta\gamma}$ (in Case I.2) = $\mathbf{\tilde{F}}^\beta \mathbf{\tilde{F}}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A4.
In this case, $\mathbf{F}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{A}$ and three isochoric simple shear deformations $\mathbf{F}^\alpha$, $\mathbf{F}^\beta$ and $\mathbf{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
a & a \alpha & c \beta \\
b & b & c \gamma \\
c & c & c \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 & 1 & 0 & 0 & c \\
b & 0 & 0 & 0 & 1 & 0 & c \\
c & 0 & 0 & 0 & 1 & 0 & c \\
\end{bmatrix}.
$$

(A5)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A5) for this case are the same as those given in Eqs. (5.17) and (5.18) for Case I.2 and in Eqs. (A4) and (5.18) for Case II.10. This shows that $\tilde{\mathbf{F}}^\beta$ (in Case I.2) = $\tilde{\mathbf{F}}^\gamma$ (in Case II.10) = $\tilde{\mathbf{F}}^\beta\tilde{\mathbf{F}}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A5.
In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{A}$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
\mathbf{F}
\end{bmatrix} = \left[ \mathbf{A} \right] \left[ \tilde{\mathbf{F}}^\gamma \right] \left[ \tilde{\mathbf{F}}^\alpha \right] \left[ \tilde{\mathbf{F}}^\beta \right] =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & c \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a & a \alpha & a \beta \\
1 & 0 & \beta \\
0 & 1 & 0
\end{bmatrix} =
\begin{bmatrix}
a & a \alpha & a \beta \\
0 & b & b \gamma \\
0 & 0 & c
\end{bmatrix}.
$$

(A6)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A6) for this case are the same as those given in Eqs. (5.15) and (5.16) for Case I.1, in Eqs. (A2) and (5.16) for Case II.8 and in Eqs. (A3) and (5.16) for Case II.9. This indicates that $\tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha$ (in Case I.1) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha$ (in Case II.8) = $\tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha$ (in Case II.9) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\beta$ (in the current case).

This decomposition is schematically shown in Fig. A6.
A.7 Case II.13: $\tilde{F} = \Lambda\tilde{F}^\alpha\tilde{F}^\gamma\tilde{F}^\beta$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{F}^\alpha$, $\tilde{F}^\beta$ and $\tilde{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
    a & 0 & 0 & 1 & \alpha & 0 & 1 & 0 & 0 & 1 & 0 & \beta \\
    0 & b & 0 & 1 & 0 & 0 & 1 & \gamma & 0 & 1 & 0 \\
    0 & 0 & c & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \gamma \\
\end{bmatrix}
= 
\begin{bmatrix}
    a & a\alpha & a\alpha\gamma + a\beta \\
    0 & b & b\gamma \\
    0 & 0 & c \\
\end{bmatrix}.
$$
(A7)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A7) for this case are the same as those given in Eqs. (5.21) and (5.22) for Case I.4. This shows that $\tilde{F}^\beta$ (in Case I.4) = $\tilde{F}^\gamma\tilde{F}^\beta$ (in the current case).

This decomposition is schematically shown in Fig. A7.
A.8 Case II.14: $\tilde{F} = \tilde{F}^\alpha \Lambda \tilde{F}^\gamma \tilde{F}^\beta$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{F}^\alpha$, $\tilde{F}^\beta$ and $\tilde{F}^\gamma$ in the following form:

$$
\tilde{F} = \left[ \begin{array}{cccc}
1 & \alpha & 0 & 0 \\
0 & 1 & 0 & b \\
0 & 0 & 1 & c
\end{array} \right] \left[ \begin{array}{cccc}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0
\end{array} \right] \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array} \right] = \left[ \begin{array}{cccc}
a & b\alpha & a\beta + b\alpha\gamma & 0 \\
0 & b & b\gamma & 0 \\
0 & 0 & c & 0
\end{array} \right].
$$

Note that the values of $a, b, c, \alpha, \beta$ and $\gamma$ in Eq. (A8) for this case are the same as those given in Eqs. (5.19) and (5.20) for Case I.3. This means that $\tilde{F}^{\beta\gamma}$ (in Case I.3) = $\tilde{F}^{\gamma} \tilde{F}^{\beta}$ (in the current case).

This decomposition is schematically shown in Fig. A8.
A.9 Case II.15: \( \mathbf{F} = \mathbf{\Lambda} \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\gamma \)

In this case, \( \tilde{\mathbf{F}} \) is decomposed into a product of matrices for one tri-axial stretch \( \mathbf{\Lambda} \) and three isochoric simple shear deformations \( \tilde{\mathbf{F}}^\alpha \), \( \tilde{\mathbf{F}}^\beta \) and \( \tilde{\mathbf{F}}^\gamma \) in the following form:

\[
[\mathbf{F}] = [\mathbf{\Lambda}] [\tilde{\mathbf{F}}^\beta] [\tilde{\mathbf{F}}^\alpha] [\tilde{\mathbf{F}}^\gamma] = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & \alpha \alpha \gamma + a \beta \\ 0 & b & b \gamma \\ 0 & 0 & c \end{bmatrix}.
\]

(A9)

Note that the values of \( a, b, c, \alpha, \beta \) and \( \gamma \) in Eq. (A9) for this case are the same as those given in Eqs. (5.21) and (5.22) for Case I.4 and in Eqs. (A7) and (5.22) for Case II.13. This indicates that \( \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\beta \) (in Case I.4) = \( \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\gamma \) (in Case II.13) = \( \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma \) (in the current case).

This decomposition is schematically shown in Fig. A9.
In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{F}^\alpha$, $\tilde{F}^\beta$ and $\tilde{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
F
\end{bmatrix} = \begin{bmatrix}
\alpha & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha & a\alpha & a\alpha + a\beta \\
b & b & b
\end{bmatrix}.
$$

(A10)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A10) for this case are the same as those given in Eqs. (5.21) and (5.22) for Case I.4, in Eqs. (A7) and (5.22) for Case II.13, and in Eqs. (A9) and (5.22) for Case II.15. This shows that $\tilde{F}^\alpha\tilde{F}^\beta\tilde{F}^\gamma$ (in Case I.4) = $\tilde{F}^\alpha\tilde{F}^\beta\tilde{F}^\gamma$ (in Case II.13) = $\tilde{F}^\beta\tilde{F}^\alpha\tilde{F}^\gamma$ (in Case II.15) = $\tilde{F}^\alpha\tilde{F}^\beta\tilde{F}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A10.
A.11 Case II.17: $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\alpha \Lambda \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
a & b\alpha & a\beta + b\alpha\gamma \\
0 & b & b\gamma \\
0 & 0 & c
\end{bmatrix}.
$$

(A11)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A11) for this case are the same as those given in Eqs. (5.19) and (5.20) for Case I.3 and in Eqs. (A8) and (5.20) for Case II.14. This means that $\tilde{\mathbf{F}}^\beta$ (in Case I.3) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\beta$ (in Case II.14) = $\tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A11.
Fig. A11 Decomposition of $\mathbf{F}$: Case II.17

A.12 Case II.18: $\mathbf{F} = \mathbf{F}^\alpha \mathbf{F}^\beta \Lambda \mathbf{F}^\gamma$

In this case, $\mathbf{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\mathbf{F}^\alpha$, $\mathbf{F}^\beta$ and $\mathbf{F}^\gamma$ in the following form:

$$\mathbf{F} = \mathbf{F}^\alpha \mathbf{F}^\beta \Lambda \mathbf{F}^\gamma$$

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A12) for this case are the same as those given in Eqs. (5.37) and (5.38) for Case II.6. This shows that $\mathbf{F}^\beta \mathbf{F}^\alpha$ (in Case II.6) = $\mathbf{F}^\gamma \mathbf{F}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A12.
A.13 Case II.19: $\tilde{F} = \tilde{F}^\gamma \tilde{F}^\beta \tilde{F}^\alpha \Lambda$

In this case, $\tilde{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{F}^{\alpha}$, $\tilde{F}^{\beta}$ and $\tilde{F}^{\gamma}$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta & 1 & \alpha & 0 \\
0 & 1 & \gamma & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b \alpha & 0 & 0 \\
c \beta & 0 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
a & b \alpha & c \beta \\
b & 0 & c \gamma \\
c & 0 & 0 \\
\end{bmatrix}
$$

(A13)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A13) for this case are the same as those given in Eqs. (5.23) and (5.24) for Case I.5. This indicates that $\tilde{F}^{\beta\gamma}$ (in Case I.5) = $\tilde{F}^{\beta\gamma}$ (in the current case).

This decomposition is schematically shown in Fig. A13.
A.14 Case II.20: $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha \mathbf{\Lambda}$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{\Lambda}$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a & b\alpha & c\beta \\
b & c\gamma & 0 \\
c & 0 & 0
\end{bmatrix}
$$

(A14)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A14) for this case are the same as those given in Eqs. (5.23) and (5.24) for Case I.5 and in Eqs. (A13) and (5.24) for Case II.19. This means that $\tilde{\mathbf{F}}^\alpha$ (in Case I.5) = $\tilde{\mathbf{F}}^\beta$ (in Case II.19) = $\tilde{\mathbf{F}}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A14.
A.15 Case II.21: $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\beta \Lambda$

In this case, $\tilde{\mathbf{F}}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\tilde{\mathbf{F}}^\alpha$, $\tilde{\mathbf{F}}^\beta$ and $\tilde{\mathbf{F}}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
b & 0 & 0 \\
c & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
a & b\alpha & c\beta \\
b & 0 & c\gamma \\
c & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & \gamma \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} 
= 
\begin{bmatrix}
a & b\alpha & c\beta \\
b & 0 & c\gamma \\
c & 0 & 0
\end{bmatrix}.
$$

(A15)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A15) for this case are the same as those given in Eqs. (5.23) and (5.24) for Case I.5, in Eqs. (A13) and (5.24) for Case II.19 and in Eqs. (A14) and (5.24) for Case II.20. This shows that $\tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha$ (in Case I.5) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\alpha$ (in Case II.19) = $\tilde{\mathbf{F}}^\beta \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha$ (in Case II.20) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\beta$ (the current case).

This decomposition is schematically shown in Fig. A15.
A.16 Case II.22: $\mathbf{F} = \tilde{\mathbf{F}}^\alpha \tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\beta \Lambda$

In this case, $\mathbf{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\mathbf{F}^\alpha$, $\mathbf{F}^\beta$ and $\mathbf{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
\mathbf{F} \\
\end{bmatrix} = \begin{bmatrix}
\mathbf{F}^\alpha & \mathbf{F}^\beta & \mathbf{F}^\gamma \\
\end{bmatrix} \begin{bmatrix}
\mathbf{A} \\
\end{bmatrix} = \begin{bmatrix}
1 & \alpha & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & \gamma \\
0 & 0 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & \beta & a & 0 & 0 \\
1 & 0 & 1 & b & 0 & 0 \\
0 & 1 & 0 & 0 & b & 0 \\
\end{bmatrix} = \begin{bmatrix}
a & b \alpha & c \alpha \gamma + c \beta \\
b & c \gamma & c \\
0 & 0 & c \\
\end{bmatrix} \\
\text{(A16)}
$$

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A16) for this case are the same as those given in Eqs. (5.25) and (5.26) for Case I.6. This indicates that $\mathbf{F}^\beta$ (in Case I.6) = $\tilde{\mathbf{F}}^\gamma \tilde{\mathbf{F}}^\beta$ (in the current case).

This decomposition is schematically shown in Fig. A16.
A.17 Case II.23: $\mathbf{F} = \mathbf{F}^\alpha \mathbf{F}^\beta \mathbf{F}^\gamma \Lambda$

In this case, $\mathbf{F}$ is decomposed into a product of matrices for one tri-axial stretch $\Lambda$ and three isochoric simple shear deformations $\mathbf{F}^\alpha$, $\mathbf{F}^\beta$, and $\mathbf{F}^\gamma$ in the following form:

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
a & c \\
b & c \\
c & c
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(A17)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A17) for this case are the same as those given in Eqs. (5.25) and (5.26) for Case I.6 and in Eqs. (A16) and (5.26) for Case II.22. This means that $\mathbf{F}^\alpha \mathbf{F}^\beta$ (in Case I.6) $= \mathbf{F}^\alpha \mathbf{F}^\gamma \mathbf{F}^\beta$ (in Case II.22) $= \mathbf{F}^\beta \mathbf{F}^\alpha \mathbf{F}^\gamma$ (in the current case).

This decomposition is schematically shown in Fig. A17.
Fig. A17 Decomposition of $\mathbf{F}$: Case II.23

(A.18) Case II.24: $\mathbf{F} = \mathbf{F}^{\alpha} \mathbf{F}^{\beta} \mathbf{F}^{\gamma} \mathbf{\Lambda}$

In this case, $\mathbf{F}$ is decomposed into a product of matrices for one tri-axial stretch $\mathbf{\Lambda}$ and three isochoric simple shear deformations $\mathbf{F}^{\alpha}$, $\mathbf{F}^{\beta}$ and $\mathbf{F}^{\gamma}$ in the following form:

$$
\begin{bmatrix}
1 & \alpha & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \beta \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c \\
\end{bmatrix}
\begin{bmatrix}
a & b\alpha & c\alpha + c\beta \\
0 & b & c\gamma \\
0 & 0 & c \\
\end{bmatrix}
$$

(A.18)

Note that the values of $a$, $b$, $c$, $\alpha$, $\beta$ and $\gamma$ in Eq. (A18) for this case are the same as those given in Eqs. (5.25) and (5.26) for Case I.6, in Eqs. (A16) and (5.26) for Case II.22, and in Eqs. (A17) and (5.26) for Case II.23. This shows that $\mathbf{F}^{\alpha} \mathbf{F}^{\beta} \mathbf{F}^{\gamma}$ (in Case I.6) = $\mathbf{F}^{\alpha} \mathbf{F}^{\gamma} \mathbf{F}^{\beta}$ (in Case II.22) = $\mathbf{F}^{\gamma} \mathbf{F}^{\alpha} \mathbf{F}^{\beta}$ (in Case II.23) = $\mathbf{F}^{\alpha} \mathbf{F}^{\beta} \mathbf{F}^{\gamma}$ (in the current case).

This decomposition is schematically shown in Fig. A18.
This completes the analyses of the 18 non-independent cases of the second type of decomposition of $\tilde{F}$. 

Fig. A18 Decomposition of $\tilde{F}$: Case II.24