Nonlinear Photonics in Twisted and Nonlocal Structures

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NONLINEAR PHOTONICS
IN TWISTED
AND NONLOCAL STRUCTURES

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NONLINEAR PHOTONICS
IN TWISTED
AND NONLOCAL STRUCTURES

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in Twisted
and Nonlocal Structures

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We provide a theoretical framework for the observed confinement of light modes within a twisted coreless photonic crystal fiber. Asymptotic methods are applied through ray theory and field theory in both the linear and nonlinear regime. We find the modes have a radially symmetric chirp and the envelope will decay away from the axis of propagation. Secondly, we study the stability and singularity formation of unidirectional beams as described by the Schrödinger equation. We propose a novel extension to the modeling equation to include a fractional Laplacian in one spatial dimension and a standard second derivative in a second dimension. The goal is to explore dynamics and stability properties as a function of the degree of fractionality. Numerically, we use a time-splitting Fourier pseudo-spectral method which accounts for nonlocal interactions from the fractional Laplacian and is applicable to the linear and nonlinear cases. We find minimal values of the fractional parameter where singularities form for various power levels and see that symmetry does not always hold near blowup in the fractional case.
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Chapter 1

Introduction

Many of the great advances in our recent history are characterized by a rapid development of electrical and digital technologies. These breakthroughs are an outcome of a deeper understanding of particular properties of physical materials. Advances in electrical and semiconductor physics paved the pathway for transistors and today’s modern computers. By effectively controlling and guiding electromagnetic waves, entirely new and creative innovations have been made possible. In the last few decades, new horizons have appeared where the goal is to understand and control the optical properties of materials. Immense potential can be imagined if we are able to design materials that regulate light of specific frequencies, propagate desired modes in isolated directions, or perfecting reflecting them, and confining the energy to a chosen region.

An early example of a mechanism that functions as an optical waveguide is given by total internal reflection (TIR) and has been dated back to 1841 during a demonstration by Jean-Daniel Colladon using a jet of water. In 1880, Alexandar Graham Bell introduced a significant breakthrough in optical communication called “the photophone.” This could transmit a person’s voice a few hundred meters on a beam of light [39]. Since the 1950’s, fiber optics has rapidly developed starting with the invention of the fiberscope in 1956. This device is able to transmit images and was created concurrently by Brian O’Brian and Narinder Kapany as well as colleagues at the American College of Science and Technology in London [91]. In 1958, Arthur Schawlow and Charles Townes invented the laser and published a paper describing the basic principles of light amplification by stimulated emission of radiation [83]. Semiconductor lasers were introduced in 1962 and are the most widely used
in fiber optics today. Light has a known capacity of carrying over 10,000 times the amount of information as the highest radio frequencies in use. A disadvantage is that it is more affected by environmental conditions and is not suited for open-air transmissions [91].

Another breakthrough in fiber optic technology that leads into a study performed in this manuscript is the invention of the photonic crystal fiber (PCF) by Philip Russell in 1996 [52, 79]. PCFs provide an innovative platform for research in light-matter interaction and nonlinear optics; both solid and hollow core variants have been demonstrated and received substantial attention. PCFs have since been applied to several fields of research such as high-powered lasers [64, 94], particle guidance [32, 84], supercontinuum generation [75, 24], light-gas interaction [80], and biological [90] and chemical sensing [19]. Chapter 4 will focus on analyzing the confinement provided by a coreless PCF that is helically twisted along the axis of propagation.

In high-capacity fiber optic communication systems the transmitted signal is amplified periodically using optical amplifiers to reduce residual fiber loss. As a result, nonlinear effects present in the fiber accumulate over long distances and should be considered. Chapter 2 provides a brief review of how the nonlinear Schrödinger equation (NLSE) may be derived in a pulse propagation model starting with Maxwell’s equations. The NLSE describes the propagation of a wave through a nonlinear medium. In contrast to the typical theory of evolutionary partial differential equations (PDEs), the NLSE in this manuscript uses the spatial variable, \( z \), that denotes the axis of propagation along the fiber as the evolution variable. The NLSE typically includes a second derivative (1D) or Laplacian (2D) component to represent spatial diffraction. In 2000, Nikolai Laskin published a series of papers generalizing the Feynman path integral originally over Brownian paths to integrate over Levy paths. The result, which Laskin coined, is referred to as fractional quantum mechanics. Using this generalization, the Laplacian component in the Schrödinger equation can be extended to a fractional power, as opposed to the usual integer power, which leads to the fractional
(nonlinear) Schrödinger equation (FSE or FNLSE).

The FSE has been shown to be the continuum limit in the weak sense of the discrete NLS with long range lattice interactions [48]. Ground states of the FLS have been studied in [25, 26] and the dynamics during propagation given in [49, 7]. For a thorough review and additional details on the various representations and current numerical methods used for the fractional Laplacian, we refer to [65]. One particular application that will be highlighted is by Longhi in 2015 [67]. Longhi provides an optical realization of the FSE based on transverse light dynamics in aspherical optical cavities. A laser implementation of the fractional quantum harmonic oscillator is presented, which allows the generation of dual Airy beams under off-axis longitudinal pumping.

In this research we wish to model short optical pulses and integrate temporal effects that influence the wave function. We therefore include a standard (integer ordered) second derivative in time denoting temporal dispersion. To the best of our knowledge, there is no published research on modeling equations involving mixed fractional and integer ordered PDEs such as this at this time. As such we will explore the ground states, stability, and dynamics of propagation of this type of equation within the linear and nonlinear regime.

1.1. Structure of the Dissertation

In this dissertation, the focus is emphasized on two different topics both of which concerning the nonlinear Schrödinger equation. The first is that of a pulse propagation model within a twisted coreless PCF. This research was inspired by a recent article [12] that shows this particular geometry of fiber realizes confinement of light and that the degree of confinement is related to the degree of twist of the fiber. Chapter 4 discusses two approaches that attempt to rigorously show why this relation to confinement exists. The first approach is through ray theory and the second approach is through field theory. We propose the phenomenon of confinement related to a high twist rate is analogous to the stability that is attained in a mechanical pendulum by rapidly vibrating its base. This effect is sometimes known as the
“Kapitza effect” and appears in the pendulum when the rate of vibration is within a specific range. Our results show that the allowed modes will have a radially symmetric chirp and the envelope of the field should decay away from the axis of rotation.

The second topic of this dissertation is on the study of a nonlinear Schrödinger type equation that includes the one dimensional fractional Laplacian in the spatial $x$ direction along with a standard time dispersion component. Chapter 5 provides a brief overview of the one and two dimensional NLS and the motivation to generalize the Laplacian to include non-integer powers. An overview of how the fractional Laplacian and fractional Schrödinger equation are formulated is given through the generalization of path integrals over Brownian motion into path integrals over Lévy paths. In chapter 6, a linear stability analysis for our extended model is performed and the regions of stability are established. We see that the defocusing case is similar to the classical case in that it is always stable. Section 6.2 discusses the time-splitting Fourier pseudo-spectral numerical method used throughout this study. The method was proposed by Kirkpatrick and Zhang [49] in order to solve the time-dependent FNLS. The derivation is reviewed and then extended to our problem. In section 6.4, we verify the regions of linear stability given in section 6.1 and explore the long term behavior of unstable perturbations.

Chapter 7 examines the ground states for our model in an infinite potential well. A method of finding the stationary states of the fractional Schrödinger equation is reviewed and then extended to fit our problem. This method was first proposed in [25] and is called the fractional gradient flow with discrete normalization (FGFDN) which itself is analogous to the normalized gradient flow used to find the stationary states of the standard Schrödinger equation [9, 17]. Section 7.2 shows the ground states found for our model. In accordance with the results of [25], boundary layers form as the fractional parameter is decreased.

In chapter 8, we analyze the formation of singularities. We find the fractional parameter and minimal power thresholds to cause the field to initially contract as opposed to spreading.
and examine the effects as both the fractional parameter and power are increased. Next the parameters required for singularity formation are discussed. In section 8.4, we investigate the behavior near blow up when the initial data is not symmetric in $x$ and $t$. In the low power simulations with a non-integer Laplacian, we find that the field does not necessarily form a radially symmetric profile near blowup as is the case when the standard Laplacian is used. If the power is much higher, the equation becomes more nonlinear and the nonlinear component then becomes the dominant characteristic near blowup at which symmetry is regained at the final collapse. Section 8.6 shows the results on the relationship between the power and the near-blowup symmetry. Lastly section 8.7 investigates the possibility that the solutions are self similar. We do not find conclusive evidence to prove this property however it may be suggested for fractional powers that are near the standard Laplacian.
Chapter 2
Maxwell’s Equations

In this chapter we provide a brief explanation of how the nonlinear Schrödinger equation is derived in optics starting from Maxwell’s equations and give a physical background to some of the parameters. Macroscopic electromagnetism is governed by the Maxwell equations. In SI units, the equations are

\begin{align*}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \text{Faraday’s Law,} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \text{Ampere’s Law,} \\
\nabla \cdot \mathbf{B} &= 0 & \text{Magnetic Gauss’ Law,} \\
\nabla \cdot \mathbf{D} &= \rho & \text{Electric Gauss’ Law,}
\end{align*}

where \( \mathbf{E} \) and \( \mathbf{H} \) are the macroscopic electric and magnetic fields, \( \mathbf{D} \) and \( \mathbf{B} \) are the displacement and magnetic induction fields, and \( \rho \) and \( \mathbf{J} \) are the free charge and current densities, respectively. A precise derivation from the microscopic counterparts is provided in [41].

Relationships exist between each of the fields and their flux densities. The densities arise in response to the electric and magnetic fields propagating inside the medium and are related by the constitutive relationships given by [21]

\begin{align*}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + \mathbf{P}, \\
\mathbf{B} &= \mu_0 \mathbf{H} + \mathbf{M},
\end{align*}

where \( \varepsilon_0 \approx 8.854 \times 10^{-12} \) Farad/m is the vacuum permittivity and \( \mu_0 = 4\pi \times 10^{-7} \) Henry/m is the vacuum permeability. In mediums that are nonmagnetic, we may set \( \mathbf{M} = 0 \) which
holds for optical fibers. We may also set $\mathbf{J} = 0$ and $\rho = 0$ for optical fibers since there are no free charges in the medium. Using these relationships along with equations (2.1) and (2.2), we can form a general wave equation of the electric field for nonlinear optics as

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} = 0$$  \hspace{1cm} (2.7)

where $\mu_0 = 1/(\varepsilon_0 c^2)$ and $c$ is the speed of light in a vacuum. The next step is to relate the polarization $\mathbf{P}$ to the electric field. In general, the polarization is a nonlinear response and can be found through quantum-mechanical methods however, in the study of wavelengths in the range of $0.5 - 2 \mu m$, we may approximate the polarization to third order nonlinear effects and split $\mathbf{P}$ into two parts such that

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_L(\mathbf{r}, t) + \mathbf{P}_{NL}(\mathbf{r}, t).$$  \hspace{1cm} (2.8)

The linear component $\mathbf{P}_L$ is determined from the electric field by [86, 85]

$$\mathbf{P}_L(\mathbf{r}, t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}(t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt'$$  \hspace{1cm} (2.9)

and the nonlinear component $\mathbf{P}_{NL}$ is determined by

$$\mathbf{P}_{NL}(\mathbf{r}, t) = \varepsilon_0 \int \int \int_{-\infty}^{\infty} \chi^{(3)}(t - t_1, t - t_2, t - t_3) \times \mathbf{E}(\mathbf{r}, t_1)\mathbf{E}(\mathbf{r}, t_2)\mathbf{E}(\mathbf{r}, t_3) dt_1 dt_2 dt_3$$  \hspace{1cm} (2.10)

where $\chi^{(j)}$ is the $j$th order susceptibility. The linear susceptibility $\chi^{(1)}$ represents the dominant contribution to $\mathbf{P}$. The second-order susceptibility $\chi^{(2)}$ is ignored here since it is only nonzero for media that lack an inversion symmetry at the molecular level [86] and $SiO_2$ is a symmetric molecule.

Next, we will make a few simplifying assumptions in order to reduce the complexity of equation (2.7). First, since the nonlinear changes in the refractive index are less than $10^{-6}$ in practice, $\mathbf{P}_{NL}$ is viewed as a small perturbation to $\mathbf{P}_L$. Second, the optical field generally maintains polarization along the fiber axis so the term may be reduced to a scalar. Third, the pulse spectrum is assumed to have a spectral width $\Delta \omega$ such that $\Delta \omega/\omega_0 \ll 1$, where $\omega_0$ is
the center of the spectrum and typically around $10^{15}$ s$^{-1}$ so this assumption holds for pulses as short as 0.1 ps. The goal of this derivation is to model the slowly varying envelope and so we will separate the highly oscillatory part by writing the electric field and polarization terms in the form of

$$
E(r, t) = \frac{1}{2} [E(r, t)e^{-i\omega_0 t} + c.c.] \hat{x}, \quad (2.11)
$$

$$
P_L(r, t) = \frac{1}{2} [P_L(r, t)e^{-i\omega_0 t} + c.c.] \hat{x}, \quad (2.12)
$$

$$
P_{NL}(r, t) = \frac{1}{2} [P_{NL}(r, t)e^{-i\omega_0 t} + c.c.] \hat{x}. \quad (2.13)
$$

The linear and nonlinear components of the polarization may be obtained by substituting equations (2.12) and (2.13) into (2.9) and (2.10), respectively. The linear component is then given by

$$
P_L(r, t) = \epsilon_0 \int_{-\infty}^{\infty} \chi^{(1)}_{xx}(t-t') E(r, t') e^{i\omega_0(t-t')} dt' \quad (2.14)
$$

$$
= \frac{\epsilon_0}{2\pi} \int_{-\infty}^{\infty} \tilde{\chi}^{(1)}_{xx}(\omega) \tilde{E}(r, \omega - \omega_0) e^{i(\omega-\omega_0)t} d\omega \quad (2.15)
$$

where $\tilde{E}(r, \omega)$ is the temporal Fourier transform of $E(r, t)$. To provide clarity moving forward in this derivation the specific variation of the Fourier transform used is explicitly given as

$$
f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(x, \omega - \omega_0) e^{-i(\omega-\omega_0)t} d\omega, \quad (2.16)
$$

$$
\hat{f}(x, \omega - \omega_0) = \int_{-\infty}^{\infty} f(x, t) e^{i(\omega-\omega_0)t} dt. \quad (2.17)
$$

If we assume the nonlinear response is instantaneous the nonlinear term can be approximated in a much simpler form by [3]

$$
P_{NL}(r, t) \approx \epsilon_0 \epsilon_{NL} E(r, t) \quad (2.18)
$$

where $\epsilon_{NL} = \frac{3}{4} \chi^{(3)}_{xxxx} |E(r, t)|^2$. As we aim to find the wave equation for the slowly varying amplitude $E(r, t)$, we choose to work in the Fourier domain. As another simplification, the nonlinear intensity dependence within $\epsilon_{NL}$ is treated as constant [6, 36]. This is a reasonable
assumption since $P_{NL}$ is viewed as a small perturbation. Using these formulations, we can now substitute equations (2.11)-(2.13) into (2.7) and find that the Fourier transform $\tilde{E}(r, \omega - \omega)$ must satisfy the Helmholtz equation

$$\Delta \tilde{E} + \epsilon(\omega)k_0^2\tilde{E} = 0,$$

(2.19)

where $k_0 = \omega/c$. The dielectric constant $\epsilon$ is given by

$$\epsilon(\omega) = 1 + \tilde{\chi}^{(1)}_{xx}(\omega) + \epsilon_{NL}.$$  

(2.20)

It is common to introduce two new terms, the refractive index $\tilde{n}$ and the absorption constant $\tilde{\alpha}$ given by

$$\tilde{n} = n + n_2|E|^2, \quad \tilde{\alpha} = \alpha + \alpha_2|E|^2,$$

(2.21)

$$n_2 = \frac{3}{8n}Re(\chi^{(3)}_{xxxx}), \quad \alpha_2 = \frac{3\omega}{4nc}Im(\chi^{(3)}_{xxxx}).$$

(2.22)

which allows an alternative form for the dielectric constant as

$$\epsilon = (\tilde{n} + i\frac{\tilde{\alpha}}{2k_0})^2.$$  

(2.23)

The $n_2$ constant is a measure of the fiber nonlinearity while the $\alpha_2$ term is small for silica fibers and typically neglected. With this form, we then approximate the dielectric constant as $\epsilon \approx n^2 + 2n\Delta n$, where $\Delta n = n_2|E|^2 + i\alpha/(2k_0)$ is small.

Equation (2.19) may be approximated through separation of variables. First we give the ansatz of

$$\tilde{E}(r, \omega - \omega_0) = F(x, y)\tilde{A}(z, \omega - \omega_0)e^{i\beta_0 z},$$

(2.24)

where $\tilde{A}$ is a slowly varying function of $z$ and $\beta_0$ is an undetermined wave number. Plugging in this form for the solution equation (2.19) gives two equations

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + [\epsilon(\omega)k_0^2 - \tilde{\beta}^2(\omega)]F = 0,$$

(2.25)

$$2i\beta_0 \frac{\partial \tilde{A}}{\partial z} + (\tilde{\beta}^2(\omega) - \beta_0^2)\tilde{A} = 0,$$

(2.26)
where the second \( z \) derivative of \( \tilde{A} \) has been neglected and \( \tilde{\beta} \) is a dummy variable inserted and becomes the eigenvalue for equation (2.25). The first step is to solve equation (2.25) through perturbation theory to recover \( F \) and \( \tilde{\beta} \). The eigenvalue is given by

\[
\tilde{\beta}(\omega) = \beta(\omega) + \Delta \beta,
\]

where

\[
\Delta \beta = \frac{k_0 \int \int_{-\infty}^{\infty} \Delta n |F(x, y)|^2 \, dx \, dy}{\int \int_{-\infty}^{\infty} |F(x, y)|^2 \, dx \, dy}.
\] (2.27)

The next step is to work with equation (2.26). We can use the approximation \( \tilde{\beta}^2 - \beta_0^2 \approx 2\beta_0 (\tilde{\beta} - \beta_0) \) to rewrite (2.26) as

\[
\frac{\partial \tilde{A}}{\partial z} = i[\beta(\omega) + \Delta \beta - \beta_0] \tilde{A}.
\] (2.28)

We now would like to expand \( \beta(\omega) \) into a Taylor series about the frequency \( \omega_0 \) at which the pulse spectrum is centered

\[
\beta(\omega) = \beta_0 + \beta_1 (\omega - \omega_0) + \frac{1}{2} \beta_2 (\omega - \omega_0)^2 + \frac{1}{6} \beta_3 (\omega - \omega_0)^3 + \ldots
\] (2.29)

where

\[
\beta_m = \left( \frac{d^m \beta}{d \omega^m} \right)_{\omega = \omega_0} \quad (m = 1, 2, 3, \ldots).
\] (2.30)

The parameters \( \beta_1 \) and \( \beta_2 \) are related to the refractive index \( n \) and its derivatives through the following relations

\[
\beta_1 = \frac{1}{v_g} = \frac{n_g}{c} = \frac{1}{c} \left( n + \omega \frac{dn}{d\omega} \right),
\] (2.31)

\[
\beta_2 = \frac{1}{c} \left( 2 \frac{dn}{d\omega} + \omega \frac{d^2 n}{d\omega^2} \right),
\] (2.32)

where \( n_g \) is the group index and \( v_g \) is the group velocity [3]. The physical relevance of \( \beta_1 \) is that the envelope of the optical pulse moves at the group velocity. The \( \beta_2 \) represents dispersion of the group velocity and affects pulse broadening. This is known as the group-velocity dispersion (GVD).

When the spectral width \( \Delta \omega \) is small, the cubic and higher order terms can be ignored. We now take the inverse Fourier transform of (2.28) which gives

\[
\frac{\partial A}{\partial z} = -\beta_1 \frac{\partial A}{\partial t} - i\beta_2 \frac{\partial^2 A}{\partial t^2} + i\Delta \beta A.
\] (2.33)
Substituting the definition for $\Delta n$ into the definition for $\Delta \beta$ (equation (2.27)) and rewrite (2.33) as

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i \beta_2 \frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2} A + i \gamma |A|^2 A,$$  \hspace{1cm} (2.34)

where the coefficient in front of the nonlinearity $\gamma$ is given by

$$\gamma = -\frac{n_2 \omega_0}{c A_{eff}}.$$  \hspace{1cm} (2.35)

$$A_{eff} = \left( \frac{\int \int_{-\infty}^{\infty} \Delta n |F(x, y)|^2 dxdy}{\int \int_{-\infty}^{\infty} \Delta n |F(x, y)|^4 dxdy} \right)^{1/2}.$$  \hspace{1cm} (2.36)

The pulse amplitude is assumed to be normalized such that $|A|^2$ represents the optical power. The parameter $A_{eff}$ is known as the effective core area and depends on fiber characteristics such as core radius and the core-cladding index difference. If the fundamental fiber mode is approximated by a Gaussian distribution of the form $F(x, y) \approx \exp\left[-(x^2 + y^2)/w^2\right]$. Equation (2.36) can be reduced to $A_{eff} = \pi w^2$. The width parameter $w$ can be determined by fitting the exact distribution to a Gaussian form. Equation (2.34) represents the propagation of picosecond optical pulse in single-mode fibers. It includes the effects of fiber losses through $\alpha$, of fiber nonlinearity through $\gamma$, and of chromatic dispersion through $\beta_1$ and $\beta_2$ [3]. In order to write (2.34) in a common form seen for the nonlinear Schrödinger equation (NLSE), we perform a change of variables into a time frame moving with the signal group velocity, $T = t - z/v_g$. Equation (2.34) then leads to

$$i \frac{\partial A}{\partial z} = \beta_2 \frac{\partial^2 A}{2 \partial T^2} - i \frac{\alpha}{2} A + \gamma |A|^2 A.$$  \hspace{1cm} (2.37)

The electric field $E(r, t)$ can be recovered after solving for $A$ and $F$ by

$$E(r, t) = \frac{1}{2} \{ F(x, y) A(z, t) \exp[i(\beta_0 z - \omega_0 t)] + c.c \} \hat{x}.$$  \hspace{1cm} (2.38)
3.1. Optical Waveguides and Fibers

An optical waveguide is a structure that guides electromagnetic waves in the optical spectrum. Optical waveguides may be classified on the basis of their material (glass, polymer, semiconductor), geometry (planar, strip, or fiber), mode structure (single-mode, multi-mode), and refractive index distribution (step or gradient index) [98]. An optical fiber is a thin, flexible, transparent fiber often made of glass (silica) or plastic. The fibers are most commonly used to transmit light between the two ends as a means of transferring information. They are often wrapped in bundles and can be used for illumination or imaging by carrying light into or out of small confined spaces. A variety of applications have come from the use of this technology which include fiber optic sensors, fiber lasers [61] and fiberscopes to name a few. There are several benefits of using fiber optic cables as opposed to metal wires. Optical fibers can travel over longer distances and at higher bandwidths. Signals in fibers travel with less attenuation (gradual loss of flux intensity through a medium) and don’t experience electromagnetic interference [69].

In the early years, the traditional optical fiber used in most industrial applications utilized a guiding core that is surrounded by a cladding of material with a lower refractive index [3]. The refractive index is a dimensionless number specific to the type of medium and describes how fast light travels through the material. The simplified definition for the refractive index is commonly given as

\[ n = \frac{c}{v}, \]  

(3.1)
where \( c \) is the speed of light in a vacuum and \( v \) is the \textit{phase velocity} of light in the medium. The phase velocity is velocity at which the phase of any one frequency component of the wave travels. The phase velocity is given by

\[
v = \frac{\lambda}{T},
\]

(3.2)

where \( \lambda \) is the wavelength and \( T \) is the time period. As an example, the refractive index of water is approximately 1.33 which implies water travels 1.33 times faster in a vacuum than it travels in water. The traditional guiding core in a fiber revolves around the idea of \textit{total internal reflection} (TIR) where light that encounters an interface of materials with different indices of refraction at a low enough angle will reflect and not enter into the new region due to Snell’s law. This requires the material with the lower refractive index to be on the outside of the fiber. This concept involving TIR was modified by the introduction of a \textit{photonic crystal fiber} (PCF) which has a core but is filled by periodic-like air holes. PCFs will be discussed in more detail in section 3.2. The typical material used to fabricate low-loss fibers is pure silica glass synthesized by fusing \( \text{SiO}_2 \) molecules.

If the refractive index of the core is kept constant, the fiber is called a \textit{step-index fiber} and if the refractive index of the core increases gradually from the core boundary into the center it is called a \textit{graded-index fiber}. Two parameters that characterize an optical fiber are the relative index difference (\( \Delta \)) and the \( V \)-parameter. The relative index difference is given by

\[
\Delta = \frac{n_1 - n_2}{n_1},
\]

(3.3)

where \( n_1 \) is the index for the core and \( n_2 \) is the index for the cladding. Typically \( \Delta \ll 1 \). The \( V \)-parameter is given by

\[
V = \frac{2\pi a}{\lambda} \sqrt{n_1^2 - n_2^2},
\]

(3.4)

where \( a \) is the core radius. The \( V \)-parameter indicates the number of \textit{modes} that are supported by the fiber. A single mode fiber has a value around \( V \approx 2.4 \). The number of modes
(\(M\)) for much higher values of \(V\) can be approximated by \(M \approx V^2/2\). The modes describe how the wave is distributed in space.

### 3.2. Photonic Crystal Fibers

Before introducing photonic crystal fibers, it is useful to first discuss what differentiates photonic crystals and their applications in optics from other crystals used in electrical engineering. A standard crystal is some structure of a periodic arrangement of atoms and molecules. The pattern that is repeated in space is known as the crystal lattice. The various components, materials, and geometry of the lattice describe the conduction properties. As electrons propagate through the crystal, the periodic structure encountered also formulates the potential. Quantum mechanics asserts that electrons travel as waves and waves that meet certain conditions are able to travel through a periodic potential without scattering. On the other hand, the potential may also prevent certain waves from propagating through the crystal. This is an important concept and one that is exploited in wave guiding techniques. This means that electrons of certain energies are not allowed within the crystal which is often referred to as a gap in the energy band of the crystal. These gaps may be designed to a specific direction of propagation. If the potential is strong enough, the gap may include all possible directions, creating a complete band gap. Semiconductors are an example that have a complete band gap between the valence and conduction energy bands.

Now turning to optics, the photonic crystal is the analogue where we consider photons propagating through the crystal instead of electrons. Here, we think of macroscopic media with varying dielectric constants replacing the atoms or molecules and the periodic potential may be referred to as the periodic dielectric function or periodic index of refraction [42]. As a function to guide and control light, photonic crystals are constructed with photonic band gaps, allowing or preventing light in certain directions with certain frequencies. As metallic waveguides are useful tools for guiding electromagnetic waves in the microwave region, photonic crystals can be thought of as a generalization to include a wider range of
frequencies, including that of visible light.

If for some frequency range, the photonic crystal prevents incoming waves from all directions with any polarization and any source, the crystal is said to have a complete photonic band gap at this frequency range. To create a complete photonic band gap, the dielectric lattice typically is periodic in three dimensions although exceptions to this rule have been made. A small amount of disorder within a periodic medium does not remove the band gap [27, 78] and a highly disordered medium may still prevent propagation through what is called Anderson localization [43].

For photonic crystals, there is no fundamental length scale and so the equations are scale invariant. This implies that the solution at one length scale will determine the solution at all other scales. Changes in frequency or mode profile are determined through a simple scaling relationship. For example, if we double the dielectric constant throughout one system, the frequencies of the harmonic modes will be scaled by a factor of $\frac{1}{4}$.

The simplest type of a photonic crystal is a multilayer film which consists of alternating layers of material in one dimension ($z$) with different refractive indices, while homogeneous in the two transverse directions ($x$ and $y$). This type of crystal can act as a mirror (a Bragg mirror) for light within a specific frequency range and may localize light if there are any defects in the structure, commonly used in dielectric mirrors and optical filters [37]. In the multilayer film scenario, the refractive index is periodic over some discrete distance $l$. This allows for a translational symmetry and creates a $z$-dependence within the propagating wave. We now imagine the wave as a product of a plane wave and some periodic function of $z$

$$H(x, y, z) \propto u(z)e^{ik_zz} \quad (3.5)$$

The function $u(z)$ is periodic in the sense that $u(z + nl) = u(z)$. In physics literature, the result is known as Bloch’s theorem and the wave form (3.5) is known as a Bloch state or Bloch wave [50] and in mechanics literature it is known as a Floquet mode [72]. We will refer to it as a Bloch wave. One key result is that the Bloch wave with wave vector $k_z$ is identical
to the wave with wave vector \( k_z + np \) where \( p = 2\pi/l \). Hence, we see the mode frequencies are also periodic in \( k_z \) such that \( \omega(k_z) = \omega(k_z + np) \). This allows us to reduce our focus and consider only \( k_z \) in the range \(-\pi/l < k_z < \pi/l \). This region of nonredundant values of \( k_z \) is called the Brillouin zone. This same argument can be generalized to symmetries in two and three dimensions. For a thorough examination of the details, please see [42].

Photonic crystal fibers can be broadly classified into a few different categories based on whether they employ index guiding or band gaps for the optical confinement, and whether the structure is periodic in one or two dimensions. Figure 3.1 shows examples of the three main categories. PCFs that function to confine light via a band gap are known as Photonic-bandgap fibers and have many advantages. They guide light within a hollow core which minimizes effects from losses and unwanted nonlinearities. Figure 3.1(A) displays a band-gap fiber designed with concentric circles of alternating dielectric constants providing 1D periodicity. These are known as Bragg fibers and additional details may be found in [104]. Figure 3.1(B) shows a 2D periodic band-gap fiber called a holey fiber. It is composed of air filled holes and is the most commonly used [53].

Figure 3.1. (A) 1D periodic cladding with alternating dielectric constants forming concentric layers (Bragg fiber). (B) 2D periodic fiber. The white holes are filled with air (holey fiber). (C) Holey fiber that utilizes index guiding for confinement within a solid core. [42]
Figure 3.1(C) shows a holey fiber that utilizes index-guiding as its confinement mechanism along with a solid core. The periodic design forms an effective low-index cladding around the core. This may provide stronger confinement by allowing for a higher dielectric contrast than can be achieved with entirely solid materials. Another possible configuration is known as a fiber Bragg grating which is constructed with a periodic modulation along the fiber’s axis of propagation (not to be confused with a Bragg fiber). A version of a fiber Bragg grating will be the focus of study in chapter 4 in which we analyze a fiber with a cross section similar to 3.1(C), except without a solid core, that is then helically twisted in the direction of propagation. As in many similar research explorations, one encounters new propagation properties due to topological features, leading to new applications and a better understanding of effects proposed in other fields such as quantum mechanics.
4.1. Introduction

Guiding light by ways of manipulating the transverse index of refraction profile is a classic well studied topic. Its relevance stems from applications to optical communications, laser systems, sensors and many more. In this chapter, we review and provide additional details to the calculations made in [18]. We also extend the analysis to include an argument formulated through ray theory. We theoretically examine light confinement in helical photonic crystals. In particular, we extend the work of [12] to include nonlinear effects. The structure studied is that of a twisted coreless PCF with features similar to that of [12] [Fig. 1], shown to display a variety of interesting phenomena. Of these phenomena, we will closely examine light confinement in helical fibers in the absence of a guiding core.

Helical fibers such as this one or those with a central core have gained interest in part due to their applications in optical communications. The twisted geometry has been shown to preserve the chirality of orbital angular momentum (OAM) modes of the same order [103]. The OAM birefringence in multicore PCFs could then be useful in transmitting information over long ranges through light’s orbital degrees of freedom [11, 101]. Additionally, the spiral twisting provides methods for controlling the loss, dispersion, and polarization [102] and those with off-axis cores can be used to eliminate higher order modes in fiber lasers [95].

In [31], the transmission characteristics of helically twisted PCFs are investigated with off-axis cores through a beam-propagation method formulated for helical waveguides. It is shown that an arbitrary polarization state can be achieved by either adjusting the position
of the core, the twist rate, and the rotation direction. One of the trade offs in adjusting these parameters is that the loss is higher for off-axis core PCFs and that loss increases as the distance from axis increases. This effect is stronger for shorter wavelengths. The analysis of Bloch dynamics in helical wave-guide arrays has been shown to provide a classical wave optics analog of quantum dynamics in an classical electromagnetic field [66].

Helicity in optical waveguides has been analyzed in a variety of situations [66, 5]. In [66], an analysis is performed in one and two dimensions where the index of refraction is higher in the channels than the surrounding medium. The one dimensional case is shown to impose confinement through a gradual twist and is analogous to the orbits of the classical pendulum. The two dimensional case is shown to mimic the effect of a uniform magnetic field, superimposed to a harmonic electrostatic force. The trajectory found in the cross section follows a flower-like path. The purpose of this chapter is to investigate the confinement where the index of refraction is inverted from [66], i.e. the medium around the hollow channels has a higher index of refraction.

We consider a photonic crystal structure with a refractive index profile similar to the one fabricated and used in experiments from [12] whose structure consists of hollow chambers with radii spaced around 5 µm apart and grouped together in the shape of a hexagon. The fiber is then twisted around its axis during the drawing process. In their experiments, light propagated through this fiber for around 20 cm and was shown to have confinement proportional to the twist rate of the fiber. Here, we will verify and show by asymptotic analytical means through two approaches, ray optics and field theory, how this type of confinement can occur due to the geometry of the fiber. The authors analyze the phenomenon by looking at the Bloch wave characteristics in these types of periodic structures. A result of the twisting is that the effective axial refractive index is increased in proportion to the square of the radius \( \rho : \Delta n_{\text{eff}}(r) \approx n_{SM} \alpha^2 \rho^2 / 2 \) where \( \alpha = 2\pi / L \) is the twist rate, \( L \) is the helical pitch, and \( n_{SM} \) is the refractive index of the space-filling mode in an untwisted fiber.
The paths taken by Bloch wave in graded photonic crystals can be calculated using Hamiltonian optics, as in [81]. The governing equations take the form

\[
\frac{dx}{d\sigma} = \nabla_k H(k, x), \quad \frac{dk}{d\sigma} = \nabla_x H(k, x),
\]

where \( H(k, x) \) is the Hamiltonian, \( \sigma = -ct/n_{LO} \), \( n_{LO} \) is the index at the bottom of the passband, \( x = (x, y, z, -ct) \) and \( k = (k_x, k_y, k_z, \omega/c) \) are, respectively, the space-time and wavevector-frequency vectors. The optical frequency is denoted by \( \omega \) and \( c \) is the speed of light in a vacuum. To illustrate how the Hamiltonian formalism can be applied, the authors in [12] make an approximation of the lower edge of the dispersion surface to a parabola, writing \( H \) as

\[
H = -k_z + (1 + \alpha^2 \rho^2/2)\omega n_{LO}/c + A(k_x^2 + k_y^2) = 0,
\]

where \( A \) is a constant with units of length. Assuming \( \alpha^2 \rho^2 << 1 \) and substituting \( H \) into the Hamiltonian equations yields

\[
\frac{\partial x}{\partial \sigma} \approx (2Ak_x, 2Ak_y, -1, n_{LO}),
\]
\[
\frac{\partial k}{\partial \sigma} \approx -\frac{\alpha^2 n_{LO} \omega}{c}(x, y, 0, 0).
\]

This shows the Bloch wave will oscillate harmonically within the potential well formed by the twist

\[
(\ddot{x}, \ddot{y}) = -(x, y)(\alpha^2 \omega c A/n_{LO}) = -(x, y)\Omega^2,
\]

where we use the dot notation, \( \dot{x} \), to represent \( dx/dz \). Equation 4.3 has solutions if \( A > 0 \) as

\[
x(z) = x_0 \cos[(z - z_0)\Omega n_{LO}/c], \quad (4.4)
\]
\[
y(z) = \sqrt{\hat{\rho}^2 - x_0^2} \cos[(z - z_0)\Omega n_{LO}/c + \phi], \quad (4.5)
\]

where \( \hat{\rho} = \alpha^{-1}\sqrt{(n_m/n_{LO})^2 - 1} \) is the critical value of the radius where the Bloch waves become evanescent, causing them to reflect back toward the axis. Hence we see a dependence on \( \alpha^{-1} \) which aligns with the experimental results showing a tighter confinement with a higher
The purpose of this work is to generate new results and study these properties through alternate theoretical approaches by assuming scaling properties suitable for the use of asymptotic techniques.

We suggest the dynamics of the rapid twisting is analogous to that of the rapid vibrating base in the classical Kapitza pendulum. Previous investigations into periodic modulations of the transverse refractive index have been connected to this "Kapitza effect" and can also provide confinement and guiding of light [4], [74]. The Lagrangian for a pendulum with a vibrating base can be computed and then analyzed under the Euler-Lagrange equations as

$$\ddot{\theta} + \left[ \frac{A_0 \omega^2}{l} \cos^2(\omega t) - \frac{g}{l} \right] \sin(\theta) = 0,$$

where $\theta$ is the angle away from vertical position, $l$ is the length and $\omega$ and $A_0$ are the frequency and amplitude of the vibrating base, respectively. We can then imagine splitting $\theta$ into its slow and rapid components due to the swing and vibration. After averaging over the period of rapid movement, the effective potential can be shown as

$$U_{\text{eff}} = -\frac{g}{l} \cos(\theta) + \left[ \frac{A_0 \omega}{2l} \sin(\theta) \right]^2,$$

which allows for a trapping region to appear if the frequency of vibration is higher than some critical value, $\omega \geq \sqrt{\frac{2g}{A_0}}$. We assert the rapid motion of the pendulum’s base is analogous to the rapid twisting of the fiber and thus creating a trapping region for light.

### 4.2. Fiber Model

As a first step in our study, we look at a cross section of the fiber structure in the $x$-$y$ plane. The entire cross section is in the shape of a hexagon and contains a pattern of equally spaced circles as is shown in [12] [Fig. 1]. The black dots represent hollow air chambers and would have a lower refractive index than the grey material surrounding them. For this cross section, we assume the refractive index profile takes the form

$$n(\hat{x}, \hat{y}) = 1 - \delta \cos(k\hat{x}) \cos(k\hat{y}),$$

where $\delta$ is the twist rate.
where $\delta$ is a small positive constant and $k$ is a parameter that defines the spacing between holes such that $\frac{1}{k}$ is on the order of $\mu m$. The origin is taken as the center of the fiber and the refractive index will then oscillate above and below 1 as we move in the $\hat{x}$ or $\hat{y}$ directions. Note the negative sign is chosen so that the origin will have a lower index to match the air channel in the middle of the fiber. This model gives a continuous, smooth form of the refractive index as opposed to the discrete form in the experimental case.

To incorporate the twisting of the fiber in the $z$ direction, we view this as applying the rotation matrix

$$
\begin{bmatrix}
\hat{x} \\
\hat{y}
\end{bmatrix} = \begin{bmatrix}
\cos(\alpha z) & -\sin(\alpha z) \\
\sin(\alpha z) & \cos(\alpha z)
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
$$

(4.9)

where $\alpha$ is a parameter that defines the twist rate. An increase in $\alpha$ will result in a tighter twist. This brings us to the final model for the refractive index of the fiber.

$$
n(x, y, z) = 1 - \delta \cos[kx \cos(\alpha z) - ky \sin(\alpha z)] \cos[kx \sin(\alpha z) + ky \cos(\alpha z)]
$$

(4.10)

### 4.3. Euler-Lagrange Equation

In ray optics, Fermat’s principle states that light travels between two points along the path that requires the least time as compared to nearby paths. It is important to note that light propagates slower through media of higher refractive index. This implies that we may be able to derive the path of light by minimizing the following integral among all possible paths from point $A$ to point $B$

$$
\int_A^B n(x, y, z) ds.
$$

We can minimize this integral over nearby paths by analyzing the Euler-Lagrange equations of the integrand. To bring the problem into a form where we can use the Euler-Lagrange equation, the integrand and the bounds are transformed into a function of $z$. Hence, we will let $x$ and $y$ depend on $z$ as well. There are two parameters in this integral, $k$ and $\alpha$, and the variables have spatial dimensions. First, we will change the integral into a non-dimensional form.
Looking at the dimensions of the variables and parameters: \( x, y, \) and \( \frac{1}{k} \) are on the order of \( \mu m \) while \( z \) and \( \frac{1}{\alpha} \) are on the order of mm. We will introduce the non-dimensional variables

\[
X = kx, \quad Y = ky, \quad Z = \alpha z,
\]

(4.11)

\[
\dot{x} = \frac{\alpha}{k} \dot{X}, \quad \dot{y} = \frac{\alpha}{k} \dot{Y}, \quad dz = \frac{dZ}{\alpha},
\]

(4.12)

where the \( \dot{f} \) notation is used to represent \( \frac{df}{dz} \). By calling the dimensionless parameter \( \epsilon = \alpha/k \), the integral becomes

\[
\int_{Z_A}^{Z_B} \left[ 1 - \delta \cos[X \cos(Z) - Y \sin(Z)] \cos[X \sin(Z) + Y \cos(Z)] \right] \sqrt{1 + \epsilon^2 (\dot{X}^2 + \dot{Y}^2)} \frac{1}{\alpha} dZ.
\]

(4.13)

From here, we use the Euler-Lagrange equations and find an approximate solution for \( X \) and \( Y \). This will give an approximation for the path of light rays that propagate in \( Z \). We expand \( X \) and \( Y \) in terms of \( \epsilon \) and solve for the leading order terms and the first correction terms.

\[
X = X_0 + \epsilon X_1 + ...
\]

(4.14)

\[
Y = Y_0 + \epsilon Y_1 + ...
\]

(4.15)

\[
\Rightarrow \quad \dot{X}^2 = \dot{X}_0^2 + 2\epsilon \dot{X}_0 \dot{X}_1 + ...
\]

(4.16)

\[
\dot{Y}^2 = \dot{Y}_0^2 + 2\epsilon \dot{Y}_0 \dot{Y}_1 + ...
\]

(4.17)

To ease the notation, we also introduce the rotational variables

\[
U = X \cos(Z) - Y \sin(Z), \quad U_0 = X_0 \cos(Z) - Y_0 \sin(Z),
\]

(4.18)

\[
V = X \sin(Z) + Y \cos(Z), \quad V_0 = X_0 \sin(Z) + Y_0 \cos(Z),
\]

(4.19)
and Taylor expand the following

\[
\cos(U) \approx \cos(U_0) - \sin(U_0)(\epsilon U_1),
\]

(4.20)

\[
\cos(V) \approx \cos(V_0) - \sin(V_0)(\epsilon V_1),
\]

(4.21)

\[
\sqrt{1 + \epsilon^2(\dot{X}^2 + \dot{Y}^2)} \approx 1 + \frac{\epsilon^2}{2}(\dot{X}^2 + \dot{Y}^2).
\]

(4.22)

In order to balance the equations, we let \( \delta \) be on the order of \( \epsilon^2 \). We can then rewrite the Lagrangian in ascending orders of \( \epsilon \) as

\[
L = 1 - \delta \cos(U_0) \cos(V_0) + \frac{1}{2} \epsilon^2(\dot{X}_0^2 + \dot{Y}_0^2)
+ \delta\epsilon[V_1 \cos(U_0) + U_1 \sin(U_0) \cos(V_0)] + \epsilon^3(\dot{X}_0 \dot{X}_1 + \dot{Y}_0 \dot{Y}_1) + \mathcal{O}(\epsilon^4).
\]

(4.23)

Similarly we will expand the Euler-Lagrange equations as

\[
\frac{\partial L}{\partial X} - \frac{d}{dZ} \frac{\partial L}{\partial \dot{X}} \approx \frac{\partial L}{\partial X_0} - \frac{d}{dZ} \frac{\partial L}{\partial \dot{X}_0} + \epsilon(\frac{\partial L}{\partial X_1} - \frac{d}{dZ} \frac{\partial L}{\partial \dot{X}_1}).
\]

Solving the equations at \( \mathcal{O}(1) \) and at \( \mathcal{O}(\epsilon) \) are trivial. At \( \mathcal{O}(\epsilon^2) \) we find

\[
\ddot{X}_0 = \sin(U_0) \cos(V_0) \cos(Z) + \cos(U_0) \sin(V_0) \sin(Z),
\]

(4.24)

\[
\ddot{Y}_0 = -\sin(U_0) \cos(V_0) \sin(Z) + \cos(U_0) \sin(V_0) \cos(Z).
\]

(4.25)

At \( \mathcal{O}(\epsilon^3) \) we find the equations for the first correction terms

\[
\ddot{X}_1 = -\sin(U_0) \sin(V_0)(V_1 \cos(Z) + U_1 \sin(Z))
+ \cos(U_0) \cos(V_0)(V_1 \sin(Z) + U_1 \cos(Z)),
\]

(4.26)

\[
\ddot{Y}_1 = -\sin(U_0) \sin(V_0)(U_1 \cos(Z) - V_1 \sin(Z))
- \cos(U_0) \cos(V_0)(U_1 \sin(Z) - V_1 \cos(Z)).
\]

(4.27)

The suggested analogy with the Kapitza pendulum is that these equations show parametric forcing that, for \( \alpha >> 1 \), are fast, and for that instance a follow up averaging can be considered. We will instead perform numerical simulations highlighting how the derived dynamical system predicts confinement.
Figure 4.1. The figure on the left shows the $X$ and $Y$ coordinates for the path of light as a function of $Z$. The figure on the right shows the parametric curve for $X$ and $Y$ evolving as both are functions of $Z$. Both are given a starting position of $(X = 6.2, Y = 5.1)$.

4.4. Numerical Results

Using Matlab’s ODE45 solver to integrate forward in $Z$, we see how this model predicts the path of light. Figure 4.1 shows how the $X$ coordinate and $Y$ coordinate evolve as we move along the $Z$ axis. Each coordinate oscillates slightly, about 1 $\mu$m, around its starting position while also rotating on a larger scale about the $Z$ axis. The smaller oscillations could resemble light moving within a hollow tunnel as the tunnel itself rotates about the axis. The light circles around and remains roughly a constant distance from the center. The behavior displayed is reminiscent of a particle in a Paul trap. An example of this motion is that of a particle in a rotating saddle potential [14, 13]. The motion in a Paul trap has been shown to be a result of a hidden Coriolis-like force in [47]. A single ion in a quadrupole trap may be described by Mathieu equations and further investigations into if these equations are related to (4.24) and (4.25) may provide further evidence for confinement.

Fig. 4.2 demonstrates how adjusting $\alpha$ or the starting position influences the confinement of the rays. The three sub-figures examine several initial positions at three different values of $\alpha$. The circles represent starting positions of rays that remained confined after propagating
Figure 4.2. Circles representing initial starting positions that remained confined after propagating for about 20 cm. From left to right, $\alpha = 0.5$ rad/mm, $\alpha = 0.75$ rad/mm, and $\alpha = 1$ rad/mm, respectively. As $\alpha$ is increased, more rays are confined. Initial momentum of 0.2 in $X$ and $Y$.

for about 20 cm. Rays starting in the area without circles diverged away from the region being examined. We notice a pattern in the regions in which rays were not confined. This pattern repeats when moving away from the origin which is likely due to the approximation in the model. These areas are where the model describes an intermediate value for the refractive index whereas the index in the experimental case assumes two different values (air vs. silica). The rays are given either zero or a small initial momentum. The lack of symmetry in figure 4.2 is due to the initial momentum given for $X_0$ and $Y_0$ in those specific simulations. If the initial momentum is increased in either the $X$ or $Y$ direction, fewer rays are confined. This is a reasonable outcome since if there was already a strong outward momentum in either direction, a larger influence would be necessary to slow the momentum and keep the paths confined.

The figures also show the change in how many rays are confined in this region if the twist rate is increased from 0.5 radians/mm (left) to 0.75 radians/mm (middle) and then to 1 radian/mm (right). The area that loses confinement starts at the region that already diverges and grows from there. This is in agreement with the experimental results showing a stronger confinement for a higher twist rate.
4.5. Extended Field Theory

In this section more details are provided as a basis for section 2, (Field Theory - Analysis) in [18]. We start the analysis from the Helmholtz equation for an electric field $E$.

$$\Delta E + \left(\frac{n\omega}{c}\right)^2 E = 0$$  \hspace{1cm} (4.28)

We assume the field depends on the spatial variables $x$, $y$, and $z$ and that it takes the form $E = F(x, y, z)e^{i\beta z}$. This form for an ansatz is given since we assume there will be oscillations along the direction of propagation in the fiber, the $z$ axis, and oscillations in time. The function $F$ is the envelope function. To change the equation into a dimensionless form we use the same non-dimensional variables, $X = kx$, $Y = ky$, and $Z = \alpha z$ from section 4.3 where $k$ and $\alpha$ are the parameters that define the spacing between channels and the rate of twist, respectively, as given by the fiber’s refractive index in equation (4.10). We then expand equation (4.28) to find that $F$ must satisfy

$$\alpha^2 F_{ZZ} + 2i\beta \alpha F_Z + k^2 \Delta_{XY} F - (\beta^2 - \frac{\omega^2}{c^2} n^2) F = 0.$$  \hspace{1cm} (4.29)

Since we are investigating the effects of a twisted geometry along the $z$ axis we would like to view the problem in a rotating frame of reference. This will allow our view of the cross section of the fiber to remain unchanged as we move along the $z$ axis. Therefore, we change into the rotational variables $U$, $V$, and $Z$ from section 4.3. Note that the Laplacian in $X$ and $Y$ is invariant under this change as shown below.

$$F_X = F_U \cos(Z) + F_V \sin(Z)$$  \hspace{1cm} (4.30)

$$F_Y = -F_U \sin(Z) + F_V \cos(Z)$$  \hspace{1cm} (4.31)

$$F_{XX} = F_{UU} \cos^2(Z) + 2F_{UV} \sin(Z) \cos(Z) + F_{VV} \sin^2(Z)$$  \hspace{1cm} (4.32)

$$F_{YY} = F_{UU} \sin^2(Z) - 2F_{UV} \sin(Z) \cos(Z) + F_{VV} \cos^2(Z)$$  \hspace{1cm} (4.33)

$$\implies \Delta_{XY} = \Delta_{UV}$$  \hspace{1cm} (4.34)
Next, expanding the first partial derivative with respect to \( Z \),

\[
F_Z = F_Z + F_V [-X \sin(Z) - Y \cos(Z)] + F_V [X \cos(Z) - Y \sin(Z)]
\]

\[
= F_Z - VF_U + UF_V.\quad (4.35)
\]

We ignore the term involving the second partial derivative with respect to \( Z \) since it is of smaller order and arrive at

\[
2i \frac{\beta}{\alpha} (F_Z - VF_U + UF_V) + \left( \frac{k}{\alpha} \right)^2 \Delta_{UV} F - \frac{1}{\alpha^2} \left( \beta^2 - \frac{\omega^2}{c^2 \alpha^2} \right) F = 0. \quad (4.36)
\]

Since \( n = 1 - \delta \cos(U) \cos(V) \), we can rewrite (4.37) as

\[
2i \frac{\beta}{\alpha} (F_Z - VF_U + UF_V) + \epsilon^{-2} \Delta_{UV} F - \frac{\omega^2 \delta}{c^2 \alpha^2} [\mu + f(U, V)] F = 0,
\]

where \( \epsilon = \frac{\alpha}{k} \), \( \mu = \frac{\beta^2 e^2 - \omega^2}{\delta \omega} \) and \( f(U, V) = 2 - \delta \cos^2 U \cos^2 V \). We assume \( \frac{\omega^2 \delta}{c^2 \alpha^2} \) is \( \mathcal{O}(\epsilon^{-2}) \), and can therefore analyze the last two terms on the same order.

Continuing with a perturbative analysis, \( F \) can be expanded as \( F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + ... \)

and then we find at \( \mathcal{O}(\epsilon^{-2}) \),

\[
L(F_0) = \Delta_{UV} F_0 - \mu F_0 - 2 \cos(U) \cos(V) F_0 = 0. \quad (4.39)
\]

We take the ansatz one step further and assume \( F_0 \) is a Bloch wave described as

\[
F_0 = A(Z, \epsilon(U^2 + V^2)) F_B(U, V) e^{i\psi(Z, \epsilon(U^2 + V^2))} \quad (4.40)
\]
\[
= A(Z, b) F_B(U, V) e^{i\psi(Z, b)}
\]
where \( b = \epsilon(U^2 + V^2) \), implying a rather small dependence on the transverse directions.

Expanding each term of equation (4.39) with the (4.40) gives

\[
\frac{\partial F_0}{\partial Z} = \left( \frac{\partial A}{\partial Z} F_B + iAF_B \frac{\partial \psi}{\partial Z} \right) e^{i\psi},
\]

\[
\frac{\partial F_0}{\partial U} = \left( 2\epsilon U \frac{\partial A}{\partial b} F_B + 2i\epsilon U AF_B \frac{\partial \psi}{\partial b} + A \frac{\partial F_B}{\partial b} \right) e^{i\psi},
\]

\[
\frac{\partial^2 F_0}{\partial U^2} = \left[ 2\epsilon \left( \frac{\partial A}{\partial b} F_B + U \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + iAF_B \frac{\partial \psi}{\partial b} + iU \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + U \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + iUA \frac{\partial F_B}{\partial U} \right) \right.
\]

\[
+ 4\epsilon^2 \left( U^2 \frac{\partial^2 A}{\partial b^2} F_B + iU^2 \frac{\partial A}{\partial b} F_B \frac{\partial \psi}{\partial b} + iU^2 \frac{\partial A}{\partial b} F_B \frac{\partial \psi}{\partial b} - U^2 AF_B \left( \frac{\partial \psi}{\partial b} \right)^2 + iU^2 AF_B \frac{\partial^2 \psi}{\partial b^2} \right)
\]

\[
+ A \frac{\partial^2 F_B}{\partial U^2} \right] e^{i\psi}.
\]

The first and second partial derivatives with respect to \( V \) are similar since the functions \( A \) and \( \psi \) are radially symmetric. At \( \mathcal{O}(\epsilon^{-1}) \), we find the next order equation given by

\[
L(F_1) = \Delta_{UV} F_1 - \mu F_1 - 2 \cos(U) \cos(V) F_1 =
\]

\[
-2i\bar{\beta}e^{i\psi} \left( \frac{\partial A}{\partial Z} F_B + iAF_B \frac{\partial \psi}{\partial Z} - V \frac{\partial A}{\partial U} \frac{\partial F_B}{\partial U} - U \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} \right)
\]

\[
-2e^{i\psi} \left[ \frac{\partial A}{\partial b} F_B + U \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + iAF_B \frac{\partial \psi}{\partial b} + iU \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + U \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial U} + iUA \frac{\partial F_B}{\partial U} \right.
\]

\[
+ \frac{\partial A}{\partial b} F_B + V \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial V} + iAF_B \frac{\partial \psi}{\partial b} + iV \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial V} + V \frac{\partial A}{\partial b} \frac{\partial F_B}{\partial V} + iVA \frac{\partial F_B}{\partial V} \left. \right] .
\]

For equation (4.41) to be solvable, we verify the solvability condition that the right hand side must be orthogonal to the null space of the \( L \) operator. Since \( F_B \) is in the null space of \( L \), we multiply the right hand side by \( F_B \) and integrate over one period. This integral must be equal to zero. The real and imaginary parts of the resulting equation are given below where \( \bar{\beta} = \beta/k \).

**Imaginary Part:**

\[
\frac{\partial A}{\partial Z} \int_p \bar{\beta} F_B^2 dU dV - A \int_p \bar{\beta} F_B (V \frac{\partial F_B}{\partial U} - U \frac{\partial F_B}{\partial V}) dU dV
\]

\[
+ A \frac{\partial \psi}{\partial b} \int_p 2F_B (F_B + U \frac{\partial F_B}{\partial U} + V \frac{\partial F_B}{\partial V}) dU dV = 0 \quad (4.42)
\]
Real Part:

\[
A \frac{\partial \psi}{\partial Z} \int_p \bar{\beta} F_B^2 dUdV + \frac{\partial A}{\partial b} \int_p 2F_B (F_B + U \frac{\partial F_B}{\partial U} + V \frac{\partial F_B}{\partial V}) dUdV = 0 \quad (4.43)
\]

We have pulled the terms involving \( A \) and \( \psi \) outside of the integrals since we assume they are very slowly dependent on \( U \) and \( V \). If we define the constants

\[
C_1 = \int_p \bar{\beta} F_B^2 dUdV, \quad (4.44)
\]
\[
C_2 = \int_p \bar{\beta} F_B (V \frac{\partial E_B}{\partial U} - U \frac{\partial F_B}{\partial V}) dUdV, \quad (4.45)
\]
\[
C_3 = \int_p 2F_B (F_B + U \frac{\partial F_B}{\partial U} + V \frac{\partial F_B}{\partial V}) dUdV, \quad (4.46)
\]

equations (4.42) and (4.43) become the following system of differential equations

\[
\frac{\partial A}{\partial Z} C_1 - AC_2 + A \frac{\partial \psi}{\partial b} C_3 = 0 \quad (4.47)
\]
\[
A \frac{\partial \psi}{\partial Z} C_1 + \frac{\partial A}{\partial b} C_3 = 0 \quad (4.48)
\]

which is solvable by the solution given in \cite{18} as

\[
A = A_0 e^{-\epsilon (U^2 + V^2)}, \quad \psi = \frac{C_2}{C_3} \epsilon (U^2 + V^2) + \frac{C_3}{C_1} Z + \psi_0. \quad (4.49)
\]

This indicates the allowed Bloch waves have a radially symmetric Gaussian envelope profile as well as a radially symmetric chirp as an effective correction of the stationary phase.

### 4.6. The Nonlinear Case

In this section, more detail is provided on the motivation for using the specific coefficients in the NLSE that is used for section 3 (Fied Theory - Numerical) of \cite{18}. As a secondary analysis, we also explore the effects of decreasing the coefficient in front of the transverse Laplacian numerically. We start by considering equation (4.28) and assume the form \( E = F(x, y, z)e^{i\beta z} \). We apply the same non-dimensional change of variables from section 4.3 for \( x \) and \( y \) however, we make a slight adjustment to the scalar multiple of \( z \) for computational
simplicity and let $\tilde{Z} = \beta z$. Using this change of variables, equation (4.28) is expanded as

$$\beta^2 F_{\tilde{Z}\tilde{Z}} + 2i\beta^2 F_{\tilde{Z}} + k^2 \Delta_{XY} F - (\beta^2 - \frac{\omega^2}{c^2} n^2) F = 0. \tag{4.50}$$

The dimensions and scale for each of the physical constants specific to this problem are given by

$$\omega = \frac{2\pi}{\lambda}, \tag{4.51}$$
$$\lambda \approx 1.5 \times 10^{-6} m, \tag{4.52}$$
$$\beta = \frac{2\pi}{\lambda} \approx 4.19 \times 10^6 m^{-1}, \tag{4.53}$$
$$k = \frac{2\pi}{7.9 \mu m} \approx 0.8 \times 10^6 m^{-1}, \tag{4.54}$$
$$\frac{k}{\beta} \approx \frac{1}{5}. \tag{4.55}$$

By ignoring the $F_{\tilde{Z}\tilde{Z}}$ term, (4.50) can be approximated by

$$i F_{\tilde{Z}} + \frac{1}{50} \Delta_{XY} F - \frac{1}{2} (1 - n^2) F = 0. \tag{4.56}$$

It is important to note that the equation used in [18](section 3) differs from (4.56) by the addition of the term $\gamma |E|^2 E$ used to account for nonlinear effects. The full equation integrated in that paper is given by

$$i E_z = \eta \Delta_{xy} E + n^2 E + \gamma |E|^2 E. \tag{4.57}$$

Since the coefficient in front of the transverse Laplacian in equation (4.56), call this $\eta$, is quite small, an additional numerical simulation is offered here to investigate how a smaller level of diffraction will affect the confinement and role of the twist during propagation. Let us refer to the twist parameter used in [18] as $\alpha'$. The choice of $\alpha' \in [0, 0.8]$ is made to be consistent with the analysis in section 4.3, specifically so that $\delta$ is $O(\epsilon^2)$. We see this after substituting the nondimensional variables into the index of refraction

$$n(X,Y,\tilde{Z}) = 1 - \delta \cos [(X \cos(\frac{\alpha}{\beta} \tilde{Z}) - Y \sin(\frac{\alpha}{\beta} \tilde{Z})] \cos [X \sin(\frac{\alpha}{\beta} \tilde{Z}) + Y \cos(\frac{\alpha}{\beta} \tilde{Z})]. \tag{4.58}$$
The largest value of $\alpha$ used in experiments of [12] is $\pi$ radians/mm, which gives $\alpha' = \frac{\alpha}{\beta} \approx 7.5 \times 10^{-4}$. Since $\frac{\alpha}{\beta}$ is $O(\frac{\alpha}{k} = \epsilon)$, if we were to choose this value of $\alpha'$, $\delta$ would be very small. The difference in index of refraction of air and silica is roughly 0.5, so we choose $\delta = 0.2$ to be aligned with the physical experiment. Since $\delta$ is $O(\epsilon^2)$, we choose the largest value of $\alpha'$ to be 0.8. Further numerical experiments were performed at $\alpha' = 0.9$ and $\alpha' = 1.0$ although the results were not significantly different from that of 0.8. The numerical simulations carried out here utilize an operator splitting method proposed in [49] which will be discussed in detail in section 6.2.

Figure 4.3. Mode profiles for increasing rates of twist after propagating a length of 12 rotations for $\alpha' = 0.8$. $\gamma = 1$, and $\eta = 1/5$. 
Figure 4.4. Full width at half maximum measurement as the fields propagate along the $z$ axis. $\gamma = 1$, and $\eta = 1/5$.

The results given in [18] are from setting $\eta = 1$. Figures 4.3 and 4.4 show the results for $\eta = \frac{1}{5}$. Likewise, the initial field profile was given by $E = \bar{A}\exp[-(x^2 + y^2)/s]$ where our simulations set $\bar{A} = 0.8$. The solution in the case without twist exhibits breather like behavior. The rate of twist in which the mode quickly spreads is shifted from 0.5 when $\eta = 1$ to 0.1 when $\eta = \frac{1}{5}$. Other than when $\alpha' = 0.1$, all of the twisted cases have a smaller peak than the non-twisted yet they have a narrower width at half maximum. The same behavior also occurs for the linear and nonlinear defocusing case with slightly smaller modes, respectively. Figures 4.5 and 4.6 show the results imply, as we investigate the effects nonlinearity, that the twisted structure has a more dominant influence on the field at such intensity levels.
Figure 4.5. Mode profiles for increasing rates of twist after propagating a length of 12 rotations for $\alpha' = 0.8$. The linear case is on the left, $\gamma = 0$, and the defocusing case is on the right, $\gamma = -1$, and $\eta = 1/5$.

Figure 4.6. Full width at half maximum measurement as the fields propagate along the $z$-axis. The linear case is on the left, $\gamma = 0$, and the defocusing case is on the right $\gamma = -1$, and $\eta = 1/5$.

In order to verify this numerical method, we simulate the case where no twist is present and in a homogeneous medium, i.e., $\delta = 0$. Figure 4.7 displays the simulation for various strengths of nonlinearity. Again an initial Gaussian field profile was given by $E = 2.8 \exp[-(x^2 + y^2)/160]$. All of the focusing cases result in a blowup and as the strength of focusing is increased, the blowup occurs sooner. The linear and defocusing cases all diffract and as the strength of defocusing is increased, the diffracting occurs sooner, as expected.
Figure 4.7. Maximum height during propagation along the z-axis. Each line represents a different strength of nonlinearity. Positive nonlinearity represents focusing while negative represents defocusing along the z axis. Each of the focusing cases blowup while the linear and defocusing cases diffract.
Chapter 5
Fractional Nonlinear Schrödinger Equation

5.1. Review of 1D and 2D NLS

We begin by reviewing the nonlinear Schrödinger equation where $u$ is a function of time, $t$, and one spatial dimension, $x$. This equation differs from the linear equation with the addition of the nonlinear term which may provide a balance to the linear dispersion that tends to spread the wave spatially. This case is entirely integrable as shown by Zakharov and Shabat [105], and can be solved by finding a Lax pair and then using the inverse scattering transform. A common and simplified form of the 1D NLS is given as

$$iu_t + u_{xx} + c|u|^2u = 0.$$  \hspace{1cm} (5.1)

The Lax pair makes use of the operators $L$ and $A$ such that (5.1) can be rewritten as $\partial L/\partial t = i[L, A]$. The operators take the form

$$L = i \begin{pmatrix} 1 + p & 0 \\ 0 & 1 - p \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \quad c = \frac{2}{1 - p^2},$$  \hspace{1cm} (5.2)

$$A = -p \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} |u|^2/(1 + p) & iu_x^* \\ -iu_x^* & -|u|^2/(1 - p) \end{pmatrix},$$  \hspace{1cm} (5.3)

where $u^*$ represents the complex conjugate of $u$. The cubic nonlinearity produces a focusing effect from self interaction of the wave with itself. In the case of a positive $c$, the equation is called “focusing” since the self interaction causes a positive gain and allows for bright soliton solutions as well as breather solutions. In the case of a negative $c$, the equation is called
“defocusing” and allows for dark soliton solutions. Zakharov and Shabat provide particular solutions of (5.1) as

\[
u(x, t) = \sqrt{2cn} \frac{\exp[-4i(\xi^2 - \eta^2)t - 2i\xi x + i\psi]}{\operatorname{ch}[2\eta(x - x_0) + 8\eta\xi t]}\]

which is called a soliton. The soliton is a single wave packet with velocity \(v = -4\xi\) and propagates with no distortion. It is characterized by 4 constants: \(\eta, \xi, x_0,\) and \(\psi\) of which \(\eta\) and \(\xi\) are independent and can be chosen arbitrarily. The soliton solution is a simple representation of a family of exact solutions which have an explicit formulation.

We now view the nonlinear Schrödinger equation where \(u\) is a function of \(z\), the direction of propagation, and two transverse spatial dimensions, \(x\) and \(y\). The 2D NLS is given as

\[
iu_z + \Delta_\perp u + cu^2u = 0, \quad u(0, x, y) = u_0(x, y)
\]

where \(\Delta_\perp\) represents the Laplacian in the transverse directions of propagation \((\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\) and \(c\) is commonly \(\pm 1\). This can be used to model propagation of a laser beam through a medium with Kerr nonlinearity. The Kerr effect applies a change in the medium's refractive index in response to an applied electric field. In optics, this applied electric field is due to the light itself. This causes a variation of refractive index which is proportional to the local irradiance of light [93]. The balance between the diffraction and the nonlinearity is sensitive and has been shown to lead to singularities, which was first shown in 1965 by P. Kelley [45]. Kelley showed that for optical beams whose power is above some critical value, the self focusing effect overtakes the spreading effect from diffraction. The wave dynamics that occur in this situation are known as a wave collapse (or blowup). The process during a wave collapse is an energy localization into small scales in finite-time. Additional physical effects such as nonlinearity saturation or damping eventually become more influential and stop the collapse [76, 15]. Wave collapse has been suggested as a mechanism of the optical pulse compression [89].
Two important invariants of (5.5) are the $L^2$ norm (power)

$$N = N(0) = \frac{1}{2\pi} \int |u|^2 dx dy,$$

and the Hamiltonian

$$H = H(0) = \frac{1}{2\pi} \left( \int |\nabla u|^2 dx dy - \frac{c}{2} \int |u|^4 dx dy \right).$$

A solution is known to exist at $z$ if its $H_1$ norm is finite, i.e.

$$||u(z, \cdot)||_{H^1} < \infty, \quad ||u||_{H^1} = \left( \int |u|^2 dx dy + \int |\nabla u|^2 dx dy \right)^{1/2}.$$

The solution will blowup at $z = z_c$ if the norm exists for $0 \leq z < z_c$ and $\lim_{z \to z_c} ||u||_{H^1} = \infty$. From the theory for local existence of solutions for equation (5.5), if $||u(z, \cdot)||_{H^1}$ is bounded, the solution exists for all $z$ [33, 44]. It is also known that in the defocusing case ($c < 0$), the $H^1$ norm is bounded and the solution exists globally, hence for the remainder of this section we will restrict our attention to the case of the focusing NLS with $c = 1$. Equation (5.5) has radially symmetric waveguide solutions

$$u(z, r) = R(r)e^{iz},$$

where $r = \sqrt{x^2 + y^2}$ and $R$ satisfies the nonlinear boundary value problem

$$\Delta_{\perp} R - R + R^3 = 0, \quad R'(0) = 0,$$

$$\lim_{r \to \infty} R(r) = 0, \quad \Delta_{\perp} = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right).$$

Equations (5.10) and (5.11) have several possible solutions [76]. Let us focus on the solution with the lowest power, called the *Townes soliton*. The Townes soliton is positive, monotonically decreasing, has the exact power needed for blowup [99] and its Hamiltonian is equal to zero. Thus, the borderline case for blowup solution (5.9) is unstable.

Interestingly, Gaussians may look similar to the Townes soliton however there is no Gaussian that satisfies the properties of having the exact power needed for blowup and a
Hamiltonian of zero. Therefore it is unreliable to use Gaussians to capture the balance between diffraction and the focusing nonlinearity. This created a challenge in study of the blowup rate near the singularity of which a significant amount of research was dedicated. The structure and dynamics of the wave function near blow up was solved by Fraiman [30] and, independently though in a different manner, by Landman et al. [56] and Lemesurier et al. [62], who showed that the beam follows the loglog law as it approaches the singularity. This was a breakthrough however, it is important to note that the loglog behavior is quite difficult to observe numerically. Additionally, the model loses validity much earlier as the field intensity reaches the threshold for material breakdown.

5.2. Motivation to Generalize into Fractional

Over the past few decades, several studies have been conducted into the possible extensions of quantum mechanics to include fractional operators. Space-fractional quantum mechanics (SQFM), coined by Laskin [59, 58], allows for a generalization of quantum mechanics that emerges when the Brownian trajectories in Feynman path integrals are replaced by Lévy flights. A Lévy flight is also a random walk however the step lengths have a probability distribution that is heavy tailed. Lévy flights are widely used to model a variety of processes such as turbulence [51], chaotic dynamics [106], plasma physics [110], financial dynamics [71], anomalous diffusion [70], biology and physiology [100].

Lévy and Khintchine proved it is possible to generalize the central limit theorem and discovered a class of non-Gaussian Lévy \( \alpha \)-stable probability distributions [46, 63]. In this generalization the parameter \( \alpha \) typically ranges between 0 and 2 (or in some texts 0 to 1) and is often called a stability index or Lévy index. Brownian motion is then a special case of Lévy motion when \( \alpha = 2 \) [28]. Following from the generalization, since the path integral over Brownian trajectories leads to the Schrödinger equation, the path integral over the generalized Lévy trajectories leads to the fractional Schrödinger equation [59, 58]. The key difference being the spatial derivative of order \( \alpha \) in place of the second order (\( \alpha = 2 \)) spatial
Laskin has also proven the hermiticity of the fractional Hamiltonian operator which is given by $H_{\alpha} = D_{\alpha}(-\hbar^2 \Delta)^{\alpha/2} + V(r, t)$. As physical applications of the time-independent fractional Schrödinger equation, Laskin has found the energy spectrum and equation for the radius of the orbits of a hydrogren-like atom (or fractional ”Bohr atom”) as well as the energy spectrum of the 1D fractional operator in the semiclassical approximation [60].

5.3. Path Integral over Lévy Paths

In this section we review the path integral formulation over Lévy paths and then move into how it leads to the fractional Schrödinger equation. The formulation in this section is a summarization from Longhi in [67]. Let the path of a particle be given by $r(t)$ and suppose the particle starts at point $a$ at time $t_a$ and moves to point $b$ at time $t_b$ such that $r(t_a) = r_a$ and $r(t_b) = r_b$. In quantum mechanics, an amplitude, often called a kernel, is assigned as we move from point $a$ to point $b$. This is the weighted sum over all trajectories that go between each end point. If the particle is moving in a potential $V(r)$ the the fractional quantum-mechanical amplitude $K_L(r_b | r_a t_b t_a)$ is given as [58]

$$K_L(r_b | r_a t_b t_a) =$$

$$\int_{r_a}^{r_b} Dr(\tau) \int Dp(\tau) \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\tau [p(\tau) \dot{r}(\tau) - H_{\alpha}(p(\tau), r(\tau))] \right\},$$

where $\hbar$ is Planck’s constant, $\dot{r}$ denotes the time derivative, $H_{\alpha}(p(\tau), r(\tau))$ is the fractional Hamiltonian given by

$$H_{\alpha}(p, r) = D_{\alpha}|p|^\alpha + V(r), \quad 1 < \alpha \leq 2,$$
where \( \{ \mathbf{p}(\tau), \mathbf{r}(\tau) \} \) is the particle trajectory in phase space. The quantity \( D_\alpha \) has dimensions \( \text{erg}^{1-\alpha}\text{cm}^\alpha \text{sec}^{-\alpha} \). The phase space path integral in (5.12) is defined by

\[
\int_{\mathbf{r}_a}^{\mathbf{r}_b} D\mathbf{r}(\tau) \prod \int D\mathbf{p}(\tau) \cdots = \lim_{N \to \infty} \int_{-\infty}^{\infty} d\mathbf{r}_1 \cdots d\mathbf{r}_{N-1} \frac{1}{(2\pi \hbar)^{3N}} \int_{-\infty}^{\infty} d\mathbf{p}_1 \cdots d\mathbf{p}_N \times \exp\left\{ i \frac{\mathbf{p}_1(\mathbf{r}_1 - \mathbf{r}_a) - D_\alpha \zeta |\mathbf{p}_1|^\alpha}{\hbar} \right\} \times \cdots \times \exp\left\{ i \frac{\mathbf{p}_N(\mathbf{r}_b - \mathbf{r}_{N-1}) - D_\alpha \zeta |\mathbf{p}_N|^\alpha}{\hbar} \right\},
\]

where \( \zeta = (t_b - t_a)/N \). If we let \( \alpha = 2 \) and \( D_\alpha = 1/(2m) \), \( m \) being the mass of the particle, equation (5.13) becomes the familiar Hamiltonian with kinetic energy \( p^2/(2m) \) and (5.12) becomes the Feynman path integral in the phase space representation.

5.4. Fractional Schrödinger Equation

In this section we use the kernel given in section 5.3 to derive the fractional Schrödinger equation. In this process we will make use of the Fourier and inverse Fourier transforms given as

\[
\psi(\mathbf{r}, t) = \frac{1}{(2\pi \hbar)^3} \int d\mathbf{p} \psi(\mathbf{p}, t) \exp\left( \frac{i\mathbf{p}\mathbf{r}}{\hbar} \right), \quad \psi(\mathbf{p}, t) = \int d\mathbf{r} \psi(\mathbf{r}, t) \exp\left( -\frac{i\mathbf{p}\mathbf{r}}{\hbar} \right). \quad (5.15)
\]

The 3D quantum Riesz fractional derivative will be used and defined by [77]

\[
(-\hbar^2 \Delta)^{\alpha/2} \psi(\mathbf{r}, t) = \frac{1}{(2\pi \hbar)^3} \int d\mathbf{p} |\mathbf{p}|^\alpha \psi(\mathbf{p}, t) \exp\left( \frac{i\mathbf{p}\mathbf{r}}{\hbar} \right), \quad (5.16)
\]

The kernel \( K_L \) defined in equation (5.12) describes the advancement of the quantum mechanical system from point \( \mathbf{r}_a \) to point \( \mathbf{r}_b \) starting from time \( t_a \) and ending at time \( t_b \) by

\[
\psi(\mathbf{r}_b, t_b) = \int d\mathbf{r}_a K_L(\mathbf{r}_b, t_b | \mathbf{r}_a, t_a) \cdot \psi(\mathbf{r}_a, t_a). \quad (5.17)
\]

We look at a small perturbation away from some time \( t \) and apply equation (5.17). Let \( \epsilon \) be a small positive constant such that \( \epsilon << 1 \).

\[
\psi(\mathbf{r}, t + \epsilon) = \int d\mathbf{r}' K_L(\mathbf{r}, t + \epsilon | \mathbf{r}', t) \cdot \psi(\mathbf{r}', t) \quad (5.18)
\]
By using (5.12) and Feynman’s approximation such that $\int_{t}^{t+\epsilon} d\tau V(\mathbf{r}(\tau), \tau) \approx \epsilon V(\frac{\mathbf{r}+\mathbf{r}'}{2}, t)$ we find the right hand side of (5.18) becomes

$$
\int d\mathbf{r'} \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} \exp\{\frac{i}{\hbar}[\mathbf{p}(\mathbf{r}' - \mathbf{r}) - D_\alpha \epsilon|\mathbf{p}|^\alpha - \epsilon V(\frac{\mathbf{r}+\mathbf{r}'}{2}, t)]\} \cdot \psi(\mathbf{r}', t). \tag{5.19}
$$

We may then expand both sides of (5.18) in power series as

$$
\psi(\mathbf{r}, t) + \epsilon \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = \int d\mathbf{r'} \frac{1}{(2\pi\hbar)^3} \int d\mathbf{p} \exp\{\frac{i}{\hbar}[\mathbf{p}(\mathbf{r}' - \mathbf{r})] \cdot (1 - \frac{i}{\hbar} D_\alpha \epsilon|\mathbf{p}|^\alpha) \times [1 - \frac{i}{\hbar} \epsilon V(\frac{\mathbf{r}+\mathbf{r}'}{2}, t)] \cdot \psi(\mathbf{r}', t). \tag{5.20}
$$

At this point we will utilize the Fourier transforms and the definition for the Riesz fractional derivative to rewrite (5.20) as

$$
\psi + \epsilon \frac{\partial \psi}{\partial t} = \psi - \frac{i}{\hbar} \epsilon D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi - \frac{i}{\hbar} \epsilon V \psi, \tag{5.21}
$$

where $\psi = \psi(\mathbf{r}, t)$ and $V = V(\mathbf{r}, t)$. We now see that equation (5.21) is true to order $\epsilon$ if $\psi$ satisfies

$$
i\hbar \frac{\partial \psi}{\partial t} = D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi + V \psi. \tag{5.22}
$$

Here the spatial Laplacian is of fractional order $\alpha$ and this is the form we will refer to as the fractional Schrödinger equation.

### 5.5. The Limiting Case of the Discrete (Nonlocal) NLS

The authors of [48] show how the nonlinear Schrödinger equation with the fractional Schrödinger Laplacian can be rigorously derived from the continuum limit of discrete nonlinear Schrödinger equations on lattices $h\mathbb{Z}$ as $h \to 0$. They state that fractional powers of $1 < \alpha < 2$ arise when long-range interactions are in effect and that the classical Laplacian ($\alpha = 2$) is found when only short-range interactions are used in the model. This is of particular usefulness in the field of optics as nonlinear coupled arrays of optical resonators are often a topic of interest. These coupled resonators may have various degrees of interaction with their neighbors. This section provides a brief outline of the main theorem of [48] in which the continuum limit
gives the FNLS. We start by introducing the one-dimensional discrete wave function on the lattice \( h\mathbb{Z} \) satisfying the initial value problem

\[
\frac{i}{\beta} \frac{du(t, x_m)}{dt} = \sum_{n \neq m} J_{|n-m|} [u(t, x_m) - u(t, x_n)] \pm |u(t, x_m)|^2 u(t, x_m),
\]

\[ u(0, x_m) = v(x_m), \]

where \( x_m = mh, m \in \mathbb{Z} \), and \( h > 0 \). The initial data is given in \( v(x) \) and \( \beta = \beta(h) > 0 \) denotes the scaling factor that is dependent on the lattice spacing \( h \). The cubic nonlinearity may arise in physical models as quantum particles on a lattice with a three wave interaction. In this scenario, the + sign is used for repulsive self-interaction while the – sign is used for attractive self-interaction. If used to model DNA, the nonlinearity may represent self-interaction for a base pair of the strand and the summation term can represent interaction with other base pairs decaying by inverse power of the distance along the strand [73].

The notation for the kernel, \( J_n = J(n)_{n=1}^{\infty} \), is used to denote a sequence of positive numbers that we can classify as either an \( s \)-kernel \( (0 < s < \infty) \) or an \( \infty \)-kernel. If \( \lim_{n \to \infty} n^{1+2s} J_n = A \) for \( 0 < s < \infty \) and \( 0 < A < \infty \) the sequence is called an \( s \)-kernel. If \( A = 0 \) the sequence is called an \( \infty \)-kernel.

If \( J_n \) is an \( s \)-kernel, (5.23) may be shown as globally well-posed in the space \( L^2_h \) defined as \[ L^2_h = \{v_h \in \mathbb{C}^{h\mathbb{Z}}: (v_h, v_h)_L^2 := h \sum_{m \in \mathbb{Z}} |v_h(x_m)|^2 < \infty \}. \] Similarly conservation of energy and conservation of the discrete \( L^2 \) mass can be shown.

Given a locally integrable function \( f : \mathbb{R} \to \mathbb{C} \), we will define its discretization \( f_h : h\mathbb{Z} \to \mathbb{C} \) by

\[
f_h(x_m) = \frac{1}{h} \int_{x_m}^{x_{m+1}} f(x) dx,
\]

where \( x_m = mh, \) and \( m \in \mathbb{Z} \). The piecewise linear interpolation of a given lattice function \( f_h \) is denoted by \( p_h : \mathbb{R} \to \mathbb{C} \),

\[
p_h f_h(x) = f_h(x_m) + (D^+_h f_h)(x_m)(x - x_m), \quad x \in [x_m, x_{m+1}],
\]
where $D_h^+$ represents the discrete right-hand derivative on $h\mathbb{Z}$ defined as
\begin{equation}
(D_h^+ f_h)(x_m) = \frac{f(x_{m+1}) - f(x_m)}{h}.
\end{equation}
(5.27)
Lastly, we will introduce the NLS-type initial value problem in the continuous form with $u : [0, T) \times \mathbb{R} \to \mathbb{C}$ as
\begin{equation}
i \partial_t u = c(-\Delta)^{\alpha/2} u \pm |u|^2 u,
\end{equation}
(5.28)
\begin{equation*}
u(0, x) = v(x).
\end{equation*}
where $c > 0$ is some fixed constant and $(-\Delta)^{\alpha/2}$ is defined by the pseudo-differential in Fourier space with $\alpha \in (0, 2]$. This is introduced here since the wave function that solves (5.28) will be the limiting case (in a weak sense) of the discrete version as $h \to 0$. The key result of [48] (Theorem 2.1) is now given in theorem 5.1.

**Theorem 5.1 (Continuum Limit)** Let $J = (J_n)_{n=1}^\infty$ be an $s$-kernel for some $s > \frac{1}{2}$, where we assume $J_1 > 0$. Also let
\begin{equation*}
\alpha = \begin{cases} 
2s, & \frac{1}{2} < s < 1, \\
2, & 1 \leq s.
\end{cases}
\end{equation*}
Now suppose $v \in H^{\alpha/2}(\mathbb{R})$ and consider its discretization $v_h : h\mathbb{Z} \to \mathbb{C}$ defined in (5.25). Let $u_h = u_h(t, x_m)$ denote the corresponding global solution to (5.23) with initial datum $v_h \in L^2_h$ and choose
\begin{equation}
\beta(h) = \begin{cases} 
h^{2s}, & \frac{1}{2} < s < 1, \\
-h^2 \log(h), & s = 1, \\
h^2, & 1 < s.
\end{cases}
\end{equation}
(5.29)
Then, for every $0 < T < \infty$, we have the weak convergence [48]
\begin{equation}
p_h u_h \rightharpoonup u \text{ in } L^\infty([0, T]; H^\alpha(\mathbb{R})) \text{ as } h \to 0^+,
\end{equation}
(5.30)
where $u \in C^0([0, \infty) ; H^{\alpha/2}(\mathbb{R})$ is the unique global solution of the initial value problem (5.28) with $\alpha > 1$ defined above and some constant $c > 0$ that only depends on $J$. 

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For a detailed proof of Theorem 5.1, we refer to [48].

5.6. An Optical Realization of the FNLS

In this section, we review a particularly interesting publication by Longhi in 2015 [67]. Longhi displays an interesting application of space fractional quantum mechanics (SFQM) where transverse light dynamics in aspherical optical resonators are considered. The study involves an effective one-dimensional optical resonator with transverse spatial direction $x$. An example of where such assumptions hold is that of an optical cavity with astigmatic optical elements in which separation of variables in the two transverse spatial dimensions, $x$ and $y$, can be applied and the lowest-order transverse mode in the $y$ direction is excited. A schematic and description of the one-dimensional Fabry-Perot optical resonator is given in [67]. Two flat end mirrors and two converging lenses of focal length $f$ are set in a $4f$ configuration. Two thin phase masks with transmission functions $t_1(x) = \exp[-if(x)/2]$ and $t_2(x) = \exp[-iV(x)/2]$ where $f(x) = \beta|x|^\alpha$ are also included. In view of the scalar and paraxial approximations, propagation during successive transits inside the cavity can be obtained from the generalized Huygens integral [87]. Let $F^{(n)}(x) = i^n \psi^{(n)}(x)$ denote the field envelope at the $n$-th round trip at a given reference plane. If gain/loss elements are not present and finite aperture effects are neglected we can write

$$\psi^{(n+1)}(x) = \int K(x, \theta) \psi^{(n)}(\theta) d\theta, \tag{5.31}$$

where the kernel of the integral transformation is given by

$$K(x, \theta) = \frac{1}{\lambda f} \int \exp \left[ -i f(\xi) - iV(x) + \frac{2\pi i \xi (x - \theta)}{\lambda f} \right] d\xi, \tag{5.32}$$

where $\lambda = c/\nu$ is the optical wavelength ($c$ being the speed of light in a vacuum and $\nu$ the frequency). The Fredholm equation

$$\sigma_n \psi_n(x) = \int K(x, \theta) \psi_n(\theta) d\theta \tag{5.33}$$
can be used to find the transverse modes TEM$_n$ and corresponding resonance frequencies of the resonator [87]. We consider the limiting case where $V(x), f(x) \to 0$ so that the field is nearly self-imaging and slowly changes each successive trip. By expanding the exponential in the kernel as $e^{-if(\xi) - iV(x)} \approx 1 - if(\xi) - iV(x)$ and using the definition for the one-dimensional Riesz fractional Laplacian given in this section by

$$
\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} \psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |p|^{\alpha} \psi(\xi) \exp[ip(x - \xi)] dp d\xi
$$

(5.34)

we can obtain the approximation for $\psi^{(n+1)}$ as

$$
\psi^{(n+1)}(x) \approx \left[ 1 - iV(x)\psi^{(n)}(x) - iD_\alpha \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}\right] \psi^{(n)}(x)
$$

(5.35)

where $D_\alpha = \beta(\lambda f/2\pi)^{\alpha}$. Next introducing the time variable $t$, normalized by the round-trip time $T_R$ of photons in the cavity, and setting $\psi(x, t) = \psi^{(n=0)}(x), \frac{\partial \psi}{\partial t} \approx \psi^{(n+1)}(x) - \psi^{(n)}(x)$, we can see the time evolution of the transverse field satisfies the FSE [67]

$$
i \frac{\partial \psi}{\partial t} = \left[D_\alpha \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} + V(x)\right] \psi(x).
$$

(5.36)

### 5.7. Fractional Quantum Harmonic Oscillator Example

This section provides an example of the space fractional Schrödinger equation as an optical realization of the fractional quantum harmonic oscillator. This can be found using the tools constructed in the previous section and by replacing the phase mask and flat mirror by a spherical mirror with radius of curvature $R$ (see [67] [Fig.1b]) such that

$$
V(x) = \frac{2\pi x^2}{\lambda R}
$$

(5.37)

The transverse modes and corresponding frequencies of the resonator can be found from the FSE eigenvalue equation in momentum space. Note that the Fourier transform of the solution to (5.36) at the Fourier plane located right before the phase mask may be given as

$$
\hat{\psi}(\xi, t) = \sqrt{\frac{i}{\lambda f}} \int_{-\infty}^{\infty} \psi(x, t) \exp \left(\frac{2\pi i x \xi}{\lambda f}\right) dx,
$$

(5.38)
which itself satisfies the FSE in momentum space [38]

\[
\frac{i}{\hbar} \hat{\psi}_t = \beta |\xi|^\alpha \hat{\psi} + \frac{1}{\lambda f} \int \int_{-\infty}^{\infty} V(x) \hat{\psi}(\mu, t) \exp \left[ \frac{2\pi ix(\xi - \mu)}{\lambda f} \right] dx d\mu.
\] (5.39)

If we assume the function is separable as \( \hat{\psi}(x, t) = \psi(x)e^{-iEt} \) and plug this into equation (5.39) while using the potential given in (5.37) we arrive at an eigenvalue equation

\[
E \hat{\psi}(x) = \beta |x|^\alpha \hat{\psi}(x) - \left( \frac{\lambda f^2}{2\pi R} \right) \frac{d^2 \hat{\psi}}{dx^2}.
\] (5.40)

Longhi highlights a special case for \( \alpha = 1 \) which simplifies (5.40) to the Airy equation. The analytical solution that arises then is seen physically by using a one-dimensional refractive axicon in the Fourier plane before the phase mask. This is the case of the massless relativistic harmonic oscillator. Additional studies of this topic can be found in [54, 68]. When \( \alpha = 1 \) the eigenfunctions and corresponding energies are given by

\[
\hat{\psi}_n(x) = \left( \frac{x}{|x|} \right)^n \text{Ai}(\kappa|x| + r_n),
\] (5.41)

\[
E_n = -\frac{\lambda f^2 \kappa^2 r_n}{2\pi R},
\] (5.42)

where \( \text{Ai}(x) \) is the Airy function and \( \kappa = \left[ \frac{2\pi \beta R}{\lambda f^2} \right]^{1/3} \). When \( n \) is even, \( r_n \) denote the roots of \( \frac{d\text{Ai}}{dx} \), and when \( n \) is odd \( n \) denotes the roots of \( \text{Ai}(x) \). The dual Airy beams may be described by (5.41) for sufficiently large \( n \) [67]. These are characterized by two displaced Airy beams that accelerate in opposite directions [40]. Longhi’s example highlights the field of optics as an interesting arena where fractional operators can be experimentally observed. Inspired by the work demonstrated here, in the next chapter we propose a Schrödinger type equation that involves a one-dimensional fractional operator in space and the classical temporal dispersion operator. We suggest this type of model can be useful for short pulse lasers which have a relatively larger dependence on time.
6.1. Linear Stability Analysis

Here we perform a linear stability analysis of the continuous wave (CW) solutions for the nonlinear Schrödinger type equation

\[ iu_z + Lu + \beta u_{tt} + \gamma |u|^2 u = 0, \]  

(6.1)

where \( L \) is the one-dimensional fractional spatial Laplacian operator in \( x \) represented as \((-\frac{\partial^2}{\partial x^2})^{\alpha/2}\) with \( \alpha \in [0, 2] \). An exact CW solution of (6.1) is \( u = Ae^{i\gamma A^2 z} \). We slightly perturb the CW solution to \( u = (A + a)e^{i\gamma A^2 z} \) with \( a = a(x, t, z) \). Plugging the perturbed ansatz into (6.1) and expanding we find

\[ ia_z + La + \beta a_{tt} + \gamma A^2 a + \gamma A^2 a^* = 0, \]

where higher orders of \( a \) have been ignored. We assume \( a(x, t, z) \) is complex and therefore we can split the function into its real and imaginary components as \( a = f + ig \). This leads to a system of equations given by

\[ f_z + Lg + \beta g_{tt} = 0, \]  

(6.2)

\[ g_z - Lf - \beta f_{tt} - 2\gamma A^2 f = 0. \]  

(6.3)

To handle the \( L \) operator, we view the Fourier transform of equations (6.2) and (6.3) while noting that \( \mathcal{F}\{(\frac{\partial^2}{\partial x^2})^{\alpha/2}a(x, t, z)\}(\xi) = |\xi|^\alpha \hat{a}(\xi, t, z) \). This leads to

\[ \hat{f}_z + |\xi|^\alpha \hat{g} + \beta \hat{g}_{tt} = 0, \]  

(6.4)

\[ \hat{g}_z - |\xi|^\alpha \hat{f} - \beta \hat{f}_{tt} - 2\gamma A^2 \hat{f} = 0. \]  

(6.5)
The result is a standard system of coupled PDEs. Now, we attempt an ansatz through separation of variables such as \( \hat{f} = \hat{p}_1(\xi)e^{i(\lambda z + \mu t)} \) and \( \hat{g} = \hat{p}_2(\xi)e^{i(\lambda z + \mu t)} \). This ansatz leads to

\[
\begin{bmatrix}
  i\lambda & |\xi|^\alpha - \beta \mu^2 \\
  -|\xi|^\alpha + \beta \mu^2 - 2\gamma A^2 & i\lambda
\end{bmatrix}
\begin{bmatrix}
  \hat{p}_1 \\
  \hat{p}_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (6.6)

which will have nontrivial solutions if

\[
\lambda^2 = |\xi|^{2\alpha} - 2|\xi|^\alpha(\beta \mu^2 - \gamma A^2) + \beta^2 \mu^4 - 2\gamma A^2 \beta \mu^2,
\]

\[
\lambda^2 = (|\xi|^\alpha - (\beta \mu^2 - \gamma A^2))^2 - \gamma^2 A^4.
\] (6.7)

For the solutions to be stable, \( \lambda \) must be real and so we search for values of the parameters which make this possible. An instability arises when \( \lambda^2 < 0 \) or when

\[
-(\gamma + |\gamma|)A^2 < |\xi|^\alpha - \beta \mu^2 < (|\gamma| - \gamma)A^2.
\] (6.8)

Depending on the signs of \( \beta \) and \( \gamma \) the dynamics will vary and we can thus split the analysis into four different cases.

**Case 1:** \( \beta < 0, \gamma > 0 \) (defocusing) \( \lambda \) is always real

**Case 2:** \( \beta > 0, \gamma > 0 \) \( -2\gamma A^2 < |\xi|^\alpha - \beta \mu^2 < 0 \)

**Case 3:** \( \beta > 0, \gamma < 0 \) \( 0 < |\xi|^\alpha - \beta \mu^2 < 2|\gamma|A^2 \)

**Case 4:** \( \beta < 0, \gamma < 0 \) (focusing) \( 0 < |\xi|^\alpha - \beta \mu^2 < 2|\gamma|A^2 \)
Figure 6.1. Instability region (blue) for case 2, $\beta > 0, \gamma > 0$; $\alpha = 2$ (left), $\alpha = 1$ (right).

Figure 6.2. Instability region (blue) for case 3, $\beta > 0, \gamma < 0$; $\alpha = 2$ (left), $\alpha = 1$ (right).
6.2. Numerical Propagation Method

In this section, the numerical method that is implemented in order to propagate a field that evolves according to our Schrödinger-like equation (6.18) is presented. To continue investigating the stability analysis, we would like to choose various parameters from the stable or unstable regions given above and understand the short and long term behavior. The method adapted to our modeling equation was developed by Kirkpatrick and Zhang in [49] for integrating the one-dimensional FNLS equation (6.13). They propose an explicit method that has high order spatial accuracy and little computational cost. The method has spectral-order accuracy in space and second-order accuracy in time. To begin, the
right hand side of (6.13) is split into two parts, one containing the fractional Laplacian and one containing the potential and nonlinear components. Each are numerically solved independently and then coupled using the second-order Strang method [88]. The Fourier pseudospectral method is used to discretize the Riesz fractional Laplacian.

\[
i \partial_t \psi(x, t) = c(-\Delta)^{\alpha/2} \psi + V(x) \psi + \gamma |\psi|^2 \psi
\]  

(6.13)

The outline of the numerical method to advance a state from \(t = t_n\) to \(t = t_{n+1}\) is given below.

\[
\psi_j^{(1)} = \psi_n \exp \left\{ - \frac{k}{2} [V(x_j) + \gamma |\psi_n|^2] \right\},
\]  

(6.14)

\[
\psi_j^{(2)} = \sum_{l=-J/2}^{J/2-1} \hat{\psi}_l^{(1)} \exp(-i k |\xi_l| x_j - \alpha) e^{i \xi_l (x_j - a)},
\]  

(6.15)

\[
\psi_{n+1}^{(2)} = \psi_j^{(2)} \exp \left\{ - \frac{k}{2} [V(x_j) + \gamma |\psi_j^{(2)}|^2] \right\},
\]  

(6.16)

where \(\psi_n\) denotes the numerical approximation of \(\psi(x_j, t_n)\), \(k\) is the time step size, \(\hat{\psi}\) is the Fourier transform of \(\psi\), \(\alpha \in [0, 2]\) and \(\xi_l = \frac{2l}{b-a}\) for \(-J/2 \leq l \leq J/2 - 1\).

An alternative method, although not used for the results presented in this work, is known as the split-step Galerkin method [97]. This method also uses the second-order Strang split-step method for handling the potential and nonlinear terms but uses the Legendre spectral Galerkin method for approximating the Riesz space-fractional derivative. The outline for this method is given below. The steps given are for a Schrödinger equation with a fractional Laplacian in both the \(x\) and \(y\) spatial dimensions with independent degrees of fractionality, \(\alpha_1\) and \(\alpha_2\) respectively. The first and third steps are the same as in (6.14) and (6.16) however the middle step is replaced with

\[
i (\delta_t \psi_j^{(2)}, \nu_j) - \frac{\beta \tau}{2} \mathcal{A}(\psi_j^{(2)}, \nu_j) = i (\delta_t \psi_j^{(1)}, \nu_j) + \frac{\beta \tau}{2} \mathcal{A}(\psi_j^{(1)}, \nu_j)
\]  

(6.17)

for all \(\nu_j \in V_j^0(\Omega)\) where \(V_N^0(\Omega) = (P_N(I_x) \otimes P_N(I_y)) \cap H_0^{\alpha_{\text{max}}}(\Omega)\) and \(P_N\) are the Jacobi polynomials of degree \(N\). Further details of the \(\mathcal{A}\) operator and definitions of the Riemann-
Liouville derivatives can be found in [97] and [107]. The numerical results given in [97] show that this method is more accurate and has a higher convergence order than other available methods. Two other methods of comparison for computing these fractional type Schrödinger equations are known as the fractional centered difference (FCD) method [96] and compact difference method [109].

6.3. Extending the Method to Our Problem

In this section we describe how the method proposed in [49] is expanded to be applicable to our model. We wish to integrate forward in $z$ the following equation:

$$i\frac{\partial}{\partial z}\psi(x, t, z) = (\left(-\frac{\partial_{xx}}{2} - \partial_{tt} - \gamma|\psi|^2\right)\psi(x, t, z).$$

(6.18)

In this model $z$ represents the direction of propagation, $x$ is the transverse spatial direction upon which we will explore the fractional diffraction properties, $t$ represents time and we have included the temporal dispersion operator. The operator on the right hand side is split into two parts and we integrate each independently.

$$i\frac{\partial}{\partial z}\psi(x, t, z) = \left(-\frac{\partial_{xx}}{2} - \partial_{tt}\right)\psi(x, t, z)$$

(6.19)

$$i\frac{\partial}{\partial z}\psi(x, t, z) = -\gamma|\psi|^2\psi(x, t, z)$$

(6.20)

The two-dimensional computational domain for the problem is defined to be $\Omega = [a, b] \times [t_a, t_b]$. The fractionality parameter $\alpha \in [0, 2]$ and $J$ and $R$ be a positive even integers.

Define the mesh sizes $h_x = (b - a)/J$ and $h_t = (t_b - t_a)/R$ and grid points $x_j = a + jh_x$ for $0 \leq j < J$, and $t_r = t_a + rh_t$ for $0 \leq r < R$. Integrating equation (6.19) first we can assume the ansatz

$$\psi(x, t, z) = \sum_{m=-R/2}^{R/2-1} \sum_{l=-J/2}^{J/2-1} \hat{\psi}_{l,m}(z) \exp\{i[\xi_l(x-a) + \mu_m(t-t_a)]\},$$

(6.21)
where \( \hat{\psi}_{l,m}(z) \) represents the 2D Fourier transform of \( \psi(x,t,z) \) at frequencies \( l \) and \( m \), and

\[
\xi_l = \frac{2l\pi}{b-a}, \quad -\frac{J}{2} \leq l \leq \frac{J}{2} - 1, \\
\mu_m = \frac{2m\pi}{t_b-t_a}, \quad -\frac{R}{2} \leq m \leq \frac{R}{2} - 1. 
\]

Substituting (6.21) into (6.19) and using the orthogonality of the Fourier basis functions, the following equation must be satisfied for each combination of \( l \) and \( m \).

\[
i\partial_z \hat{\psi}_{l,m}(z) = (|\xi_l|^\alpha + \mu_m^2) \hat{\psi}_{l,m}(z), \quad z \in [z_n, z_{n+1}] 
\]

(6.24)

We are able to integrate (6.24) exactly in \( z \) which gives

\[
\hat{\psi}_{l,m}(z) = \hat{\psi}_{l,m}(z_n) \exp[-i(|\xi_l|^\alpha + \mu_m^2)(z - z_n)], \quad z \in [z_n, z_{n+1}]. 
\]

(6.25)

Next we focus on integrating the second part of the split operator, equation (6.20). It is useful to first multiply both sides by \( \psi^* \) and then subtract the complex conjugate of the resulting equation.

\[
i\psi^* \partial_z \psi = \gamma |\psi|^4 \\
-i\psi \partial_z \psi^* = \gamma |\psi|^4 
\]

(6.26)

(6.27)

The right hand side disappears and the left hand side is found simply be the partial \( z \) derivative of the modulus squared.

\[
i(\psi^* \partial_z \psi + \psi \partial_z \psi^*) = 0 \\
\partial_z |\psi|^2 = 0 
\]

(6.28)

(6.29)

This implies the modulus is \( z \)-invariant and can thus be integrated exactly in \( z \).

\[
\psi(x,t,z) = \psi(x,t,z_n) \exp[-i\gamma |\psi|^2(z - z_n)], \quad z \in [z_n, z_{n+1}] 
\]

(6.30)

The numerical computations have been performed in C++. We have used the FFTW version 3.3.8 subroutine library to compute the discrete Fourier and inverse Fourier Transforms. These routines can handle transforms in one or multiple dimensions.
The outline of the method to advance a state at $\psi(z_n)$ to $\psi(z_{n+1})$ is as follows. The state at each grid point $\psi_{h,j}(z_n)$ is first integrated forward half a step of width $k$ from the nonlinear contribution to give $\psi_{h,j}^{(1)}$. The result is then transformed by the 2D discrete Fourier transform in $x$ and $t$ into $\hat{\psi}_{l,m}^{(1)}(z_n)$. Each of these are then integrated forward in Fourier space a full step of width $k$ from the contributions of both the fractional second derivative in $x$ and the standard second derivative in $t$. The inverse 2D discrete Fourier transform is performed to give $\psi_{h,j}^{(2)}$. Finally the second integration forward by half a step of width $k$ is calculated to complete the contribution from the nonlinear term to give $\psi_{h,j}(z_{n+1})$.

$$\psi_{h,j}^{(1)} = \psi_{h,j}^n \exp \left\{ -\frac{i}{2}k^2\gamma |\psi_{h,j}^n|^2 \right\}$$ (6.31)

$$\psi_{h,j}^{(2)} = \sum_{m=-R/2}^{R/2-1} \sum_{l=-J/2}^{J/2-1} \hat{\psi}_{l,m}^{(1)} \exp(-ik(|\xi_l|\alpha + \mu_m)) e^{i(\xi_l(x_h-a)+\mu_m(t_j-t_a))}$$ (6.32)

$$\psi_{h,j}^{n+1} = \psi_{h,j}^{(2)} \exp \left\{ -\frac{i}{2}k^2\gamma |\psi_{h,j}^{(2)}|^2 \right\}$$ (6.33)

It is important to note that this numerical method is used for several simulations throughout this work including those of sections 4.6 and 6.4, and all of chapter 8. As analytical results for the non-integer ordered equations are not available we first verify that the case of $\alpha = 2$ matches known solutions when available and then lower the value of $\alpha$ to explore new dynamics. For all of the data presented, we also sample the simulations at $2\times$ and $4\times$ the given resolution to verify the accuracy of what is described.

### 6.4. Linear Stability Numerical Results

In this section, the numerical method developed in the previous section is used to verify the stability regions from section 6.1. We also explore the behavior that occurs in the unstable scenarios outside of the linear regime. The initial condition throughout the tests in this section is $\psi(x,t,0) = A + ae^{i(\xi x + \mu t)}$ where $A = 1.0$, $a = 0.005$ and $|\xi| = |\mu| = 1$. This is done so that the initial condition is roughly flat with very small waves on the diagonal of
the $x - t$ plane. Various points from the stable and unstable regions in each case have been sampled. The two-dimensional boundary is set to be periodic in both $x$ and $t$. The domain is accordingly set as $\Omega = [0, 4\pi/\xi] \times [0, 4\pi/\mu]$ with $1,024 \times 1,024$ grid points.

For every sampled point from the stable regions, the small perturbation appearing as waves added to the CW solution would either travel maintaining the same amplitude and speed, travel with a variable speed, or they may oscillate slightly up and down. This kind of behavior appeared for all values of $\alpha \in [0, 2]$. Figure 6.4 shows an example of a stable case for $\alpha = 1$. The yellow areas are slightly higher than the blue. The left figure shows the initial condition at $z = 0$ and the right shows the propagated field at $z = 20$.

![Figure 6.4. Example of a stable perturbation: $\alpha = 1$, $\xi = 1.2$, $\mu = 0.5$, $\gamma = -1$, $\beta = 1$.](image)

Figure 6.5 provides an example of an unstable case also with $\alpha = 1$. The excited modes grow quickly and excite a few higher modes as well. Around $z = 7.5$ the initially excited mode has reached its peak and begins to come back down. This shows how a small perturbation of $a = 0.005$ grows to much higher orders of magnitude.
Figure 6.5. Example of an unstable perturbation: $\alpha = 1$, $\xi = 1.2$, $\mu = 1.2$, $\gamma = -1$, $\beta = 1$.

Examining the effects when we move outside of the linear regime, we notice higher wave number modes become excited as well. Since the computational domain is periodic in $x$ and $t$, only integer multiples of the initial perturbation are allowed. We search for patterns in the behavior as $\alpha$ is decreased to see if the nonlocal operator plays any special part however no definitive conclusion can be reached. The most dominant effect that is noticeable is realized from the equation for $\lambda$ in (6.7). As the imaginary part of $\lambda$ becomes larger, the amplitude grows much faster as can be seen in figure 6.6. The figure shows the amplitude of the perturbed mode as the wave propagates in $z$. The different lines represent $\alpha = 2.0$, 1.5, 1.0, and 0.5 with respective values of $\lambda^2 \approx -0.79$, $-0.68$, $-0.5$, and $-0.16$. The amplitudes for the additional modes that are excited as a result are shown in figure 6.7. The figure is only a snapshot taken at the first peak (left) and second peak (right) for the first excited mode. The figure displays the wave numbers in $x$ however the wave numbers in $t$ behave the exact same. The snapshots at the first and second peaks are shown as a point of numerical validation and confirming the solution is a breather. The modes excited move to nearly identical heights at the peaks. It is the case that with the current code, we could not run the simulation beyond two cycles. If larger numerical error was present we would expect to
see larger differences between the two left and right graphs.

![Figure 6.6. Evolution of the amplitude for the first excited mode ($\xi = 0.5, \mu = 1.1, \gamma = -1, \beta = -1$).](image)

Figure 6.6. Evolution of the amplitude for the first excited mode ($\xi = 0.5, \mu = 1.1, \gamma = -1, \beta = -1$).

![Figure 6.7. Amplitudes of the additional excited modes. Snapshots taken at the first (left) and second peak (right) for the first excited mode ($\alpha = 1, \xi = 0.5, \mu = 1.1, \gamma = -1, \beta = -1$).](image)

Figure 6.7. Amplitudes of the additional excited modes. Snapshots taken at the first (left) and second peak (right) for the first excited mode ($\alpha = 1, \xi = 0.5, \mu = 1.1, \gamma = -1, \beta = -1$).

The behavior displayed in figure 6.6 is fairly smooth and a similar recurrence appears towards the end of the simulation. This kind of response was seen for all the points sampled from the focusing case $\beta = \gamma = -1$. If the signs of the derivatives in $x$ and $t$ are opposite the behavior is more erratic as can be seen in figures 6.8 and 6.9. Also it should be noted that energy has moved into several higher mode levels.
Figure 6.8. Evolution of the amplitude for the first excited mode ($\xi = 1.1, \mu = 0.7, \gamma = -1, \beta = 1$).

Figure 6.9. Amplitudes of the additional excited modes. Snapshot taken at the first peak for the first excited mode ($\alpha = 1.5, \xi = 1.1, \mu = 0.7, \gamma = -1, \beta = 1$).

As a last remark, the scenario of exciting several different modes in the initial perturbation was also explored. The linear stability analysis in section 6.1 only holds for one excited mode so we are stepping away slightly and merely exploring if the dynamics change or if certain modes are favored more than others. Due to the periodicity restriction on the boundary for the numerical method, only integer multiples of the modes were able to be initially excited and while there was no clear indication of certain modes favored more than others, often the even integer multiples of the original had higher amplitudes than the odd.
7.1. Review of Existing Method

In this section, we explore a numerical method designed to find the ground and first excited states of the fractional Schrödinger equation in an infinite potential well. In physics literature, the eigenfunctions that satisfy the eigenvalue equation \( L\psi = \lambda\psi \) are commonly called the stationary states. The eigenfunction with the smallest nonzero eigenvalue is called the ground state and those with larger values are called the excited states. In the linear nonfractional case, the eigenvalues and eigenfunctions in an infinite potential well can be found exactly [34, 10]. However, knowledge of eigenvalues and eigenfunctions in the fractional case, linear and nonlinear, is not as thoroughly available. The non-locality of the fractional Laplacian makes finding analytical solutions very challenging.

In [8], Bañuelos provides an estimate of the lower and upper bounds for the smallest eigenvalue in the linear case for \( \alpha \in (0, 2] \). A more general estimate is later given in [20, 16] for all eigenvalues. Kwaśnicki gives asymptotic approximations to all eigenvalues in the linear case in a bounded domain [55]. The numerical results given by Duo and Zhang in [25] for the linear case show that the nonlocal interactions from the fractional Laplacian lead to a large scattering inside the potential well. Furthermore, a decrease in \( \alpha \) leads to more scattering. In the nonlinear case, the local interactions lead to boundary layers in the ground states with the excited states also forming inner layers.

The method we will use is called the fractional gradient flow with discrete normalization (FGFDN) and is introduced by Duo and Zhang in [25]. The derivation for the numerical
method provided here is a summary of the work provided by Duo and Zhang and a more
detailed formulation can be found in their paper. This method is analogous to the normalized
gradient flow used to find stationary states of the standard Schrödinger equation. To begin,
we consider the one-dimensional (1D) fractional Schrödinger equation of the form
\[
  i\partial_t \psi(x, t) = (-\Delta)^{\alpha/2} \psi + V(x)\psi + \gamma|\psi|^2\psi, \quad x \in \mathbb{R}, \quad t > 0.
\]
(7.1)

where \(V(x)\) represents the external potential. In our study we will consider the infinite
potential well (or box potential) defined by
\[
  V(x) = \begin{cases} 
  0, & |x| < L, \\
  \infty, & \text{otherwise},
  \end{cases} \quad x \in \mathbb{R},
\]
(7.2)

with constant \(L > 0\) defining our domain of interest such that \(\Omega := \{x : |x| < L\}\). In this
section we will define the Riesz fractional Laplacian \((-\Delta)^{\alpha/2}\) through the principle value
integral [82, 92, 23]
\[
  (-\Delta)^{\alpha/2} \psi(x) = C_{1,\alpha} \int_\mathbb{R} \frac{\psi(x) - \psi(y)}{|x - y|^{1+\alpha}} dy,
\]
(7.3)

where the normalization constant is given by \(C_{1,\alpha} = \Gamma(1 + \alpha) \sin(\alpha\pi/2)/\pi\). The function
\(\Gamma(z)\) represents the gamma function. Through a change of variables, the operator can further
be modified as is written in equation (7.17). The literature on this subject is divided based
on the definition for the fractional Laplacian. Many arguments have also been made based
on the pseudo-differential representation given as \(-(-\Delta)^{\alpha/2} \psi(x) = \mathcal{F}^{-1}[-|\xi|^\alpha \mathcal{F}(\psi)]\) where
\(\mathcal{F}\) is the Fourier transform and \(\xi\) the wave number in \(x\) and the validity of these methods
is still disputed and under consideration [59, 35, 22]. By using the principle value integral
representation, it is clear that the nonlocal interactions are taken into account.

Two conserved quantities of the fractional Schrodinger equation are the \(L^2\) norm (or mass
of the wave function),
\[
  ||\psi(\cdot, t)||^2 := \int_\mathbb{R} |\psi(x, t)|^2 dx = \int_\mathbb{R} |\psi(x, 0)|^2 dx = ||\psi(\cdot, 0)||^2 = 1, \quad t \geq 0,
\]
(7.4)
which has been normalized, and the total energy
\[ E(\psi(\cdot, t)) := \int_{\mathbb{R}} \psi^* (-\Delta)^{\alpha/2} \psi + V(x)|\psi|^2 + \frac{\gamma}{2} |\psi|^4 dx = E(\psi(\cdot, 0)), \quad t \geq 0, \quad (7.5) \]
where \( \psi^* \) denotes the complex conjugate of \( \psi \). To find the stationary states of (7.1) we assume an ansatz for the wave function as
\[ \psi(x, t) = e^{-i\mu t} \phi(x), \quad x \in \mathbb{R}, \quad t \geq 0, \quad (7.6) \]
where \( \mu \in \mathbb{R} \). Substituting (7.6) into (7.1) under the mass conservation constraint (7.4) and at the eigenvalue equation
\[ \mu \phi(x) = [(-\Delta)^{\alpha/2} + V(x) + \gamma |\phi|^2] \phi, \quad (7.7) \]
\[ ||\phi||^2 = \int |\phi(x)|^2 dx = 1. \quad (7.8) \]
Due to the infinite well potential imposing \( V(x) = \infty \) outside of \( \Omega \), the problem is reduced to finding the eigenfunction satisfying (7.7) and (7.8) within \( \Omega \) and also that \( \phi(x) = 0 \) for \( x \in \mathbb{R} \setminus \Omega \). After finding the eigenfunction \( \phi(x) \) the corresponding eigenvalue \( \mu \) can be found through
\[ \mu = \int \phi^*(-\Delta)^{\alpha/2} \phi + \gamma |\phi|^4 dx \quad (7.9) \]
For linear Schrödinger equation using the classical Laplacian \( (\alpha = 2) \) in the infinite well potential, the eigenvalues and eigenfunctions can be found exactly. The \( s \)-th eigenfunction is given by [34, 57]
\[ \phi_s(x) = \sqrt{\frac{1}{L}} \sin \left[ \frac{(s + 1)\pi}{2} \left( 1 + \frac{x}{L} \right) \right], \quad x \in \Omega, \quad (7.10) \]
and the corresponding \( s \)-th eigenvalue is
\[ \mu_s = \left[ \frac{(s + 1)\pi}{2L} \right]^2. \quad (7.11) \]
The ground state is the eigenfunction when \( s = 0 \) and the first excited state is when \( s = 1 \). For the nonlinear case, \( \gamma \neq 0 \), the eigenvalue problem cannot be solved exactly however the
solutions to the linear case give a good approximation for the weakly nonlinear regime when \( \gamma = o(1) \). For the strongly defocusing case, \( \gamma \gg 1 \), the leading order approximation can be found through the Thomas-Fermi approximation given by [108, 10]

\[
\phi_s(x) \approx \phi_x^a(x) = \sqrt{\frac{\mu_x^a}{\gamma}} \left\{ \sum_{r=0}^{[s+1]/2} \tanh \left[ L \sqrt{\frac{\mu_x^a}{2}} \left( 1 + \frac{x}{L} - \frac{4r}{s+1} \right) \right] + \sum_{r=0}^{[s/2]} \tanh \left[ L \sqrt{\frac{\mu_x^a}{2}} \left( \frac{4r+2}{s+1} - \left( 1 + \frac{x}{L} \right) \right) \right] - c_s \tanh \left( L \sqrt{\frac{\mu_x^a}{2}} \right) \right\}, \quad x \in \Omega,
\]

where \([r]\) represents the floor function and \( c_s = 1 \) for \( s \) even and \( c_s = 0 \) for \( s \) odd. The leading order approximation for the eigenvalue is then

\[
\mu_s \approx \mu_x^a = \frac{1}{L^2} \left[ \frac{\gamma L}{2} + (s+2) \sqrt{\gamma L + (s+2)^2} + (s+2)^2 \right]. \quad (7.13)
\]

Through these approximations, we can see that for \( \gamma \gg 1 \), the eigenfunctions have boundary layers and that inner layers are also formed for \( s \geq 1 \) with the \( s \) inner layers in the \( s-th \) excited state [25].

Analytical estimates for the bounds and asymptotic approximations of the eigenvalues for the fractional linear case (\( \gamma = 0 \)) have been given in [8, 16, 58]. In [8] [Corollary 2.2], an estimate is given for the smallest eigenvalue at \( s = 0 \) with \( \alpha \in (0, 2] \) as

\[
p(\alpha) \leq \mu_0 \leq p(\alpha) \frac{B(\frac{1}{2}, 1 + \frac{\alpha}{2})}{B(\frac{1}{2}, 1 + \alpha)} , \quad \text{where} \quad p(\alpha) = \frac{2^\alpha \Gamma(1 + \frac{\alpha}{2}) \Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{1}{2})} \quad (7.14)
\]

and \( B(a, b) \) denotes the Beta function of \( a \) and \( b \). In seeking bounds of any \( s \in \mathbb{N}^0 \), we may use the estimates given in [16]

\[
\frac{1}{2} \left[ \left( \frac{(s+1)\pi}{l} \right)^\alpha \right] \leq \mu_s \leq \left[ \left( \frac{(s+1)\pi}{l} \right)^\alpha \right], \quad (7.15)
\]

where \( l \) is the length of the interval for our domain. For the asymptotic approximations for the \( s-th \) eigenvalue of the fractional linear Schrödinger equation in the interval \((-1, 1)\), Laskin provided the approximation [58]

\[
\mu_s = \left[ \frac{(s+1)\pi}{2} - \frac{(2-\alpha)\pi}{8} \right]^\alpha + O\left( \frac{2-\alpha}{(s+1)\sqrt{\alpha}} \right). \quad (7.16)
\]
Note when $\alpha = 2$, (7.16) gives the exact eigenvalue for the standard linear Schrödinger equation in an infinite potential well as in equation (7.11). For numerical simulations of the eigenfunctions and eigenvalues in the fractional case, we turn to [25]. Since we are searching for the “stationary states”, we discretize the operator on the right of equation (7.7) subject to the constraint (7.8) and numerically propagate the equation until it levels off and successive iterations no longer change the solution by more than some small required value. Hence, we set up the equation as

$$\frac{\partial \phi(x, t)}{\partial t} = C_{1,\alpha} \int_0^\infty \frac{\phi(x - \xi, t) - 2 \phi(x, t) + \phi(x + \xi, t)}{\xi^{1+\alpha}} d\xi - \gamma |\phi(x, t)|^2 \phi(x, t)$$

(7.17)

for $x \in \Omega$. For $x \notin \Omega$, $\phi = 0$ due to the infinite potential. Since $\phi = 0$ outside of $\Omega$, the nonlocal component of the wave function will only interact with other points inside $\Omega$ and the operator can be discretized as shown in (7.19). Let $\phi = (\phi_1(t), \phi_2(t), \ldots, \phi_{J-1}(t))^T$ denote the solution vector at time $t$. The authors in [25] show the semi-discretization of the fractional gradient flow is given by

$$\frac{d\Phi(t)}{dt} = D\Phi(t) + F(\Phi(T)), \quad t \in [t_n, t_{n+1}].$$

(7.18)

The $D$ matrix is given by

$$D_{jk} = -\frac{C_{1,\alpha}}{\sigma h^\alpha} \begin{cases} \sum_{l=1}^{J-1} \frac{(l+1)^\sigma - (l-1)^\sigma}{l^{2-\gamma}} + \frac{J^\sigma - (J-1)^\sigma}{J^{2-\gamma}} + \frac{2\sigma h^\alpha}{\alpha(2L)^\sigma}, & k = j, \\ \frac{(k-j+1)^\sigma - (k-j-1)^\sigma}{2(k-j)^{2-\gamma}}, & k \neq j, \end{cases}$$

(7.19)

where $\gamma = 1 - \alpha/2$ and $\sigma = 2 - (\alpha + \gamma)$ are the optimal parameter values. We notice that $D$ is a symmetric Toeplitz matrix and can quickly be solved with a conjugate gradient method. The vector $F(\Phi)$ consists of the nonlinear component, $f(\phi_j) = -\gamma |\phi_j|^2 \phi_j$.

The semi-discretization of the fractional gradient flow is a system of nonlinear ordinary differential equations. It may be discretized in time by the semi-implicit Euler method. This scheme can be used for both the ground states and first excited states as well as the linear and nonlinear case. As expected, with increasing $\alpha$ the solutions approach those of
the standard Schrödinger equation. We notice a boundary layer form when \( \alpha \) is small and that the layers are more pronounced in the nonlinear case.

### 7.2. Time Dispersion Extension

In this section we build on the FGFDN and use it to find ground and first excited states for equation (7.20). The \( z \) variable is now the “time-like” variable although the main difference is the addition of the temporal dispersion component. We continue to use the infinite potential well for \( V(x) \). The equation we will work with is now

\[
i \frac{\partial \phi}{\partial z} = (-\Delta_x)^\alpha \phi + \frac{\partial^2 \phi}{\partial t^2} + \gamma |\phi|^2 \phi + V(x) \phi.
\]  

(7.20)

The initial condition is given by the \( s \)-th eigenfunction of the standard linear Schrödinger equation. The idea is that the eigenfunction for the fractional case should not be very far away and that this is a good starting point. We will use

\[
\phi(x, t, 0) = \frac{1}{L} \sin \left[ \frac{(s + 1)\pi}{2} \left( 1 + \frac{x}{L} \right) \right] \sin \left[ \frac{(s + 1)\pi t}{T} \right]
\]  

(7.21)

where again we assume \( x \in [-L, L] \) and \( t \in [0, T] \). We first analyze the ground states by adjusting the FGFDN scheme for the 1 dimensional case and add a centered finite difference discretization in \( t \) to account for the second time derivative. Preliminary results are shown in figure 7.2.

The numerical simulation is carried out in C++. The conjugate gradient method is used to solve the linear system with a tolerance of \( 1e^{-7} \). The \( z \) step length is set to \( 5e^{-6} \), and we assumed convergence at \( 1e^{3} \). The number of grid points in both \( x \) and \( t \) dimensions is 256. Ideally, a stronger convergence would be sought however the method struggled to converge with a finer resolution in the two-dimensional case. As a point of numerical validation, we first matched our 1 dimensional results with that of [25], then extending the method to 2 dimensions we matched the stationary state for \( \alpha = 2 \) with that of the analytical solution in the linear case. We also verified the results at this mesh size by doubling the resolution.
Figure 7.1. [2] Ground states for the fractional linear Schrödinger equation with temporal dispersion for $\alpha = 0.2$ (top left), 0.75 (top right), 1.1 (bottom left), and 1.9 (bottom right).

interpolating the results to the additional grid points and then using that as the starting condition for the method. After this starting solution converged, the shape was unchanged.

We see that the solution is not radially symmetric. This is expected since, in contrast to the two-dimensional Laplacian, we have split the operator to have one dimension fractional and the other to be non-fractional. Similar to one-dimensional case, boundary layers form near the $x = -1, 1$ edges and the solution converges to the solution of the standard Schrödinger equation as $\alpha$ approaches 2. Concluding this chapter with these results we also point out that future work will extend the analysis to the nonlinear cases and the first excited states.
8.1. Field Contraction

A standard nonlinear Schrödinger equation may be given as

\[ i \frac{\partial u}{\partial z} + \beta \Delta_d u + \gamma |u|^{2\sigma} u = 0, \tag{8.1} \]

where \( d \) and \( \sigma \) are positive integers and \( d \) represents the number of variables such that \( \Delta_d = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \). It is a known result that if \( d\sigma \geq 2 \), a singularity exists in the solution and if \( d\sigma < 2 \), no singularity exists. In this section, we numerically investigate the scenario that can be described in some sense where \( d \) must not necessarily be an integer but we look into the transition between \( d = 1 \) and \( d = 2 \). Our model departs from equation (8.1) by allowing a one-dimensional Laplacian of fractional order in the \( x \)-variable and a standard one-dimensional Laplacian in the \( t \)-variable. Since it is known that solutions will exist globally when \( \gamma = -1 \) (defocusing), we will restrict the numerical simulations to the focusing cases where \( \gamma = 1 \) because we are interested in the formation of singularities.

Throughout our results, several validations of the numerical method are performed. The first of which is to demonstrate a singularity formation in the standard \( \alpha = 2 \) case when the \( L^2 \) norm of the initial condition is above a critical value. Likewise we look at the case when the norm is below the critical value and observe the initial condition spread throughout the domain. The initial field profile is described as a product of two Gaussians, \( \psi_0 = A \exp(-x^2-t^2) \). Figure 8.1 shows the evolution of two different fields as they propagates along the \( z \)-axis. The only difference in the initial data is that of the coefficient of \( A = 3.5 \).
for the top row and $A = 2$ for the bottom row. When $A = 3.5$, the self-focusing effect from the nonlinearity dominates and all of the energy collapses to the center. When $A = 2$, the diffraction and dispersion dominate and cause the field to spread in $x$ and $t$. This agrees with the theory and shows the critical value of $A$ for a singularity to form is somewhere in the range of $[2, 3.5]$ in our setup.

![Figure 8.1. Top row: evolution of a field with norm above the critical value to form a singularity. Bottom row: evolution of a field with norm below the critical value.](image)

Next, we investigate how the distance traveled before the energy collapses to the center is affected if we lower the value of $\alpha$. To begin this process, we first seek the distances traveled before the peak begins to decrease. In other words, in cases where the profile would either grow or remain the same initially, the distance until it first began to shrink was calculated and will be denoted as $z_s$. This is used to as an indicator of where singularities may form.

For low energy initial profiles, the field immediately begins to spread. As the energy is increased, the first display of self-focusing overcoming the diffraction and dispersion begins around $A \approx 1.333$ for $\alpha \approx 1.36$. Figure 8.2 shows how an increase in power will increase $z_s$, and that a wider range of $\alpha$ centered about 1.36 will also have $z_s > 0$. 

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Figure 8.2. Measuring the length of propagation in $z$ before the maximum value of $|\psi|$ begins to decrease. The width of the beam initially contracts or remains the same around $\alpha = 1.38$ for $A \in [1.332, 1.338]$.

This behavior continues as $A$ is increased further until the field does not immediately spread for all values of $\alpha \in [0, 2]$. The next change in the dynamics shown in figure 8.2 appears slightly after $A = 1.954$. Figure 8.3 shows a spike in the graph (red line) around $\alpha = 1.16$. This peak for $z_s$ continues to move to the left as $A$ is increased until around $A = 2$. At $A = 2$ (yellow line) a much larger spike in $z_s$ occurs around $\alpha = 1.035$. This is the first resemblance of a singularity. Peaks of $z_s$ discussed previously would show only small growth in $|\psi|$ at the center of a few tenths however at this point, the maximum peak reaches 20 before either the numerical method is no longer valid or $|\psi|$ begins to spread throughout the domain. We also observe a second kink in the graph of $z_s$ around $\alpha = 0.96$. For ease of reference we will label these two kinks as $\alpha_a$ (left) and $\alpha_b$ (right). The possible blowups occur between these two values of $\alpha$. 
Figure 8.3. Measuring the length of propagation in $z$ before the maximum value of $|\psi|$ begins to decrease. As $A$ is increased, more values of $\alpha$ allow the field to contract for longer distances. The large spike in $A = 2$ (yellow line) is the first appearance of a blowup in the solution.

Figure 8.4 shows the width between $\alpha_a$ and $\alpha_b$ widen as $A$ continues to increase. While the smaller values of $\alpha$ near zero propagate a farther distance before the field blows up or begins to spread, the midrange values of $\alpha$ see a blowup or spread much sooner.

Figure 8.4. Measuring the length of propagation in $z$ before the maximum value of $|\psi|$ begins to decrease. As $A$ is increased above 2, the gap between $\alpha_a$ and $\alpha_b$ grows wider indicating a wider range of possible blowups.
Around $A = 2.75$, $\alpha_a$ has shifted left to its minimum value before it changes direction and begins to increase. Subsequently higher values of $A$ shift $\alpha_a$ to the right and the corresponding $z_s$ at that point decreases. On the other hand, $\alpha_b$ always increases as $A$ is increased and $\alpha_b = 2.0$ around $A = 2.75$. At this point the standard Laplacian is now included in the range that forms a singularity of which we will attempt to verify later.

![Figure 8.5](image)

Figure 8.5. Measuring the length of propagation in $z$ before the maximum value of $|\psi|$ begins to decrease. The minimum value of $\alpha_a$ is reached around $A = 2.75$ and then reverses direction while the corresponding value of $z_s$ drops quickly.

As $A$ is increased further passed $3.1$, the same behavior continues. The value of $z_s$ at $\alpha_a$ continues to decrease and move to the right and eventually levels off. This implies the distance needed to reach the peak amplitude becomes much smaller, which is understandable as the self-focusing component is becoming more dominant. In the next section we investigate whether or not a blowup actually occurs at these points.

### 8.2. Finding Singularities

In this section we attempt to determine if the solutions discussed in the previous section actually form a singularity or are better classified as *breather* solutions. A breather solution is a nonlinear wave in which the energy is localized in an oscillatory manner and which does not collapse to a single point. The criteria used in this section to determine if a singularity
is formed is as follows: periodic measurements of the cross section at half the maximum value of $|\psi|$ were taken throughout the simulation. If either the width or length at half max dropped to 3 discrete points of the mesh size or less, we considered this to be a blowup. Ideally the widths in both $x$ and $t$ would collapse to a single point and in most cases this did occur, however due to numerical error, or by only periodically measuring the cross section throughout the integration, we will use the above criteria as an approximation.

Figure 8.6 is in a similar format to the figures from the previous section except these lines represent the distance travelled in $z$ where the solution actually experienced a blowup as determined by the conditions specified above. The results for $\alpha < 0.3$ are not clear and while the cross section at half maximum often contracts, it merely oscillates and does not tend to a single point. The results match what might be expected from figures 8.4 and 8.5. When $A = 2.3$ the incidents of blowup take longer in $z$ to collapse than higher values of $A$ for all comparable values of $\alpha$. The curves for smaller $A$ take on a parabolic shape with the plotted $z$ initially decreasing as $\alpha$ is increased and then spiking up near the largest value of $\alpha$ included in the plot. As $A$ is increased, the maximum value of $\alpha$ that incurs a blowup is also increased.

Figure 8.6. Measuring the length of propagation in $z$ before the solution reaches a blowup. Each line represents a different level of power.

If we continue looking into the strongly nonlinear cases by increasing $A$ further, the self-focusing becomes much more dominant and the blowup occurs much faster. Figure 8.7 shows
the same measurements for $A = 7.5$ and $A = 10$. In both cases, all values of $\alpha$, other than those close to 0, blow up at nearly the same distance with subtle decreases seen in the latter half of the $\alpha$ range.

Figure 8.7. Measuring the length of propagation in $z$ before the solution reaches a blowup for $A = 7.5$ (blue) and $A = 10$ (orange).

The parameters used in all of the numerical simulations for this section and in section 8.1 are as follows: the computational domain is given by $\Omega = \{(x, t) \in [-12, 12] \times [-24, 24]\}$. The range for $t$ is twice as large since the time dispersion component caused the energy to spread in $t$ faster than the fractional diffraction spread the energy in $x$. The mesh size is $h_x = h_t = 3/64$, equivalently 512 grid points used in $x$ and 1024 used in $t$. The $z$ step is set to $h_z = 0.0001$. Finally $\gamma = 1$ is used to study the focusing nonlinear case.

8.3. The $\alpha = 0$ Case

In the simulations of this chapter we vary $\alpha$ between 2 and 0. Here, we show that the integration with $\alpha = 0$ only differs by a phase change from the case of not having a diffraction component in $x$ at all. Suppose $\alpha = 0$ is the exponent for the eigenvalue Fourier method. We start with the equation

$$
 i\partial_z \psi(x, t, z) + \left[(-\partial_{xx})^0 - \partial_{tt}\right] \psi(x, t, z) = 0.
$$

(8.2)
After taking the 2D Fourier transform in $x$ and $t$, equation (8.2) becomes

$$i \frac{\partial \hat{\psi}}{\partial z} = -(|\xi|^2 + \mu^2) \hat{\psi},$$

(8.3)

where $\hat{\psi} = \hat{\psi}(\xi, \mu, z)$ is the Fourier transform of $\psi$, and $\xi$ and $\mu$ are the wave numbers in $x$ and $t$ respectively. Integrating (8.3) from $z$ to $z + \Delta z$ gives

$$\hat{\psi}(\xi, \mu, z + \Delta z) = \hat{\psi}(\xi, \mu, z) e^{-i(1+\mu^2)\Delta z}.$$  

(8.4)

Now we want to revert back to the spatial coordinates. Applying the inverse Fourier transform in $x$ and $t$, the right hand side becomes

$$\frac{1}{4\pi^2} \int \int \hat{\psi}(\xi, \mu, z) e^{-i(1+\mu^2)\Delta z} e^{i(\xi x + \mu t)} d\xi d\mu.$$  

(8.5)

Rearranging the terms and pulling out the factor of $e^{-i\Delta z}$ gives

$$\frac{e^{-i\Delta z}}{4\pi^2} \int e^{-i(\mu^2\Delta z - \mu t)} \int \hat{\psi}(\xi, \mu, z) e^{i\xi x} d\xi d\mu,$$

$$= \frac{e^{-i\Delta z}}{2\pi} \int \hat{\psi}^{(1)}(x, \mu, z) e^{-i(\mu^2\Delta z - \mu t)} d\mu,$$

(8.6)

where $\hat{\psi}^{(1)}$ is the 1D Fourier transform in $t$. In conclusion, the result only differs by a phase change. After taking the modulus of $\psi$ the final result is unaffected by the derivative term with a zero exponent.

### 8.4. Asymmetric Initial Conditions

In this section we investigate and compare the behavior near blowup between initial conditions that are symmetric and non symmetric in $x$ and $t$. A well known result for solutions with singularities when $\alpha = 2$ (classical Laplacian) is that the self-focusing effect causes the amount of power which goes into the singularity to always be equal to the critical power needed for blowup [29]. Since the total power must be conserved, the pulse separates into two components as it propagates, $\psi = \psi_s + \psi_{\text{back}}$, where $\psi_s$ represents the high intensity inner core which self-focuses to the center axis and $\psi_{\text{back}}$ is the lower intensity outer component which propagates forward following the usual linear propagation mode [1]. It can
also be shown for cases where the initial profile is not radially symmetric, when the field is near the point of collapse, the field approaches a radially symmetric asymptotic profile [2]. In this section we numerically explore if this same phenomenon holds when the degree of fractionality in one dimension is lowered.

For this section we will consider three separate initial conditions to analyze how symmetry plays a role in the blowup. These conditions are given below with $A = 3.5$.

**Case 1**: Symmetric \[ \psi_0 = A\exp(-x^2 - t^2) \] (8.7)

**Case 2**: Stretched in $t$ \[ \psi_0 = A\exp(-2x^2 - \frac{1}{2}t^2) \] (8.8)

**Case 3**: Stretched in $x$ \[ \psi_0 = A\exp(-\frac{1}{2}x^2 - 2t^2) \] (8.9)

All three cases maintain equivalent $L^2$ norms. This holds as long as the amplitude remains the same and the product of the coefficients in front of $x^2$ and $t^2$ is the same.

As an example of how the field approaches a symmetric asymptotic profile in the classical case ($\alpha = 2$), figure 8.8 shows the behavior for the nonsymmetric initial condition of case 2. The images display the cross section of the profile at half of the maximum modulus ($\frac{1}{2}\max_{x,t} |\psi|$), as the field propagates from left to right.

Figure 8.8. Example of a non symmetric field approaching a radially symmetric asymptotic profile near blowup ($\alpha = 2$).

Figures 8.9 (case 1), 8.10 (case 2), and 8.11 (case 3) show the progression of the widths in $x$ and $t$ at half max as the solution moves towards blow up. Also displayed by the gray
The maximum modulus of the field \( \max_{x,t} |\psi| \). The orange and blue lines overlap entirely in case 1 since it starts symmetric and \( \alpha = 2 \) gives the classical Laplacian. For the non-symmetric cases we see the wider dimension initially shrink faster and catch up to the other dimension before both synchronously decreasing near the final collapse. All three cases show the maximum slightly grow as the width begins to shrink until it quickly moves towards the blowup at the end at the final collapse.

Figure 8.9. Measuring the full widths at half maximum in \( x \) and \( t \) along with the maximum modulus as \( z \) progresses (\( \alpha = 2, A = 3.5, \text{case 1} \)).

Figure 8.10. Measuring the full widths at half maximum in \( x \) and \( t \) along with the maximum modulus as \( z \) progresses (\( \alpha = 2, A = 3.5, \text{case 2} \)).
Figure 8.11. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 2, A = 3.5, \text{case 3}$).

Now we will lower the value of $\alpha$ and see how the symmetry near blowup holds. Figures 8.12-8.14 show cases 1-3 for $\alpha = 1.8$. In the symmetric case, the width in $t$ is the first to decrease, causing a slightly oval shape. About half-way to the collapse, the width in $x$ surpasses $t$ and becomes smaller which holds through to the end without becoming symmetric again. The presence of this gap near blowup between the widths in $x$ and $t$ is the feature we would like to highlight here. As mentioned before, when $\alpha = 2$, a radially symmetric blowup should occur independent of the shape of the initial condition. Here we shown that this property does not hold when the the fractional exponents in each dimension are not the same. This may align with intuition considering the balance between the nonlinearity and the second derivative in each dimension. In all three cases, the width in $x$ is smaller than in $t$ near the blowup.
Figure 8.12. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1.8$, $A = 3.5$, case 1).

Figure 8.13. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1.8$, $A = 3.5$, case 2).

Figure 8.14. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1.8$, $A = 3.5$, case 3).
Next, we continue to decrease $\alpha$ to 1 where results are shown in figures 8.15-8.17. We see similar behavior in the symmetric case although the width in $t$ drops much farther than in $x$. The point of collapse as well as the point at which the $x$ width becomes thinner are delayed in $z$ from that of $\alpha = 1.8$. Similar results are seen in the non symmetric cases.

![Case 1, $\alpha=1$](image1)

Figure 8.15. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1$, $A = 3.5$, case 1).

![Case 2, $\alpha=1$](image2)

Figure 8.16. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1$, $A = 3.5$, case 2).
Figure 8.17. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 1$, $A = 3.5$, case 3).

Further decreasing $\alpha$ to 0.7 a new behavior emerges in figures 8.18-8.20. As shown for the symmetric case, the widths initially decrease but the contraction stalls for some distance before slightly growing and then collapsing entirely. This oscillation becomes more pronounced when $\alpha$ is lowered closer to 0. Interestingly, for case 1 and 2, the width in $t$ is the first to approach a single point as opposed to $x$ as seen when $\alpha$ is larger. It’s also important to note the max of $|\psi|$ does not grow as tall as before and it is not at its max height when the width in $t$ tends to zero. With this in mind it may be inaccurate to classify any case at $\alpha = 0.7$ as a blowup.

Figure 8.18. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 0.7$, $A = 3.5$, case 1).
As \( \alpha \) moves closer to 0, the conclusion of a blow up becomes even less certain. Figures 8.21-8.23 show the results for \( \alpha = 0.3 \). Here we also notice a clear distinction between the behavior of case 2 and that of case 1 or 3. In case 2, we see larger smooth oscillations. The \( t \) width more directly evolves inversely in relation to the \( \max|\psi| \) while the \( x \)-width appears to sluggishly following along. As \( \alpha \) approaches 0, we expect the behavior to move towards that of the one dimensional NLS if we were to freeze \( x \) and only look at the evolution in the \( z-t \) plane. Cases 1 and 3 have more erratic behavior with smaller oscillations. Also note that \( \max|\psi| \) does not grow beyond 12 and appears to level off or even decrease as \( z \) progresses.
resembling the breather solutions.

Figure 8.21. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 0.3$, $A = 3.5$, case 1).

Figure 8.22. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 0.3$, $A = 3.5$, case 2).
Figure 8.23. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 0.3$, $A = 3.5$, case 3).

As $\alpha$ is moved nearly to 0, new dynamics are observed. Figures 8.24-8.26 show the results for $\alpha = 0.1$. The erratic behavior seen before in cases 1 and 3 is now much smoother. There is still a small contraction of the field with the max roughly doubling in size. Case 3 now shows the smoothest behavior seen in this section so far with hardly any noticeable oscillations. The shape moves to a nearly symmetrical profile throughout the propagation however the $x$ width remains slightly larger.

Figure 8.24. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum modulus as $z$ progresses ($\alpha = 0.1$, $A = 3.5$, case 1).
Finally, the $\alpha = 0$ cases are shown in figures 8.27-8.29. In all three cases, we see breather-like solutions. Case 1 settles into a non-symmetric pulse. Case 2 oscillates while mostly driven by the balancing act in $t$. Around $z = 2.4$, the width in $t$ does collapse however we presume this to be a numerical artifact due to reflected modes in the finite domain. Case 3 settles into a steady breather quicker than in case 1. Larger oscillations occur further along the $z$-axis although this may also be due to numerical error.

How can we gain an intuition into what occurs when $\alpha = 0$? If we imagine freezing $x$ and viewing the cross section in the $z-t$ plane; this plane would propagate according to the
one dimensional standard NLS. This explains the difference between cases 2 and 3. Case 2
starts with a much wider Gaussian curve in $t$. This implies that $\int |\psi|^2$ is much greater which
will cause stronger oscillations due to the nonlinearity.

Figure 8.27. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum
modulus as $z$ progresses ($\alpha = 0$, $A = 3.5$, case 1).

Figure 8.28. Measuring the full widths at half maximum in $x$ and $t$ along with the maximum
modulus as $z$ progresses ($\alpha = 0$, $A = 3.5$, case 2).
8.5. Strongly Nonlinear Cases

As a side remark, we briefly show the results in this section when the $L^2$ norm of the initial condition is much higher and moves the dynamics into the strongly nonlinear regime. The only change from the previous section is of the coefficient in front of the exponential, $\psi_0 = 7.5 \exp(-x^2 - t^2)$ or $\psi_0 = 10 \exp(-x^2 - t^2)$. Figure 8.30 shows the propagation when $A = 7.5$ and figure 8.31 shows the same when $A = 10$. Both figures are for $\alpha = 1$. Comparing 8.30 to figure 8.15 ($A = 3.5$), first notice the initial difference between the widths is much larger for $A$ large. There is one large swing after the field initially contracts and then the width in $x$ drops rapidly and the field becomes nearly symmetric until blowup. Interestingly, the solution has nearly regained the property of radial symmetry near blowup. A topic that will be further discussed in section 8.6. When $A = 10$, the behavior is similar although an additional oscillation emerges before heading into the final collapse.
8.6. Breaking of Symmetry

In the previous section, we noted when $\alpha$ is reduced in one dimension, we are no longer guaranteed symmetry near the point collapse. The figures in section 8.4 are for a relatively low power ($A = 3.5$), slightly above the critical power needed to form a singularity. In this section, we will view pulses with higher power and study this breaking of the symmetry property. In particular we will examine the ratio between the widths at half maximum near the blowup.
For case 1 and case 3, in all of the figures shown in section 8.4 with $\alpha > 1$, the field is non-symmetric for most of the propagation. The width in $x$ is wider than in $t$ in the beginning and at some point the $x$ width decreases past $t$ and then approaches a single point. The measurements performed here are taken near the end; from the moment it is symmetric until the blowup is determined. We measure the ratio of the widths and find that the ratios decrease as the power is increased. The ratio is decreased to where it approaches symmetry again as is proven for the classical case. This suggests that the nonlinear component is the driving force for the symmetry property. When the nonlinearity is weak, the forcing towards symmetry is reduced however when the nonlinearity plays a stronger role, the disparity introduced by the fractional derivative is diminished.

Figures 8.32 and 8.33 show the maximum of this ratio for varying $\alpha$. Each line represents a different level of power. Figure 8.32 shows the results for the symmetric initial conditions (case 1) and figure 8.33 shows the results for the initial conditions stretched in $x$ (case 3). In all scenarios, as the power is increased, the field becomes closer to symmetric near the blowup. The ratio is typically largest for $\alpha \in (1.2, 1.6)$ and is smaller in case 1 than in case 3.

Additional tests were performed at even higher powers ($A = 7.5$ and $A = 10$). In both of these cases, the field became symmetric right at the point of blowup. In this analysis, we do not consider points below $\alpha = 1$ since it is not clear if those solutions actually form a singularity.
Figure 8.32. Measuring the maximum of the ratio of widths near blowup. Each line represents a different initial condition with varying powers. Increasing the power brings the field closer to symmetric (case 1).

Figure 8.33. Measuring the maximum of the ratio of widths near blowup. Each line represents a different initial condition with varying powers. Increasing the power brings the field closer to symmetric (case 3).

8.7. Self Similar

In this section we numerically explore the possibility that the solutions are self similar. If the solutions are shown as self similar, we have another justification for determining that
the solution moves towards a wave collapse. The results seen in this section are not definite however for values of $\alpha$ near 2, it is suggested that the solution takes such a form. Let $M_z = \max_{x,t}(|\psi_z|)$, the maximum modulus of the field at a given $z$. Also let $w_t$ and $w_x$ denote the full width at half of $M_z$ in $t$ and $x$ respectively. If the solution presented is self similar, we expect to see $M_z$ evolve has a function of either width according to

$$M_z = cw_x^r,$$

$$\log(M_z) = r \log(w_x) + \log(c),$$

where $c$ and $r$ are constants.

Figures 8.34 and 8.35 graph $|r|$ as a function of $w_x$ (left) and $w_t$ (right). To clarify precisely how $r$ is calculated for these graphs, the following steps are taken: $M_z$ is first plotted as a function of $w_x$ and $w_t$ from the initial data until either $M_z$ reached its first peak and started to decline or the width moved to a single grid point. Next the slope of the log of both sides was calculated at each point that $w_x$ or $w_t$ decreased. The slope appears negative since $M_z$ increases while $w_x$ and $w_t$ decrease so the absolute value is plotted in the figures. Figure 8.34 displays the results in the weakly nonlinear regime. Here, the strength of the nonlinear component is controlled by the amount of power in the initial condition, namely by setting $A = 3.5$, in $\psi(0,x,t) = Ae^{-x^2-t^2}$. When measuring against $w_x$ (left), for $\alpha = 2$ or $\alpha = 1.5$, the value of $r$ is nearly constant towards the moment of collapse which would suggest self similarity for this portion of the simulation. As $\alpha$ is lowered to 1, $r$ becomes slightly more variable and when $\alpha = 0.5$, $r$ is obviously not constant as $w_x$ decreases.
Figure 8.34. Measuring $|r|$ as a function of $w_x$ (left) and $w_t$ (right) in the weakly nonlinear regime ($A = 3.5$). When $\alpha$ is near 2, $r$ is relatively flat near the end of the simulation compared to smaller values of $\alpha$.

Figure 8.35 shows the results in the strongly nonlinear regime, $A = 10$. The nonlinear component plays a more dominant role and even though it is more evident from the analysis in sections 8.2 and 8.4 that the field incurs a blowup, we cannot determine that the solutions are self similar. Interestingly, the $|r|$ values increase as $\alpha$ decreases when viewed as a function of $w_x$ yet the opposite is true when viewed as a function of $w_t$.

In the strongly nonlinear case, there is clearly a trend for $|r|$ to decrease and then spike upwards at the end of the simulation. This implies that $M_z$ has more incremental gain during the initial contraction. These gains slow for a brief distance and then the field rapidly focuses towards the center before either collapsing or reversing course and spreading back outward. This feature is not present in the weakly nonlinear cases.
Figure 8.35. Measuring $r$ as a function of $w_x$ (left) and $w_t$ (right) in the strongly nonlinear regime ($A = 10$).
In this work, nonlinear optics has been studied in applications to twisted and nonlocal structures. In particular, the confinement of light realized in a coreless twisted PCF has been studied through both ray theory and field theory. Outside of the traditional optical fibers that contain one or multiple cores, these results show that a twisted PCF structure provides an additional method of guiding light in the absence of a core. Through ray theory, we’ve shown numerically that the minimal path results in a transverse circular motion about the axis similar to that of a particle in a Paul trap. This behavior persists for distances of around 20 cm. An increase in twist will also pressure more rays to follow this motion. Through field theory, we’ve shown by asymptotic and perturbative methods that the propagation of the envelope of the Bloch modes through a periodic helical medium adapt to follow the twist. We see the allowed modes will have a radially symmetric chirp and the envelope will decay away from the axis. The energy is thus confined and will propagate along the axis with little energy loss. An increase in twist will increase confinement as the model predicts. The numerical results show enhanced confinement may exist within certain ranges of twist. Larger twist rates may cause the mode to quickly spread, after which even higher twist rates can allow confinement similar to the non-twisted case. We suggest that the confinement effect is due to the rapid spiralling of hollow channels and is analogous to the creation of a stable region in the classical Kapitza pendulum [18].

We’ve proposed a model for short pulse propagation by a nonlinear Schrödinger type equation with a second derivative in $t$, representing temporal dispersion, and a fractional spatial second derivative to include nonlocal interactions. To our knowledge, this is the
first study of a mixed fractional/integer ordered equation. We perform exploratory analyses of the resulting behavior. Regions of linear stability are established given a continuous wave solution. The defocusing case is similar to the standard NLSE in that it is always stable. Numerical methods previously proposed for fractional equations are extended to our model to find the ground states and to describe the behavior during propagation. We next numerically describe propagation of perturbed solutions outside the stable regions and find which modes are excited in the long term. We’ve also analyzed the delicate balance between the diffraction/dispersion and the focusing nonlinearity. Given that the fractional derivative is only on the spatial $x$ variable, the field is not radially symmetric near blowup. Approximate minimal power thresholds are found that cause the field to first contract through focusing as well as those that cause a blowup. These thresholds are given for all values of $\alpha \in [0, 2]$. Propagation of non symmetric initial conditions are investigated. A radially symmetric blowup is approached as the power is increased to shift into a more nonlinear regime. On a final note, we see the solution near blowup does not appear to follow the loglog law for most fractional exponents ($\alpha < 1.5$). This work was supported by the US National Science Foundation, grant number 1909559.
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