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NEW EFFICIENT EQUATIONS TO COMPUTE THE PRESENT VALUE OF MORTGAGE INTEREST PAYMENTS AND ACCELERATED DEPRECIATION TAX BENEFITS

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by

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ABSTRACT

A large number of capital budgeting problems depend for their solution on knowledge of the present value of interest payments or the present value of tax benefits to be derived from accelerated depreciation. The present approach to such problems requires that individual period values for interest or depreciation first be computed. This involves the time consuming task of constructing loan amortization tables or depreciation schedules. This paper derives equations for the direct computation of the present values of mortgage interest payments and tax benefits from accelerated (double declining balance and sum of the year's digits) depreciation. The equations, which can be solved on any calculator with a "Y^x" function, can significantly simplify computational procedures in real life situations, and they enable professors to discuss types of problems which previously they either omitted because of the quantity of required calculations or discussed in the context of artificially truncated time horizons.

Certain important and realistic capital budgeting analyses that make excellent teaching material are often avoided or greatly simplified by classroom instructors because of the sheer volume of computations involved. For example, the typical lease or buy decision requires determining the present value of mortgage interest payments and the present value of the tax benefits resulting from accelerated depreciation. Currently, a loan amortization schedule must be prepared before the present value of the interest payments can be computed. The accelerated depreciation methods, sum of the year's digits (SYD) and declining balance (DB), also require preparation of schedules. In both situations, preparation of the schedules can be a lengthy task for a twenty year mortgage or a long lived asset.

After the individual period values are determined, they are converted to the equivalent after tax values and then discounted. In practice, the steps described above are performed by computers. Students, however, may not use a computer for a number of reasons. The school may not have a computer, the appropriate programs may not be available, the instructor may be reluctant to require use of the computer, or the students may not know how to access the computer available to them.

In this paper, new equations for the direct computation of the present values of mortgage interest payments and accelerated depreciation tax benefits are derived. The equations do not require computation of individual period values for interest and depreciation; hence they simplify the analysis in real-life situations, and in the classroom.

The equations can be solved on any calculator with a "Y^x" function, and the time required for solutions is independent of the number of periods involved -- an advantage not only in the initial solution but also in the subsequent sensitivity analysis. The equations can also be used to develop tables for use by students if calculators are not generally available. The equations are also particularly effective when programmed on calculators or computers.

The equations along with examples are explained in the following sections with the mathematical proofs reserved for the Appendix.

Present Value of Mortgage Interest Payments

To determine the present value of mortgage interest payments before or after taxes, the following equation is used:

$$\text{MGT. INT. PV} = \frac{i \times PV}{(1+K)} + \frac{PMT}{(1+K)} \times A(n-1,K) + A(n-1,b)\left[\frac{i \times PV - PMT}{(1+K)}\right] \quad (1)$$

$$\begin{matrix}\text{AFTER TAX} \\ \text{MGT. INT. PV}\end{matrix} = (1 - TX)\ (\text{MGT. INT. PV}) \quad (2)$$

where:

MGT. INT. PV = Present value of mortgage interest payments

PV = Amount of mortgage

PMT = Equal periodic payment

i = Mortgage interest rate per payment period expressed as a decimal

n = Number of payments not to exceed remaining life of mortgage

Tx = Tax rate expressed as a decimal

K = Discount rate per payment period expressed as a decimal

$$b = [(1+K)/(1+i)] - 1 \quad (3)$$

$$a(n-1,K) = \frac{1 - (1+K)^{-n+1}}{K} \quad (4)$$

$$a(n-1,b) = \frac{1 - (1+b)^{-n+1}}{b} \quad (5)$$

Equations (4) and (5) are valid where $k \neq 0$ and $b \neq 0$, but when $b = 0$

$$a(n-1,b) = n-1 \tag{6}$$

and when $K = 0$

$$a(n-1,K) = n-1 \tag{7}$$

While Equation (1) may appear complex, it can be easily solved using a calculator with a Y^x function. Furthermore, since the two expressions $A(n-1,K)$ and $A(n-1,b)$ represent the present value of an ordinary annuity, the two values can be computed directly using a financial or programmable calculator.

The following example demonstrates the application of Equation (1). Assume a 30 year $50,000 mortgage with monthly payments of $514.31 and an annual interest rate of 12% compounded monthly. The present value of the interest paid is computed for the first twelve payments using Equation (1) and an 18% annual discount rate compounded monthly. The key values are:

$$PV = 50{,}000$$

$$PMT = 514.31$$

$$i = .12/12 = .01$$

$$n = 12$$

$$K = .18/12 = .015$$

$$b = (1.015/1.01) - 1 = .004950495 \tag{3}$$

$$a(n-1,K) = a(11,.015) = \frac{1 - (1.015)^{-11}}{.015} \tag{4}$$

$$= 10.07111779$$

$$a(n-1,b) = a(11,.004950495) \tag{5}$$

$$= \frac{1 - (1.004950495)^{-11}}{.004950495}$$

$$= 10.68015677$$

$$\text{MGT. INT. PV} = \frac{.01(50{,}000)}{1.015} + \frac{(514.31)(10.07111779)}{1.015} \tag{1}$$

$$+ (10.68015677)\left[\frac{(.01)(50000) - 514.31}{1.015}\right]$$

$$= \$5{,}445.17$$

The present value of the interest portions of the twelve payments is \$5,445.17. Assuming a 40% tax rate, the after tax value using Equation (2) is:

$$\begin{array}{l}\text{AFTER TAX} \\ \text{MGT. INT. PV}\end{array} = (1-.4)\ \$5{,}445.16 \tag{2}$$

$$= \$3{,}267.10$$

The present value of the principal payments can be found by computing the present value of the 12 payments using the 1.5% discount rate and subtracting the present value of the interest payments.

$$\begin{array}{l}\text{Present Value of} \\ \text{Principal Payments}\end{array} = \text{PMT } A(n,K) - \text{Mgt. Int. PV} \tag{8}$$

$$\text{PV of 12 Payments} = \$514.31 \times \frac{1 - 1.015^{-12}}{.015}$$

$$= \$5{,}609.84$$

$$\begin{array}{l}\text{Present Value of} \\ \text{Principal Payments}\end{array} = \$5{,}609.84 - \$5{,}445.17$$

$$= \$164.67$$

The difference of .01 between this answer and the value computed using the conventional method in Table 1 is because the Table values are rounded to the nearest penny. Only the terms $A(n-1,K)$ and $A(n-1,b)$ need to be recomputed to determine the present value of the interest and principal payments for 360 payments because the other values remain constant.

$$n = 360$$

$$A(n-1,K) = A(359,.015) \tag{4}$$

$$= 66.34854037$$

	Beginning Balance	Principal	Interest x (1-.4)=	After Tax Interest
1	50,000,00	14.31	500.00	300.00
2	49,985.69	14.45	499.86	299.92
3	49,971.24	14.60	499.71	299.83
4	49,956.64	14.74	499.57	299.74
5	49,941.90	14.89	499.42	299.65
6	49,927.01	15.04	499.27	299.56
7	49,911.97	15.19	499.12	299.47
8	49,896.78	15.34	498.97	299.38
9	49,881.44	15.50	498.81	299.29
10	49,865.94	15.65	498.66	299.20
11	49,850.29	15.81	498.50	299.10
12	49,834.48	15.97	498.34	299.00
PRESENT VALUE USING 1.5% DISCOUNT RATE		164.68	5,445.16	3,267.10

Table 1

Conventional Method of Computing
Present Value of Interest Payments

$$A(n-1,b) = A(359, .004950495) \quad (5)$$

$$= 167.690422$$

$$\text{MGT. INT. PV} = \frac{.01(50{,}000)}{1.015} + \frac{514.31(66.34854037)}{1.015} \quad (1)$$

$$+(167.690422)\left[\frac{.01(50{,}000) - 514.31}{1.015}\right]$$

$$= \$31{,}747.85$$

$$\text{Present Value of Principal Payments} = 514.31 \times \frac{(1 - 1.015^{-360})}{.015} - 31{,}747.85 \quad (8)$$

$$= 34{,}126.14 - 31{,}747.85$$

$$= \$2{,}378.29$$

Thus, for 360 payments, the present value before taxes of the interest is \$31,747.85 and the present value of the principal is \$2,378.29.

Present Value of Tax Benefit Provided by Depreciation

Variables common to all depreciation equations to follow are shown below. Variables unique to specific equations will be shown where appropriate.

C = Cost of asset

S = Asset Salvage value

L = Depreciable life

n = number of periods not to exceed asset life (L)

K = discount rate per depreciation period in decimal form

TX = Tax rate in decimal form

Sum-of-the-Year's - Depreciation (SYD)

The equation to determine the present value of the tax benefits provided by SYD depreciation is:

$$\text{PVSYD} = \frac{2(C-S)}{L(L+1)(1+K)^n}\left[(L-1/K)\ S(n,K) + \frac{n}{K}\right] \tag{9}$$

TAX BENEFIT

$$\text{PVSYD} = \text{TX (PV-SYD)} \tag{10}$$

where:

PVSYD = Present value of SYD depreciation

$$S(n,K) = \frac{(1+K)^n - 1}{K} \text{ when } K \neq 0 \tag{11}$$

$K > 0$

To illustrate Equation (9), assume an asset has a cost of \$12,000, a salvage value of \$2,000, and a life of 20 years. Using a 10% discount rate, the present value is:

$C = 12,000$

$S = 2,000$

$L = 20$

$n = 10$

$K = .1$

$$S(n,K) = S(20,.1) = \frac{(1.1^{20} - 1)}{.1} \tag{11}$$

$$= 57.27499949$$

$$\text{PVSYD} = \frac{2(12,000 - 2,000)}{20(21)(1.1)^{20}}\left[(20 - 1/.1)57.27499949 + \frac{20}{.1}\right] \tag{9}$$

$$= \$5,469.73$$

Assuming a 40% tax rate and using Equation (9)

TAX BENEFIT

$$\text{PVSYD} = .4 \times \text{(PV-SYD)} \tag{10}$$

$$= .4(5,469.73)$$

$$= \$2,187.89$$

The present value of the tax benefit is \$2,187.89. The tax benefit for a period of less than the 20 year life can also be determined using Equations

(9) and (10). To determine the present value of the first 6 year's depreciation and deduction:

$n = 6$

$$S(6,.1) = 7.71561 \tag{11}$$

$$PVSYD = \frac{2(12{,}000-2{,}000)}{20(21)(1.1)^6}\left[(20-1/.1)\ 7.71561 + \frac{6}{.1}\right] \tag{9}$$

$= \$3{,}686.72$

TAX BENEFIT

$$PVSYD = .4(3{,}686.72) \tag{10}$$

$= \$1{,}474.69$

Declining Balance (D.B.) Depreciation

Some preliminary comments on D.B. depreciation are necessary before introducing the equation to compute the tax benefit present value.

The periodic depreciation expense using the D.B. method of depreciation can be expressed as:

$$DB\ Dep_j = (F/L)\ C(1-F/L)^{j-1} \tag{12}$$

where:

$DB\ Dep_j$ = D.B. depreciation expense for period j

F = the DB factor 1, 1.25, 1.5, 2, etc.

j = the period to compute the depreciation expense.

The number of periods that DB depreciation can be computed before salvage is reached can be determined by:

$$n^* = \ln(S/C)/\ln(1-F/L) \tag{13}$$

where:

S = a positive salvage value

The term n^*, however, will normally have a fractional value. To determine the number of whole periods for D.B. depreciation requires rounding the term n^* down or taking its integer value.

The integer value of n* represents the number of periods DB depreciation can be computed before the final period depreciation expense is computed :

$$\text{Final Period DB Depreciation Expense For Period } m+1 = C\,(1-F/L)^m - S \tag{14}$$

where:

m = the integer value of n^* and is the last period to use D.B. depreciation.

When m + 1 is equal to or less than the life of the asset (L) and the salvage (S) is a positive value, the present value of the tax benefit provided by DB is:

$$PVDB = \frac{1}{1+K}\left[C - \frac{S}{(1+K)^m}\right] - \frac{K(C)}{1+K} \times A(m,b) \tag{15}$$

$$\text{TAX BENEFIT } PVDB = TX(PV\text{-}DB) \tag{16}$$

where:

PVDB = Present value of D.B. depreiciation when $m + 1 \leq L$

$$A(m,b) = \frac{1-(1+b)^{-m}}{b} \text{ when } b \neq 0$$

$$= m \text{ when } b = 0$$

$$b = (1+K)/(1-F/L) - 1 \tag{17}$$

To demonstrate Equations (15) and (16), consider the following example. An asset costing $12,000 with a salvage value of $2,000 and a 20 year life is depreciated using double declining balance depreciation. What is the present value using a 15% annual discount rate? What is the tax benefit using a 30% tax rate?

The key values are:

$C = 12{,}000$

$S = 2{,}000$

$L = 20$

$F = 2$

$K = .15$

$$n^* = \ln(2{,}000/12{,}000)/\ln(1-2/20) \tag{13}$$

$$= 17.005986$$

$m = 17$

$$b = (1.15)/(1-2/20) - 1 \tag{17}$$

$$= .277777778$$

$$A(m,b) = \frac{1 - (1.277777778)^{-17}}{.277777778}$$

$$= 3.544209289$$

Next, the PV - DB is calculated using Equation (15).

$$PVDB = \frac{1}{1.15}\left[12{,}000 - \frac{2{,}000}{(1.15)^{17}}\right] - \frac{.15(12{,}000)}{1.15}(3.544209289)$$

$$= \$4{,}725.71 \tag{15}$$

The after tax benefit using Equation (16) is:

TAX BENEFIT

$$PVDB = .3(PV-DB)$$

$$= .3(4{,}725.71)$$

$$= \$1{,}417.71 \tag{16}$$

Equation (15) is adequate to determine the PV of DB depreciation when $(m + 1) \leq L$. Another equation, however, must be used if $(m + 1) > L$ and a switch to straight line depreciation is desired to maximize the tax benefit. When the salvage value is zero, switching to straight line (SL) depreciation is also necessary to maximize the tax benefit.

When the salvage value is zero, the cross over point can be determined using,

$$m = L - L/F \tag{18}$$

where m is the last period to use DB depreciation [Bierman and Smidt, 1971]. If m is a fractional value, then round the value up.

Determining the cross over point when the salvage value is positive requires finding the first depreciation period which satisfies:

$$\frac{C(1-F/L)^{m-1} - S}{L - m + 1} \geq (F/L)C\ (1-F/L)^{m-1} \tag{19}$$

The starting estimate for m is found using Equation (18). After determining the last year for DB using Equation (19), the present value is:

$$DBSLPV = \frac{C(b-K)\ A(m,b)}{1 + K} + \frac{(C(1-F/L)^{m} - S)\ A(L-m,K)}{(L-m)(1+K)^{m}} \tag{20}$$

$$\begin{aligned}&\text{TAX BENEFIT}\\ &\quad DBSLPV = TX(DBSL\text{-}PV)\end{aligned} \tag{21}$$

where:

DBSLPV = Present value of DB depreciation with switch over to SL depreciation

m = last period to use DB depreciation

$$A(n,b) = \frac{1 - (1+b)^{-m}}{b} \text{ when } b \neq 0$$

$$= m \text{ when } b = 0$$

$$A(L-m,K) = \frac{1 - (1+K)^{-L+m}}{K} \text{ when } K \neq 0$$

$$= (L - m) \text{ when } K = 0$$

To illustrate Equation 18, assume an asset costing \$12,000 with a salvage value of \$825 is depreciated over 10 years using double DB depreciation. The discount rate is 15%, and the tax rate is 30%.

First, the number of years required to reach salvage is computed using Equation (13).

$$n^* = \ln(825/12{,}000)/\ln(1-2/10) \qquad (13)$$
$$= 11.998$$

A switch to straight line is desirable to maximize the tax benefit, because the number of years required to depreciate the asset down to the salvage value (11.998) exceeds the life of 10 years. The cross over point assuming salvage is zero is used as the starting point for m.

$$m = 10 - 2/10 \qquad (18)$$
$$= 5$$

Next, the condition expressed in Equation (19) is satisfied.

$m = 5$; SL Dep. Year 5 $\dfrac{12{,}000(1-2/10)^{5-1} - 825}{10 - 5 + 1} = 681.70$

DB Dep. Year 5 $2/10(12{,}000)(1-2/10)^{5-1} = 983.04$

$m = 6$; SL Dep. Year 6 $\dfrac{12{,}000(1-2/10)^{6-1} - 825}{10 - 6 + 1} = 621.43$

DD Dep. Year 6 $2/10(12{,}000)(1-2/10)^{6-1} = 786.432$

$m = 7$; SL Dep. Year 7 $\dfrac{12{,}000(1-2/10)^{7-1} - 825}{10 - 7 + 1} = 580.18$

DB Dep. Year 7 $2/10(12{,}000)(1-2/10)^{7-1} = 629.15$

$m = 8$; SL Dep. Year 8 $\dfrac{12{,}000(1-2/10)^{8-1} - 825}{10 - 8 + 1} = 563.86$

DB Dep. Year 8 $2/10(12{,}000)(1-2/10)^{8-1} = 503.32$

As shown above, the last year to use DB depreciation is year 7 because the SL depreciation is larger than DB depreciation in year 8. Now that the last year to use DB depreciation is established, Equation (20) is used to determine the present value.

The key values are:

$m = 7$

$L = 10$

$K = .15$

$$b = (1.15/(1-2/10)) - 1$$

$$= .4375 \tag{17}$$

$$A(m,b) = \frac{1 - (1.4375)^{-7}}{.4375}$$

$$= 2.105509231$$

$$A(L-m,K) = \frac{1 - (1.15)^{-10+7}}{.15}$$

$$= 2.283225117$$

$$DBSLPV = \frac{12{,}000(.4375 - .15)(2.105509231)}{1.15}$$

$$+ \frac{(12{,}000(1-2/10)^{7} - 825)(2.283225117)}{(10-7)(1.15)^{7}}$$

$$= \$6{,}800.52 \tag{20}$$

The tax benefit is determined using Equation (21).

TAX BENEFIT

$$\text{DBSLPV} = .3(6800.52)$$

$$= \$2{,}040.16 \tag{21}$$

Equation (20) can also be modified to compute the tax benefit of DB depreciation for situations where the analysis period is less than the straight line life.

The following modification of Equation (20) is used to compute the present value of the tax benefit when the number of periods in the analysis period is greater than the cross over point but less than the asset life.

Partial Period

$$\text{DBSLPV} = \frac{C(b-K)\ A(m,b)}{1 + K} + \frac{(C(1 - F/L)^m - S)\quad A(h-m,K)}{(L-m)(1 + K)^m} \tag{22}$$

where:

Partial Period DBSLPV = Present value of DB with switch to SL where $h > m$ and $h < L$

$h < L$

$h > m$

h = number of analysis periods for computing present value

m = last period to use DB depreciation

To illustrate Equation (22), the previous DB example will be solved using an analysis period of 9 years.

$h = 9$

$b = .4375$

$K = .15$

$m = 7$

$A(m,b) = 2.10550923$

$$A(h-m,K) = \frac{1 - (1.15)^{-9+7}}{.15}$$

$$= 1.625708885$$

Partial Period

$$DBSLPV = \frac{12{,}000(.4375 - .15)(2.10550923)}{1.15}$$

$$+ \frac{(12{,}000(1-2/10)^{7} - 825)1.625708885}{(10-7)(1.15)^{7}}$$

$$= \$6{,}661.14 \tag{22}$$

The present value of the depreciation for 9 years is $6,661.14.

When the number of analysis periods are less than the asset life and no switch over or the number of analysis periods are less than m (the last period to use DB for a switch to straight line) the equation is:

Partial Period

$$DBPV = \frac{C(b-K)\ A(h,b)}{1 + K} \tag{23}$$

Where:

Partial Period

DBPV = Present value of DB depreciation where $h < m$

$h \leq m$

$h < L$

To illustrate equation (23), the previous DB example will be solved using an analysis period of 5 years.

$h = 5$

$$A(5,b) = \frac{1 - (1.4375)^{-5}}{.4375}$$

$$= 1.913337434$$

Partial Period

$$DBPV = \frac{12{,}000(.4375 - .15)(1.913337434)}{1.15} \tag{23}$$

$$= \$5{,}740.01$$

Thus, the present value for the first five years of DB depreciation is \$5,740.01.

Converting Interest Rates

In some analysis, the number of mortgage Payments per year and/or the number of depreciation periods per year will not agree with the number of compounding periods per year for the discount rate. In such cases, the appropriate equivalent discount rate for use with the equations in this paper can be calculated using:

$$(1 + K)^f = (1+K^*)^c \qquad (24)$$

$$K = (1+K^*)^{c/f} - 1$$

where:

K^* = discount rate compounded c times per year

K = equivalent discount rate compounded f times per year. The term f represents either the number of time mortgage payments are made per year or the number of depreciation periods per year.

For example, if the discount rate is expressed as 12% annual compounded quarterly, and mortgage payments are made monthly, the discount rate to use for calculating the present value of mortgage interest payments is:

$K^* = .12/4 = .03$

$c = 4$

$f = 12$

$$K = (1.03)^{4/12} - 1 \qquad (24)$$

$$= .009901634$$

If the discount rate is 18% annual compounded monthly and the depreciation is computed yearly, the discount rate to use for calculating the present value of the Depreciation tax benefit is:

$$K^* = .18/12 = .015$$

$$c = 12$$

$$f = 1$$

$$K = (1.015)^{12/1} - 1 \tag{24}$$

$$= .195618171$$

Conclusion

The equations presented in this paper allow students to quickly calculate the before or after tax present value of mortgage payments and of depreciation tax benefits. Any calculator with a Y^x function is suitable for solving the equations. The equations when programmed are also efficient because computer time is reduced since period by period calculations are avoided.

By using these equations teachers can assign realistic capital budgeting problems such as lease or buy analysis without requiring excessive calculation time by students.

APPENDIX

Proof for Present Value of Mortgage Interest Payments

The mortgage balance at the end of payment period (j) is expressed as:

$$RB_j = PV(1+i)^j - PMT \frac{(1+i)^j - 1}{i} \tag{25}$$

where:

PV = amount of mortgage

PMT = equal periodic payment

i = mortgage interest rate per payment period expressed as a decimal

j = payment period j where j = 1,2,3,...,n.

n = number of mortgage payments

RB_j = mortgage balance at the end of period j

The present value of n interest payments can be expressed as:

$$\begin{aligned} \text{MGT. INT. PV} = {} & i(PV)/(1+K)^1 + i(RB_1)/(1+K)^2 \\ & + i(RB_2)/(1+K)^4 + \ldots + i(RB_{n-2})/(1+K)^{n-1} \\ & + i(RB_{n-1})/(1+K)^n \end{aligned} \tag{26}$$

where:

K = discount rate per mortgage payment period

Substituting for RB_j in Equation (26) gives:

$$\text{MGT. INT. PV} = \frac{i(PV)}{(1+K)} + \frac{i(PV)(1+i)^1}{(1+K)^2} - \frac{i(PMT)((1+i)^1 - 1)}{i(1+K)^2}$$
$$+ \frac{i(PV)(1+i)^2}{(1+K)^3} - \frac{i(PMT)((1+i)^2 - 1)}{i(1+K)^3}$$
$$+ \frac{i(PV)(1+i)^3}{(1+K)^4} - \frac{i(PMT)((1+i)^3 - 1)}{i(1+K)^4}$$
$$+ \ldots + \frac{i(PV)(1+i)^{n-2}}{(1+K)^{n-1}} - \frac{i(PMT)((1+i)^{n-2} - 1)}{i(1+K)^{n-1}}$$
$$+ \frac{i(PV)(1+i)^{n-1}}{(1+K)^n} - \frac{i(PMT)((1+i)^{n-1} - 1)}{i(1+K)^n} \qquad (27)$$

The next step is expressing Equation (27) as:

$$\text{MGT. INT. PV} = \frac{i(PV)}{(1+K)} + \frac{i(PV)}{(1+K)}(Y) - \frac{PMT}{(1+K)}(Z) \qquad (28)$$

where:

$$Y = \frac{1}{(1+K)^1(1+i)^{-1}} + \frac{1}{(1+K)^2(1+i)^{-2}} + \frac{1}{(1+K)^3(1+i)^{-3}}$$
$$+ \ldots + \frac{1}{(1+i)^{n-2}(1+K)^{-(n-2)}} + \frac{1}{(1+i)^{n-1}(1+K)^{-(n-1)}} \qquad (29)$$

$$Z = \frac{1}{(1+K)^1(1+i)^{-1}} - \frac{1}{(1+K)^1} + \frac{1}{(1+K)^2(1+i)^{-2}} - \frac{1}{(1+K)^2}$$
$$+ \frac{1}{(1+K)^3(1+i)^{-3}} - \frac{1}{(1+K)^3} + \ldots + \frac{1}{(1+K)^{n-2}(1+i)^{-(n-2)}}$$
$$- \frac{1}{(1+K)^{n-2}} + \frac{1}{(1+K)^{n-1}(1+i)^{-(n-1)}} - \frac{1}{(1+K)^{n-1}} \qquad (30)$$

The Y term, Equation (29), can now be expressed as:

$$Y = \frac{1}{(1+b)} + \frac{1}{(1+b)^2} + \frac{1}{(1+b)^3} + \ldots + \frac{1}{(1+b)^{n-2}} + \frac{1}{(1+b)^{n-1}} \tag{31}$$

$$Y = \frac{1 - (1+b)^{-(n-1)}}{b} \tag{32}$$

$$Y = A(n-1,b) \tag{33}$$

where:

$$1 + b = (1+K)^1(1+i)^{-1} \tag{34}$$

Equation (31) is expressed as the present value of an ordinary annuity with n-1 payments discounted at b percent per payment period.

The Z term is simplified using the same procedure as for the Y term.

$$\begin{aligned} Z = {} & \frac{1}{(1+k)^1(1+i)^{-1}} + \frac{1}{(1+K)^2(1+i)^{-2}} + \frac{1}{(1+k)^3(1+i)^{-3}} \\ & + \ldots + \frac{1}{(1+K)^{n-2}(1+i)^{-(n-2)}} + \frac{1}{(1+K)^{n-1}(1+i)^{-(n-1)}} \\ & - \left[\frac{1}{(1+K)^1} + \frac{1}{(1+K)^2} + \frac{1}{(1+K)^3} + \ldots + \frac{1}{(1+K)^{n-2}} + \frac{1}{(1+K)^{n-1}}\right] \end{aligned} \tag{35}$$

$$Z = \frac{1 - (1+b)^{-(n-1)}}{b} - \frac{1 - (1+K)^{-(n-1)}}{K} \tag{36}$$

Equation (28) is now expressed as:

$$\text{MGT. INT. PV} = \frac{i(PV)}{(1+K)} + \frac{i(PV)}{(1+K)} \; \frac{1-(1+b)^{-(n-1)}}{b}$$

$$- \frac{PMT}{(1+K)} \; \frac{1-(1+b)^{-(n-1)}}{b} + \frac{PMT}{1+K} \; \frac{1-(1+K)^{-(n-1)}}{K} \tag{37}$$

$$= \frac{i(PV)}{(1+K)} + \frac{PMT}{(1+K)} \times A(n-1,K) + \frac{A(n-1,b)}{(1+K)} \left(i(PV) - PMT\right) \tag{38}$$

where:

$$A(n-1,K) = \frac{1-(1+K)^{-(n-1)}}{K} \text{ when } K \neq 0$$

$$= n - 1 \text{ when } K = 0$$

$$A(n-1,b) = \frac{1-(1+b)^{-(n-1)}}{b} \text{ when } b \neq 0$$

$$= n - 1 \text{ when } b = 0$$

<u>Proof for Present Value of SYD Depreciation Tax Benefit</u>

The periodic SYD depreciation expense decreases each period by:

$$(C-S) \times \frac{2}{L(L+1)} \tag{39}$$

where:

C = cost of asset

S = salvage

L = Depreciable asset life

The amount of the first period's depreciation expense is:

$$\frac{L(C-S)2}{L(L+1)} \tag{40}$$

Because the depreciation expense decreases each period by a constant dollar amount, the present value can be computed using the equation for a Constant-Increment Annuity [Greynolds, Aronofsky and Frame, 1980].

$$PVSYD = (1+K)^{-n}\left[\left(PMT + \frac{\Delta P}{K}\right) S(n,K) - n\,\frac{\Delta P}{K}\right] \tag{41}$$

where:

PVSYD = present value of tax benefit provided by SYD depreciation

K = discount rate per depreciation period

n = number of depreciation periods

$n \leq L$, the depreciable asset life

PMT = first periods depreciation

ΔP = constant increment increase

$S(n,K) = \dfrac{(1+K)^n - 1}{K}$ when $K \neq 0$

$K > 0$

By setting

$$PMT = \frac{L(C-S)2}{L(L+1)} \tag{42}$$

$$\Delta P = -\frac{(C-S)2}{L(L+1)} \tag{43}$$

Equation (41) can be expressed as:

$$PVSYD = (1+K)^{-n}\left[\left(\frac{L(C-S)2}{L(L+1)} - \frac{(C-S)2}{L(L+1)K}\right) S(n,K) + \frac{n(C-S)2}{L(L+1)K}\right] \tag{44}$$

$$= \frac{2(C-S)}{L(L+1)(1+K)^{-n}} \quad [(L - 1/K)S(n,K) + n/k] \tag{45}$$

Proof for Present Value of DB Depreciation Tax Benefit

Assume the last period to use DB depreciation (m) is less than the asset's straight line life (L).

$$PVDB = \frac{C(1-F/L)^0}{(1+K)^1} - \frac{C(1-F/L)^1}{(1+K)^1} + \frac{C(1-F/L)^1}{(1+K)^2} - \frac{C(1-F/L)^2}{(1+K)^2} + \dots + \frac{C(1-F/L)^{m-1}}{(1+K)^m} - \frac{C(1-F/L)^m}{(1+K)^m} + \frac{C(1-F/L)^m}{(1+K)^{m+1}} - \frac{S}{(1+K)^{m+1}} \tag{46}$$

where:

PVDB = present value of tax benefit provided by DB depreciation

C = cost of asset

S = salvage

L = straight line asset life

F = DB factor, 1, 1.25, 1.5, 2, etc.

K = discount rate per depreciation period

m = last period to use DB depreciation

$m < L$

Next, Equation (46) can be expressed as:

$$PVDB = \frac{C(1-F)^0}{(1+K)^1} - \frac{S}{(1+K)^{m+1}} + \frac{C}{1+K}\left[\frac{(1-F/L)^1}{(1+K)^1} + \dots + \frac{(1-F/L)^m}{(1+K)^m}\right] - C\left[\frac{(1-F/L)^1}{(1+K)^1} + \frac{(1-F/L)^2}{(1+K)^2} + \dots + \frac{(1-F/L)^m}{(1+K)^m}\right] \tag{47}$$

By setting:

$$(1+b) = (1+K)/(1-F/L) \tag{48}$$

Equation (47) simplifies to:

$$\begin{aligned} PVDB &= \frac{1}{(1+K)}\left[C - \frac{S}{(1+K)^m}\right] + \frac{C}{(1+K)} A(m,b) - (C)\ A(m,b) \\ &= \frac{1}{(1+K)}\left[C - \frac{S}{(1+K)^m}\right] + (C)\ A(m,b)\left[\frac{1}{1+K} - 1\right] \end{aligned} \tag{49}$$

Setting $\frac{1}{1+K} - 1 = -\frac{K}{1+K}$ gives:

$$PVDB = \frac{1}{(1+K)}\left[C - \frac{S}{(1+K)^m}\right] - \frac{C(K)\ A(m,b)}{(1+K)}$$

where:

$$b = ((1+K)/(1-F/L)) - 1 \tag{50}$$

$$\begin{aligned} A(m,b) &= \frac{1-(1+b)^{-m}}{b} \text{ when } b \neq 0 \\ &= m \text{ when } b = 0 \end{aligned}$$

$m < L$

<u>When the number of analysis periods (h) is less than (m) the number of periods to use DB Depreciation</u>

Partial Period

$$\begin{aligned} DBPV = {} & \frac{C(1-F/L)^0}{(1+K)^1} - \frac{C(1-F/L)^1}{(1+K)^1} + \frac{C(1-F/L)^1}{(1+K)^2} \\ & - \frac{C(1-F/L)^2}{(1+K)^2} + \ldots + \frac{C(1-F/L)^{h-1}}{(1+K)^h} - \frac{C(1-F/L)^h}{(1+K)^h} \end{aligned} \tag{51}$$

where:

h = number of analysis periods

m = last period to use DB depreciation

$h \leq m$

Next, Equation (51) can be expressed as:

$$\text{Partial Period DBPV} = \frac{C(1-F/L)^0}{(1+K)^1} + \frac{C}{(1+K)^1}\left[\frac{(1-F/L)^1}{(1+K)^1} + \ldots + \frac{(1-F/L)^{h-1}}{(1+K)^{h-1}}\right] - C\left[\frac{(1-F/L)^1}{(1+K)^1} + \frac{(1-F/L)^2}{(1+K)^2} + \ldots + \frac{(1-F/L)^h}{(1+K)^h}\right] \quad (52)$$

By setting:

$$1 + b = (1+K)/(1-F/L) \quad (53)$$

$$A(h,b) = \frac{1 - (1+b)^{-h}}{b}$$

Equation (52) can be shown as:

$$\text{Partial Period DBPV} = \frac{C(1-F/L)^0}{(1+K)^1} + \left[\frac{C}{(1+K)^1} \; A(h-1,\ b)\right] - (C)\ A(h,b) \quad (54)$$

By using the relationship:

$$A(h-1,\ b) = A(h,b) - (1+b)^{-h} \quad (55)$$

Equation (54) is shown as:

$$\begin{aligned}\text{Partial Period DBPV} &= \frac{C}{(1+K)^1} + \frac{C}{(1+K)^1}\left[A(h,b) - (1+b)^{-h}\right] - (C)\ A(h,b) \quad (56)\\ &= \frac{C}{(1+K)} - \frac{C(1+b)^{-h}}{(1+K)} + (C)\ A(h,b)\left[\frac{1}{1+K} - 1\right]\\ &= \frac{C(b)}{(1+K)}\left[\frac{1-(1+b)^{-h}}{b}\right] + (C)\ A(h,b)\left[\frac{-K}{1+K}\right]\\ &= \frac{C(b)}{(1+K)}\ A(h,b) + (C)\ A(h,b)\left[\frac{-K}{1+K}\right]\\ &= \frac{C(b-K)\ A(h,b)}{(1+K)} \quad (57)\end{aligned}$$

where:

$$b = ((1+K)/(1-F/L))-1$$

$$h \leq m$$

$$A(h,b) = \frac{1 - (1+b)^{-h}}{b} \text{ when } b \neq 0$$

$$= h \text{ when } b = 0$$

Switch over to straight line depreciation from DB depreciation.

By setting h equal to m the last period to use DB depreciation, the DB portion of the present value is the same as Equation (57)

$$\text{DB Portion} = \frac{C(b-K)\ A(m,b)}{1 + K} \tag{58}$$

The straight line depreciation expense per year for L - m years is:

$$\text{SL DEP} = \frac{C(1-F/L)^m - S}{L - m} \tag{59}$$

By treating the SL depreciation as a deferred annuity, the present value is:

$$\text{PV of SL Dep Portion} = \frac{(C(1-F/L)^m - S)\ A(L-m,K)}{(L-m)\ (1+K)^m} \tag{60}$$

Combining Equations (58) and (60) results in:

$$\text{DBSLPV} = \frac{C(b-K)\ A(m,b)}{1 + K} + \frac{(C(1-F/L)^m - S) \times A(L-m,K)}{(L-m)\ (1+K)^m} \tag{61}$$

Proof for After Tax Equations

Given:

$$PV = \sum_{j=1}^{n} PMT\ (1+i)^{-j} \tag{62}$$

Then the after tax cash flow shielded by depreciation is:

$$(Tx)PV = \sum_{j=1}^{n} Tx(Dep_j)(1+i)^{-j}$$

$$(Tx)PV = (Tx) \sum_{j=1}^{n} Dep_j(1+i)^{-j} \tag{63}$$

The after tax present value of cash flows is:

$$(1-Tx)\ PV = (1-Tx) \sum_{j=1}^{n} PMT_j\ (1+i)^{-j}$$

where:

PV = present value of n payments

Dep_j = depreciation expense per period j

PMT_j = Payment per period j

Tx = Tax rate expressed as decimal

n = number of payments

j = 1,2,3, . . ., n

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