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Uncertainty Quantification of Nonreflecting Boundary Schemes

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UNCERTAINTY QUANTIFICATION OF
NONREFLECTING BOUNDARY SCHEMES

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Numerical methods have been developed to solve partial differential equations involving the far-field radiation of waves. In addition, there has been recent interest in uncertainty quantification- a burgeoning field involving solving PDEs where random variables are used to model uncertainty in the data. In this thesis we will apply uncertainty quantification methodology to the 1D and 2D wave equation with nonreflecting boundary. We first derive a boundary condition for the 1D wave equation assuming several models of the random wave speed. Later we use our result to compare to an asymptotic SDE approach, and finally we repeat our analysis for the 2D wave equation, providing numerical results for each.

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This is dedicated to my family and my advisor for their patience and support.

Chapter 1

INTRODUCTION

1.1. Far-Field Wave Propagation

Radiation to the far-field is an important feature in many applications of wave phenomena. This arises in many contexts, whether it be acoustic, electromagnetic, or quantum mechanical, and in many different geometries. The common feature is that the corresponding PDEs which describe these phenomena must be equipped with a boundary condition "at infinity" describing the eventual behavior of the waves in the far field. This is in conflict with the need to simulate such problems in a finite domain.

In order to resolve this issue, methods have been developed which introduce an artificial boundary along the region of interest, and prescribe appropriate boundary conditions to the artificial boundary. This allows one to simulate in a finite region the behavior of the wave as if a boundary were not present. Several novel methods have been introduced which limit the added computational complexity, as well as error that manifests (undesired) reflection at the artificial interface.

An overview of some of the main developments in the development of nonreflecting boundary conditions is described in detail in [11]. The most ubiquitous model of wave propagation, the scalar wave equation, is studied in detail for the cases of planar, spherical, and cylindrical artificial boundaries. Results are extended to other models of wave phenomena, including the dispersive wave equation, general first-order hyperbolic systems, Maxwell's equations and the equations of linear elasticity.

This situation is illustrated by Figure (1.1). In the figure, Ω is the finite region where the solution is to be computed, Σ is the unbounded region outside of Ω, and Γ is the computational "nonreflecting" boundary.

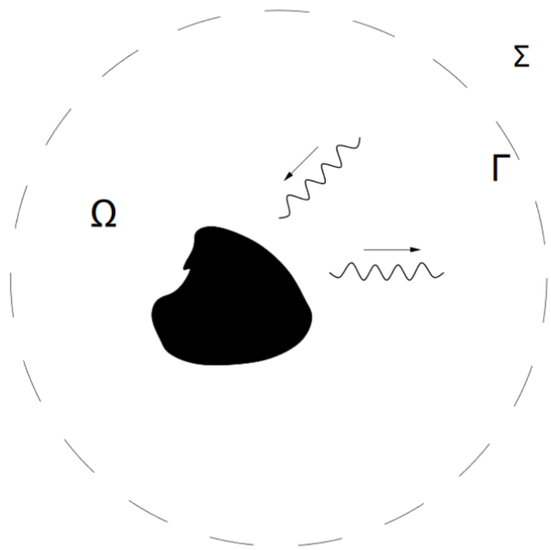

Figure 1.1. Nonreflecting Boundary Scheme

Although no physical boundary is present, it is necessary computationally to develop conditions on Γ such that waves travel through Γ without reflection. Given a second order initial-boundary value problem with Dirichlet data on Γ and zero initial conditions and forcing in the exterior region Σ, a unique causal solution u can be determined. This in turn uniquely determines the Neumann data $\frac{\partial u}{\partial n}$ on the boundary Γ. This defines the Dirichlet to Neumann (DtN) map $\mathcal{D}$. First taking the Laplace transform of u, defined for $s \in \mathbb{C}$ as

$$\hat{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt,$$

the DtN map is a linear operator parametrized by s and we write

$$\frac{\partial \hat{u}}{\partial n} = -\hat{\mathcal{D}}\hat{u}, \qquad x \in \Gamma. \tag{1.1}$$

After finding $\hat{\mathcal{D}}$ in the problem of interest, the exact radiation condition to be used or approximated in the simulation is then obtained by taking the inverse Laplace transform,

$$\frac{\partial u}{\partial n} + \mathcal{L}^{-1}(\hat{\mathcal{D}}\mathcal{L}u) = 0, \qquad x \in \Gamma. \tag{1.2}$$

The main focus of this thesis is the wave equation, which in one space dimension is given by

$$\frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial t^2} = f(x,t), \quad x \in [-L, L], \quad t \in [0, T], \tag{1.3}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x).$$

The boundary is then the points $\Gamma = \{-L, L\}$. In the case of constant coefficients, the DtN map has a simple form. Taking the Laplace transform in time, we can reduce equation(1.1) with constant c to an ODE given by

$$c^2\frac{d^2\hat{u}}{dx^2}(x,s) - s^2\hat{u}(x,s) = \hat{f}(x,s) - v_0(x) - su_0(x) \tag{1.4}$$

In the far field $|x| \geq L$ we assume the source term $\hat{f}(x,s) = 0$ and the initial conditions $u_0(x) = v_0(x) = 0$. Noting that in this simple geometry

$$\frac{\partial \hat{u}}{\partial n}(L,s) = \frac{\partial \hat{u}}{\partial x}(L,s), \qquad \frac{\partial \hat{u}}{\partial n}(-L,s) = -\frac{\partial \hat{u}}{\partial x}(-L,s),$$

we can insert equation (1.1) into equation (1.1) to obtain

$$\hat{D}(s) = \frac{s}{c}.$$

Therefore equation (1.1) for the 1D Wave equation is

$$\frac{\partial u}{\partial x}(L,t) + \frac{1}{c}\frac{\partial u}{\partial t}(L,t) = 0, \qquad \frac{\partial u}{\partial x}(-L,t) - \frac{1}{c}\frac{\partial u}{\partial t}(-L,t) = 0. \tag{1.5}$$

Exact nonreflecting boundary conditions for the wave equation in two dimensions,

$$u_{tt} = c^2 \Delta u \tag{1.6}$$

have been constructed in [2]. The solution $u(x, y, t)$ satisfies zero initial conditions on the exterior region

$$u(x, y, t) = u_t(x, y, t) = 0, \qquad t \leq 0, \quad (x, y) \in \Sigma$$

and to determine the DtN map we assume that $u(x, y, t)$ is known for $(x, y, t) \in \Gamma$ and $t > 0$. For the case where the boundary Γ (referring to Figure 1.1) are the planes $x = \pm a$, we can write the nonreflecting boundary condition by taking the Fourier transformation in the variable y,

$$u(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(x, \eta, t) e^{iy\eta} d\eta,$$

and the Laplace transformation in time

$$\tilde{u}(x, \eta, s) = \int_0^{\infty} \hat{u}(x, \eta, t) e^{-st} dt.$$

This leads to

$$\hat{D}(s, \eta) = [(s/c)^2 + \eta^2]^{1/2}$$

and therefore

$$\pm \tilde{u}_x + \sqrt{(s/c)^2 + \eta^2}\,\tilde{u} = 0, \quad x = \pm a, \tag{1.7}$$

which upon taking the inverse Laplace transform and inverse Fourier transform is the analogue of equation (1.1) in two dimensions.

1.2. Summary of Results

For the 1D Wave Equation, whose setting is illustrated in Figure (1.2), we have extended the nonreflecting boundary conditions given by equation (1.1) to equation (2.1), which contains a term which approximates the reflections brought about by small perturbations to the wave speed in the far field. The stochastic process used to represent the small fluctuations in the wave speed has expansion given by equation (2.1). Using this boundary condition, we have devised a numerical experiment to determine the accuracy of the random boundary condition in Section (2.3). In addition, we have performed a preliminary experiment to determine the mean and variance of some selected functionals of the solution.

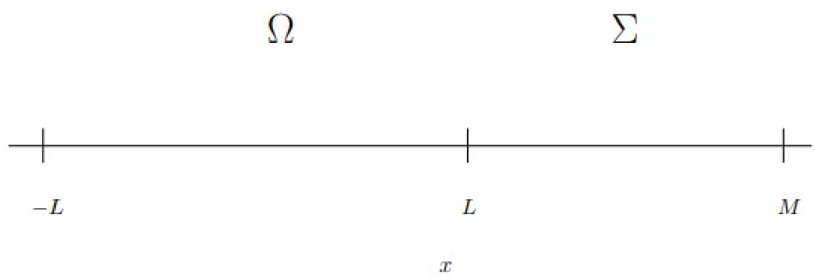

Figure 1.2. Setting for 1D Wave Equation

Similarly, for the 2D Wave Equation, whose setting is illustrated in Figure (1.3), we have extended the nonreflecting boundary condition given by equation (1.1) by calculating an additional term which approximates reflections made by small perturbations in the wave speed in the far field. The small perturbations in the far-field are modeled by the expansion in equation (5.1) and the resulting boundary condition is given by equation (5.1),

Lastly, we have devised an experiment to compare the random boundary approach developed in this thesis to an asymptotic analysis of wave propagation through a random medium in [8]. To set up the comparison, we have derived the random boundary condition in equation (3.1) using a stationary process with expansion derived in Section (3.1). An experiment to test the accuracy of this method is described in Section (3.2). A proposed experiment to

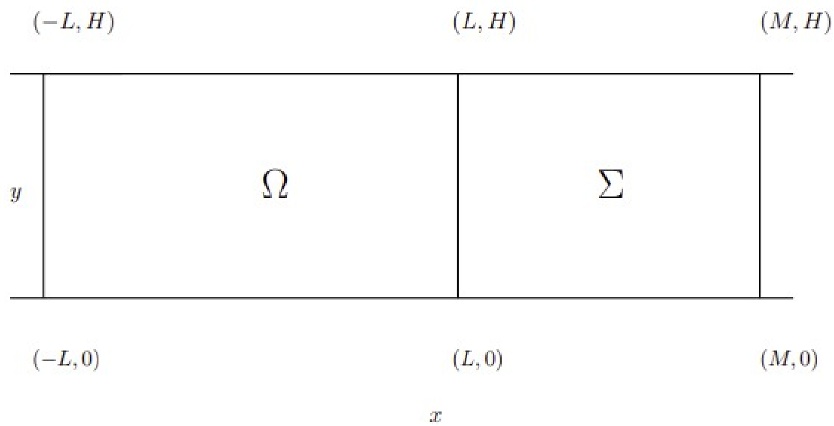

Figure 1.3. Setting for 2D Wave Equation

compare the random boundary to the asymptotic approach is set up in Section (4.5).

1.3. Uncertainty in the Far-Field

All of the results on exact radiation conditions mentioned above, in particular the calculations leading to (1.5) and (1.7), are based on the assumption that c is constant in the far-field Σ. If this assumption is relaxed very little has been done. Boundary conditions have been proposed based on high-frequency asymptotics (e.g. [6,7]) or in the case of decaying potentials [18], but there is no general theory. Moreover, in practical applications, for example wave propagation in the earth or ocean, the wave speed in the exterior region will be uncertain.

The primary contribution of this thesis is to develop a systematic approach to compute accurate radiation conditions for the wave equation where the wave speed $c(x)$ is variable in the far field. Since the precise wave speed may be unknown, we may model the wave speed as a random process $c(x,\omega)$, where $\omega \in \Omega_1$ and Ω_1 is a sample space.

We assume more specifically that the wave speed takes the following form

$$c(x,\omega) = c_\infty + \tilde{c}(x,\omega)$$

where c_∞ is the expected value of the wave speed in the far field and $\tilde{c}$ is a small perturbation. In particular we assume that almost everywhere and almost surely

$$\frac{|\tilde{c}(x,\omega)|}{c_\infty} \ll 1.$$

This in turn implies (again almost everywhere and almost surely) that there are positive constants c_0 and c_1 such that

$$c_0 \leq c(x,\omega) \leq c_1.$$

Moreover we will choose $\tilde{c}$ to be a square-integrable zero-mean stochastic process on a closed interval $[L, M]$ with covariance function $C(s,t)$ such that it may be represented in a series of eigenfunctions, see [9].

$$\tilde{c}(x,\omega) = \sum_{j=1}^{\infty} \sqrt{\nu_j}\phi_j(x)\xi_j(\omega)$$

where ν and ϕ are the eigenvalues and normalized eigenfunctions of the operator

$$\begin{aligned} T_C : L^2([L,M]) &\to L^2([L,M]) \\ f \mapsto T_C f &= \int_L^M C(s,\cdot)f(s)ds \end{aligned}$$

and ξ_j are zero-mean, uncorrelated random variables given by

$$\xi_j(\omega) = \frac{1}{\sqrt{\nu_j}} \int_L^M \tilde{c}(x,\omega)\phi_j(x)dx.$$

In the analysis that follows we will truncate the expansion to P terms and consider the fluctuations to the wave speed as simply a sum of P independent random variables:

$$\tilde{c}(x,\omega) = \sum_{j=1}^{P} \sqrt{\nu_j}\phi_j(x)\xi_j(\omega)$$

1.4. Contents

In Chapter 2, we will find boundary conditions for the one-dimensional wave equation when there are random small fluctuations in the wave speed in the far-field. A linear and quadratic approximation to a Ricatti equation will be used to obtain a closed-form result. Numerical results will then be presented to show consistency of the method and a Monte-Carlo simulation will be performed to study the variability of the solution with respect to the choice of the perturbation $\tilde{c}$.

In Chapter 3 we repeat the analysis in Chapter 2 for a different process, which has special statistical properties.

In Chapter 4 we review results presented by Papanicolaou et al. in [8] which provides a different approach to the problem of wave propagation and reflection through a random medium. The design of a numerical experiment is proposed to compare the results of Chapter 3 to the asymptotic approach.

In Chapter 5 we conclude by extending the analysis to the two-dimensional wave equation, and discuss the challenges thereof.

Chapter 2

1D WAVE EQUATION WITH RANDOM BOUNDARY CONDITIONS

2.1. Derivation of Random Boundary Condition for 1D Wave Equation

We will now develop a boundary condition to account for the far-field radiation of the 1D wave equation with variable wave speed, given by

$$\frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial t^2} = f(x,t), \quad x \in [-L, L], \quad t \in [0, T]$$

with initial conditions given by

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x).$$

Taking the Laplace transformation in time, we have

$$\frac{d}{dx}\left(c^2(x)\frac{d\hat{u}}{dx}\right) - s^2\hat{u} = \hat{f}(x) - v_0(x) - su_0(x), \quad x \in [-L, L]. \tag{2.1}$$

For simplicity we assign a Dirichlet boundary condition at $x = -L$. Therefore the exterior region is $\Sigma = [L, \infty)$ and the boundary is the point $\Gamma = \{L\}$. See Figure (2.1). We are interested in computing the DtN map for the case of variable wave speed in Σ. We also treat the random process with sample space Ω_1 so that

$$c(x,\omega) = c_\infty + \tilde{c}(x,\omega), \quad x \in \Sigma, \quad \omega \in \Omega_1$$

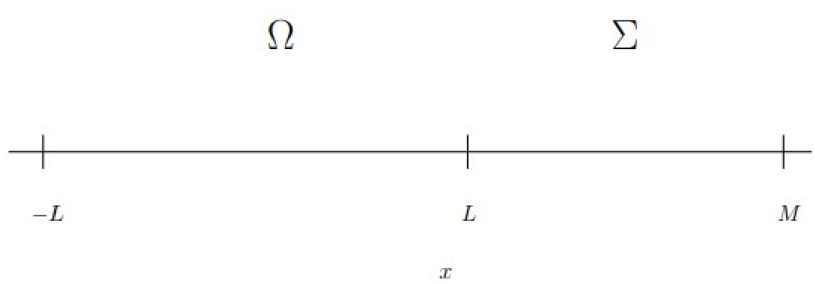

Figure 2.1. Setting for 1D Wave Equation. $\tilde{c} = 0$ for $x \in \Omega$

where $\tilde{c}(x, \omega)$ is a small random perturbation and c_∞ is deterministic and constant. The DtN map $\hat{\mathcal{D}}$ is a scalar function $\sigma(L, s, \omega)$ of the Laplace parameter s, satisfying

$$\hat{u}_x(x, s, \omega) + \sigma(x, s, \omega)\hat{u}(x, s, \omega) = 0. \tag{2.2}$$

The outside region $\Sigma = (L, \infty)$ does not contain the source f, so inserting the expression for σ into the 1D wave equation yields

$$\left(\frac{d}{dx}(c^2(x, \omega)\sigma(x, s, \omega)) - c^2(x, \omega)\sigma^2(x, s, \omega) - s^2\right)\hat{u}(x, s, \omega) = 0, \quad x \in \Sigma$$

for all radiating solutions $\hat{u}$. This implies the following Ricatti equation for σ in the exterior domain:

$$\frac{d}{dx}(c^2(x, \omega)\sigma(x, s, \omega)) = c^2(x, \omega)\sigma^2(x, s, \omega) - s^2, \quad x \in \Sigma. \tag{2.3}$$

We wish to linearize this equation in order to obtain a simple closed-form expression for σ. In the case that the wave speed $c(x) = c_\infty$ is constant, we showed in Chapter 1 that the DtN map takes the form $\sigma(s) = s/c_\infty$. Anticipating that σ will be a small perturbation of s/c_∞ for $\tilde{c}$ small we write

$$\sigma(x, s, \omega) = s/c_\infty + \tilde{\sigma}(x, s, \omega), \qquad x \in [L, M],$$

for some $M > L$. Recall from the introduction we have assumed that $c(x, \omega) \leq c_1$ almost everywhere and almost surely. Suppose now we are only interested in simulations up to some finite time T. Then if $M > L + c_1 T$ no wave can reach $x = M$ in the simulation time and we can assume that $\tilde{c} = 0$ for $x \geq M$. Then $\tilde{\sigma}(M, \omega) = 0$. Following our assumption that $\tilde{c}$ is small compared with c_∞ we will assume that $\tilde{\sigma}$ is small and, to first approximation, approximate it by linearizing equation (2.1). We seek to find $\tilde{\sigma}$, the contribution to the DtN map caused by the small perturbations $\tilde{c}$. Using the above linearization leads to the ODE for $\tilde{\sigma}$ given by

$$\begin{aligned} \frac{d\tilde{\sigma}}{dx}(x, s, \omega) - \frac{2s}{c_\infty}\tilde{\sigma}(x, s, \omega) &= -\frac{2s}{c_\infty^2}\frac{d\tilde{c}}{dx}(x, \omega) + \frac{2s^2}{c_\infty^3}\tilde{c}(x, \omega) \\ \tilde{c}(L, \omega) = \tilde{c}(M, \omega) &= 0 \\ \tilde{\sigma}(M, s, \omega) &= 0 \end{aligned} \tag{2.4}$$

which has solution

$$\tilde{\sigma}(L, s, \omega) = \frac{2s^2}{c_\infty^3}\int_L^M e^{-\frac{2s}{c_\infty}(z-L)}\tilde{c}(z, \omega)dz. \tag{2.5}$$

To complete the analysis, we need to choose a model for the random fluctuations in the wave speed. In order to obtain an analytical result we choose $\tilde{c}$ to be a stochastic process having the following expansion.

$$\tilde{c}(x, \omega) = \sum_{j=1}^{\infty}\frac{1}{\pi^2 j^r}\sin\left(\frac{j\pi(x-L)}{M-L}\right)\xi_j(\omega), \qquad \xi_j \sim U(-1, 1), \quad x \in [L, M]. \tag{2.6}$$

The boundary conditions are trivially satisfied due to the choice of the eigenfunction, and the process has negative drift, so that with high probability the process stays close to the mean value of 0. The parameter r controls the regularity of the process. For $r = 0$ we have white noise, for $r = 1$ we have a bridge process which is continuous but nowhere differentiable. In general, the process $\tilde{c}(x, \omega)$ is $r - 1$-times differentiable. The process above is therefore a convenient choice for experimentation since the regularity is controlled by a single parameter r. Further details are in [9]. Sample paths for different values of r are given below in Figure (2.2).

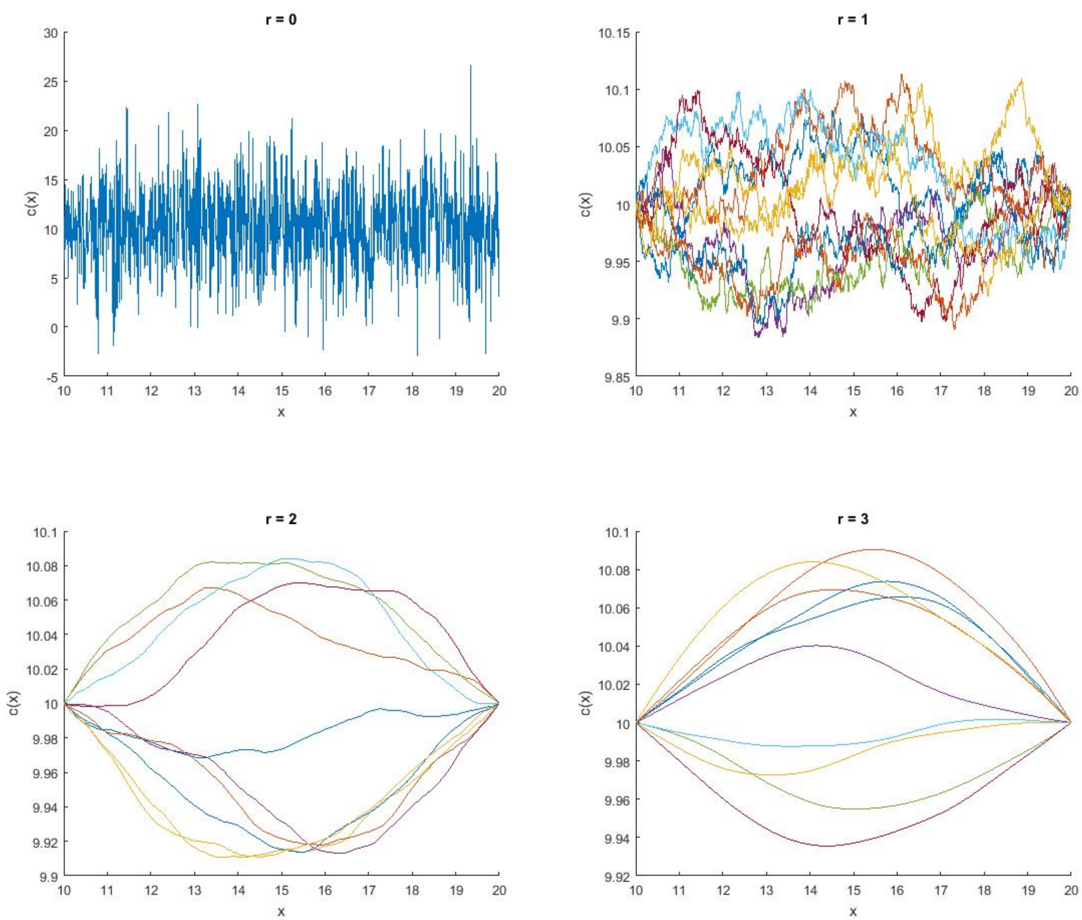

Figure 2.2. Sample Paths for Process, truncated to $P = 1000$ terms

Inserting the expression for $\tilde{c}$ into the integral in equation (2.1) for $\tilde{\sigma}$ and truncating the expansion after P terms yields

$$\tilde{\sigma}(L,s,\omega) = \frac{1}{2(M-L)c_\infty \pi} \sum_{j=1}^{P} \frac{1}{j^{r-1}} \frac{s^2}{s^2+B_j^2}[1-(-1)^j e^{-2s(M-L)/c_\infty}]\xi_j(\omega) \tag{2.7}$$

so that, in the Laplace domain, we have

$$\frac{d\hat{u}}{dx}(L,s,\omega)+\frac{s}{c_\infty}\hat{u}(L,s,\omega)+\frac{1}{2c_\infty\pi(M-L)}\sum_{j=1}^{P}\frac{1}{j^{r-1}}\frac{s^2\hat{u}(L,s,\omega)}{s^2+B_j^2}[1-(-1)^j e^{-2s(M-L)/c_\infty}]\xi_j(\omega)=0,$$

where $B_j = \frac{\pi c_\infty}{2(M-L)} j$. Since we are in the Laplace domain, we note that the $(-1)^j e^{-2s(M-L)/c_\infty}$ term can be neglected, since taking the inverse Laplace transform would invoke the identity

$$\mathcal{L}^{-1}(\hat{u}(x,s)e^{-2s(M-L)/c_\infty}) = u(x, t-2(M-L)/c_\infty) = 0. \tag{2.8}$$

for $t \leq T$ as by assumption $t - 2(M-L)/c_\infty < 0$. Therefore, taking the inverse Laplace transform to return to the time domain, we obtain

$$\frac{\partial u}{\partial x}(L,t,\omega) + \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t,\omega) + \frac{1}{2c_\infty \pi}\sum_{j=1}^{P}\frac{1}{j^{r-1}}\frac{u(L,t,\omega)-\phi_j(t,\omega)}{M-L}\xi_j(\omega) = 0, \tag{2.9}$$

where we introduce the auxiliary variable ϕ_j, satisfying

$$\begin{aligned} \frac{d^2\phi_j}{dt^2}(t,\omega) + B_j^2\phi_j(t,\omega) &= B_j^2 u(L,t,\omega), \\ \phi_j(0) = \phi_j'(0) = 0, &\qquad j = 1,2,\dots. \end{aligned} \tag{2.10}$$

The summation term in equation (2.1) approximates the contribution of the random fluctuations of the wave speed in the exterior domain to the DtN map.

2.2. Quadratic Approximation

We will also derive the DtN map $\tilde{\sigma}$ with a quadratic approximation to the random wave speed fluctuations. Starting with the Ricatti Equation derived above, equation (2.1)

$$\frac{d}{dx}(c^2(x,\omega)\sigma(x,s,\omega)) = c^2(x,\omega)\sigma^2(x,s,\omega) - s^2, \quad x \in \Sigma.$$

and writing the wave speed and DtN map as above,

$$c(x,\omega) = c_\infty + \tilde{c}(x,\omega), \qquad \sigma(x,s,\omega) = \frac{s}{c_\infty} + \tilde{\sigma}(x,s,\omega),$$

and inserting into the Ricatti equation (2.1), this time keeping the quadratic terms we obtain

$$\tilde{\sigma}_x + \frac{2s}{c_\infty^2}\tilde{c}_x + \frac{2}{c_\infty}(\tilde{c}\tilde{\sigma})_x + \frac{s}{c_\infty^3}(\tilde{c}^2)_x = \frac{2s^2\tilde{c}}{c_\infty^3} + \frac{2s}{c_\infty}\tilde{\sigma} + \frac{4s\tilde{\sigma}\tilde{c}}{c_\infty^2} + \tilde{\sigma}^2 + \frac{s^2}{c_\infty^4}\tilde{c}^2.$$

Rearranging we introduce the recursion

$$\begin{aligned}\tilde{\sigma}_x^{(n+1)} - \frac{2s}{c_\infty}\tilde{\sigma}^{(n+1)} = -\frac{2s}{c_\infty^2}\tilde{c}_x + \frac{2s^2\tilde{c}}{c_\infty^3} - \frac{2\tilde{c}\tilde{\sigma}_x^{(n)}}{c_\infty} - \frac{2\tilde{c}_x\tilde{\sigma}^{(n)}}{c_\infty} + \frac{4s\tilde{\sigma}^{(n)}\tilde{c}}{c_\infty^2} + (\tilde{\sigma}^{(n)})^2 \\ + \frac{s^2}{c_\infty^4}\tilde{c}^2 - \frac{s}{c_\infty^3}(\tilde{c}^2)_x, \quad n \geq 0\end{aligned} \tag{2.11}$$

starting with $\tilde{\sigma}^{(1)}$ satisfying the linearized problem (2.1). In the case $n = 0$ then, we have the linear case again

$$\tilde{\sigma}_x^{(1)} - \frac{2s}{c_\infty}\tilde{\sigma}^{(1)} = \frac{2s^2}{c_\infty^3}\tilde{c} - \frac{2s}{c_\infty^2}\tilde{c}_x.$$

with solution given by

$$\tilde{\sigma}^{(1)}(x,s,\omega) = \frac{2s^2}{c_\infty^3}\int_x^M e^{-2s(z-x)/c_\infty}\tilde{c}(z,\omega)dz - \frac{2s}{c_\infty^2}\tilde{c}(x,\omega)$$

Using the same representation for $\tilde{c}$ used in the previous section

$$\tilde{c}(x,\omega) = \sum_{j=1}^{P} \frac{1}{\pi^2 j^r} \sin\left(\frac{j\pi(x-L)}{M-L}\right)\xi_j(\omega), \qquad \xi_j(\omega) \sim U(-1,1), \quad x \in [L,M] \tag{2.12}$$

we calculate for $L < x < M$ that

$$\tilde{\sigma}^{(1)}(x,s,\omega) = \sum_{j=1}^{P} \frac{2s^2\xi_j}{c_\infty^3 \pi^2 j^r} \int_x^M e^{-2s(z-x)/c_\infty} \sin\left(\frac{j\pi(z-L)}{M-L}\right) dz - \frac{2s}{c_\infty^2}\tilde{c}(x,\omega).$$

Defining $A_j = \frac{j\pi(z-L)}{M-L}$,

$$\int_x^M e^{-2s(z-x)/c_\infty} \sin\left(\frac{j\pi(z-L)}{M-L}\right) dz = \frac{e^{-2s(M-x)/c_\infty}(-1)^{j+1}\frac{j\pi c_\infty^2}{4(M-L)} + \frac{c_\infty s}{2}\sin(A_j)}{s^2 + \frac{j^2\pi^2 c_\infty^2}{4(M-L)^2}} + \frac{\frac{j\pi c_\infty^2}{4(M-L)}\cos(A_j)}{s^2 + \frac{j^2\pi^2 c_\infty^2}{4(M-L)^2}}$$

Thus,

$$\tilde{\sigma}^{(1)}(x,s,\omega) = \sum_{j=1}^{P} D_j s^2 \frac{e^{-2s(M-x)/c_\infty}(-1)^{j+1}\frac{j\pi c_\infty^2}{4(M-L)} + \frac{c_\infty s}{2}\sin(A_j) + \frac{j\pi c_\infty^2}{4(M-L)}\cos(A_j)}{s^2 + C_j^2} - \frac{2s}{c_\infty^2}\tilde{c}(x,\omega),$$

where $C_j = \frac{j^2\pi^2 c_\infty^2}{4(M-L)^2}$, $D_j = \frac{2\xi_j}{c_\infty^3 \pi^2 j^r}$. Plugging in $x = L$ gives equation (2.1). Now, looking at the recursion in equation (2.2) for $n = 1$, we have

$$\tilde{\sigma}_x^{(2)} + \frac{2s}{c_\infty^2}\tilde{c}_x - \frac{2s}{c_\infty}\tilde{\sigma}^{(2)} = \frac{2s^2\tilde{c}}{c_\infty^3} - \frac{2\tilde{c}\tilde{\sigma}_x^{(1)}}{c_\infty} - \frac{2\tilde{c}_x\tilde{\sigma}^{(1)}}{c_\infty} + \frac{4s\tilde{\sigma}^{(1)}\tilde{c}}{c_\infty^2} + (\tilde{\sigma}^{(1)})^2 + \frac{s^2}{c_\infty^4}\tilde{c}^2 - \frac{s}{c_\infty^3}(\tilde{c}^2)_x$$

The solution is after some integration by parts,

$$\tilde{\sigma}^{(2)}(L,s,\omega) = \frac{2s^2}{c_\infty^3}\int_L^M e^{-2s(z-L)/c_\infty}\tilde{c}(z,\omega)dz - \int_L^M e^{-2s(z-L)/c_\infty}(\tilde{\sigma}^{(1)})^2 dz + \frac{s^2}{c_\infty^4}\int_L^M \tilde{c}^2(z,\omega)e^{-2s(z-L)/c_\infty}dz \tag{2.13}$$

The first integral of equation (2.2) has been previously evaluated:

$$\frac{2s^2}{c_\infty^3}\int_L^M e^{-2s(z-L)/c_\infty}\tilde{c}(z,\omega)dz = \frac{1}{2\pi(M-L)c_\infty}\sum_{j=1}^{P}\frac{\xi_j(\omega)}{j^{r-1}}\frac{s^2}{s^2+C_j^2}.$$

The second integral of equation (2.2) is upon expansion

$$\begin{aligned}
&\int_L^M e^{-2s(z-L)/c_\infty}(\tilde{\sigma}^{(1)})^2 dz = \\
&\sum_{j=1}^{P}\frac{D_j^2 s^4}{(s^2+C_j^2)^2}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)+E_j(-1)^{j+1}e^{-2s(M-z)/c_\infty}\right)^2 e^{-2s(z-L)/c_\infty}dz \\
&+\sum_{j\neq k}^{P}\frac{D_jD_k s^4}{(s^2+C_j^2)(s^2+C_k^2)}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)+E_j(-1)^{j+1}e^{-2s(M-z)/c_\infty}\right) \\
&\qquad\times\left(E_k\cos(A_k)+Fs\sin(A_k)+E_k(-1)^{k+1}e^{-2s(M-z)/c_\infty}\right)e^{-2s(z-L)/c_\infty}dz \qquad (2.14)\\
&-\sum_{j=1}^{P}\frac{2D_j s^3}{c_\infty^3(s^2+C_j^2)}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)+E_j(-1)^{j+1}e^{-2s(M-x)/c_\infty}\right)\tilde{c}e^{-2s(z-L)/c_\infty}dz \\
&+\frac{s^2}{c_\infty^6}\int_L^M \tilde{c}^2(z,\omega)e^{-2s(z-L)/c_\infty}dz
\end{aligned}$$

where $D_j = \frac{\xi_j}{c_\infty^4\pi^2 j^r}$, $E_j = \frac{j\pi c_\infty^2}{4(M-L)}$, $F = \frac{c_\infty}{2}$. Upon integration, the $e^{-2s(M-z)/c_\infty}(-1)^{j+1}E_j$ terms lead to the factor $e^{-2s(M-L)/c_\infty}$ which as in equation (2.1) can be neglected. Therefore, we can simplify equation (2.2) as

$$
\begin{aligned}
\int_L^M e^{-2s(z-L)/c_\infty}(\tilde\sigma^{(1)})^2 dz &= \sum_{j=1}^{P}\frac{D_j^2 s^4}{(s^2+C_j^2)^2}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)\right)^2 e^{-2s(z-L)/c_\infty}dz\\
&+\sum_{j\neq k}^{P}\frac{D_jD_k s^4}{(s^2+C_j^2)(s^2+C_k^2)}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)\right)\\
&\qquad\qquad\times\left(E_k\cos(A_k)+Fs\sin(A_k)\right)e^{-2s(z-L)/c_\infty}dz \qquad (2.15)\\
&-\sum_{j=1}^{P}\frac{2D_js^3}{c_\infty^3(s^2+C_j^2)}\int_L^M \left(E_j\cos(A_j)+Fs\sin(A_j)\right)\tilde c(z)e^{-2s(z-L)/c_\infty}dz\\
&+\frac{s^2}{c_\infty^6}\int_L^M \tilde c^2(z,\omega)e^{-2s(z-L)/c_\infty}dz
\end{aligned}
$$

Anticipating that we will use the identity (2.1) upon taking the inverse Laplace transform, we have with $a(j)=\frac{c_\infty^2 j\pi}{M-L}$, $b(j)=\frac{c_\infty^2 j^2\pi^2}{(M-L)^2}$ the following integral calculations

$$
\begin{aligned}
\int_L^M \sin(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{a(j)/4}{s^2+b(j)/4}\\
\int_L^M \cos(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{c_\infty s/2}{s^2+b(j)/4}\\
\int_L^M \sin(A_j(z))\sin(A_k(z))e^{-2s(z-L)/c_\infty}dz &= \frac{c_\infty s/4}{s^2+b(j-k)/4}-\frac{c_\infty s/4}{s^2+b(j+k)/4}\\
\int_L^M \cos(A_j(z))\cos(A_k(z))e^{-2s(z-L)/c_\infty}dz &= \frac{c_\infty s/4}{s^2+b(j-k)/4}+\frac{c_\infty s/4}{s^2+b(j+k)/4}\\
\int_L^M \sin(A_j(z))\cos(A_k(z))e^{-2s(z-L)/c_\infty}dz &= \frac{a(j+k)/8}{s^2+b(j+k)/4}+\frac{a(j-k)/8}{s^2+b(j-k)/4}\\
\int_L^M \sin(A_k(z))\cos(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{a(j+k)/8}{s^2+b(j+k)/4}-\frac{a(j-k)/8}{s^2+b(j-k)/4}\\
\int_L^M \sin^2(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{c_\infty}{4s}-\frac{c_\infty s/4}{s^2+b(2j)/4}\\
\int_L^M \cos^2(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{c_\infty}{4s}+\frac{c_\infty s/4}{s^2+b(2j)/4}\\
\int_L^M \sin(A_j(z))\cos(A_j(z))e^{-2s(z-L)/c_\infty}dz &= \frac{a(2j)/8}{s^2+b(2j)/4}
\end{aligned}
$$

$$
\int_L^M \tilde c^2(z)e^{-2s(z-L)/c_\infty}dz = \sum_{j,k=1}^{P}\frac{\xi_j\xi_k}{\pi^4 j^r k^r}\left(\frac{c_\infty s/4}{s^2+b(j-k)/4}-\frac{c_\infty s/4}{s^2+b(j+k)/4}\right)
$$

$$\int_L^M \tilde{c}(z)\sin(A_j(z))e^{-2s(z-L)/c_\infty}dz = \sum_{k=1}^{P} \frac{\xi_k}{\pi^2 k^r}\left(\frac{c_\infty s/4}{s^2+b(j-k)/4} - \frac{c_\infty s/4}{s^2+b(j+k)/4}\right)$$
$$\int_L^M \tilde{c}(z)\cos(A_j(z))e^{-2s(z-L)/c_\infty}dz = \sum_{k=1}^{P} \frac{\xi_k}{\pi^2 k^r}\left(\frac{a(j+k)/8}{s^2+b(j+k)/4} - \frac{a(j-k)/8}{s^2+b(j-k)/4}\right)$$

which upon insertion into equation (2.2) gives

$$\begin{aligned}
\int_L^M e^{-2s(z-L)/c_\infty}(\tilde{\sigma}^{(1)})^2 dz &= \sum_{j,k=1}^{P} D_j D_k \left[E_j E_k (R_1^{j-k} + R_1^{j+k}) + E_j F (R_2^{j+k} - R_2^{j-k})\right] \\
&+ \sum_{j,k=1}^{P} D_j D_k \left[E_k F (R_2^{j+k} + R_2^{j-k}) + s^2 F^2 (R_1^{j-k} - R_1^{j+k})\right] \\
&- \frac{2}{c_\infty^3}\sum_{j,k=1}^{P} \frac{D_j \xi_k}{\pi^2 k^r}\left[E_j (R_3^{j+k} - R_3^{j-k}) + F(R_4^{j-k} - R_4^{j+k})\right] \\
&+ \frac{1}{c_\infty^6}\sum_{j,k=1}^{P} \frac{\xi_j \xi_k}{\pi^4 j^r k^r}(R_5^{j-k} - R_5^{j+k})
\end{aligned}$$

where, since $b(l)/4 = C_l^2$,

$$R_1^l(s) = \frac{c_\infty s^5/4}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_l^2)}, \qquad R_2^l(s) = \frac{s^5 a(l)/8}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_l^2)} \tag{2.16}$$
$$R_3^l(s) = \frac{s^3 a(l)/8}{(s^2+C_j^2)(s^2+C_l^2)}, \qquad R_4^l(s) = \frac{c_\infty s^5/4}{(s^2+C_j^2)(s^2+C_l^2)}, \quad R_5^l(s) = \frac{c_\infty s^3/4}{s^2+C_l^2}.$$

We reduce the order of s in the numerators of equation (2.2) by

$$
\begin{aligned}
\frac{s^5}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_l^2)} &= \frac{s}{s^2+C_j^2} - (C_k^2+C_l^2)\frac{s}{(s^2+C_j^2)(s^2+C_k^2)} \\
&\quad + C_l^4 \frac{s}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_l^2)} \\
&\quad + (C_k^4+C_k^2C_l^2+C_l^4)\frac{s}{(s^2+C_j^2)(s^2+C_k^2)} \\
&\quad - C_l^6 \frac{s}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_l^2)} \\
\frac{s^3}{(s^2+C_j^2)(s^2+C_l^2)} &= \frac{s}{s^2+C_j^2} - C_l^2 \frac{s}{(s^2+C_j^2)(s^2+C_l^2)} \\
\frac{s^5}{(s^2+C_j^2)(s^2+C_l^2)} &= s - (C_j^2+C_l^2)\frac{s}{s^2+C_j^2} + C_l^4 \frac{s}{(s^2+C_j^2)(s^2+C_l^2)} \\
\frac{s^3}{s^2+C_l^2} &= s - C_l^2 \frac{s}{s^2+C_l^2}
\end{aligned}
$$

Using partial fractions, we have

$$
\begin{aligned}
\frac{1}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_{j-k}^2)} &= \frac{J_1(j,k)}{s^2+C_j^2} + \frac{J_2(j,k)}{s^2+C_k^2} + \frac{J_3(j,k)}{s^2+C_{j-k}^2} \\
\frac{1}{(s^2+C_j^2)(s^2+C_k^2)(s^2+C_{j+k}^2)} &= \frac{J_4(j,k)}{s^2+C_j^2} + \frac{J_5(j,k)}{s^2+C_k^2} + \frac{J_6(j,k)}{s^2+C_{j+k}^2} \\
\frac{1}{(s^2+C_j^2)(s^2+C_k^2)} &= \frac{J_7(j,k)}{s^2+C_j^2} + \frac{J_8(j,k)}{s^2+C_k^2} \\
\frac{1}{(s^2+C_j^2)(s^2+C_{j-k}^2)} &= \frac{J_9(j,k)}{s^2+C_j^2} + \frac{J_{10}(j,k)}{s^2+C_{j-k}^2} \\
\frac{1}{(s^2+C_j^2)(s^2+C_{j+k}^2)} &= \frac{J_{11}(j,k)}{s^2+C_j^2} + \frac{J_{12}(j,k)}{s^2+C_{j+k}^2}
\end{aligned}
$$

$$
J_1 = \frac{1}{(C_k^2-C_j^2)(C_{j-k}^2-C_j^2)}, \qquad J_2 = \frac{1}{(C_j^2-C_k^2)(C_{j-k}^2-C_k^2)}
$$

$$
J_3 = \frac{1}{(C_j^2-C_{j-k}^2)(C_k^2-C_{j-k}^2)}, \quad J_4 = \frac{1}{(C_k^2-C_j^2)(C_{j+k}^2-C_j^2)}
$$

$$
J_5 = \frac{1}{(C_j^2-C_k^2)(C_{j+k}^2-C_k^2)}, \quad J_6 = \frac{1}{(C_j^2-C_{j+k}^2)(C_k^2-C_{j+k}^2)}
$$

$$
J_7 = \frac{1}{C_k^2-C_j^2}, \quad J_8 = \frac{1}{C_j^2-C_k^2}, \quad J_9 = \frac{1}{C_{j-k}^2-C_j^2}
$$

$$
J_{10} = \frac{1}{C_j^2-C_{j-k}^2}, \quad J_{11} = \frac{1}{C_{j+k}^2-C_j^2}, \quad J_{12} = \frac{1}{C_j^2-C_{j+k}^2}
$$

When $j = l$, we have

$$\frac{1}{(s^2+C_j^2)^2(s^2+C_k^2)} = \frac{J_{13}}{(s^2+C_j^2)^2} + \frac{J_{14}}{s^2+C_j^2} + \frac{J_{15}}{s^2+C_k^2}$$
$$J_{13} = \frac{1}{C_k^2 - C_j^2}, \quad J_{14} = \frac{1 - C_k^2 J_1 - C_j^4 J_3}{C_j^2 C_k^2}, \quad J_{15} = \frac{1}{(C_j^2 - C_k^2)^2}$$

When $k = l$, we have

$$\frac{1}{(s^2+C_k^2)^2(s^2+C_j^2)} = \frac{J_{16}}{(s^2+C_k^2)^2} + \frac{J_{17}}{s^2+C_k^2} + \frac{J_{18}}{s^2+C_j^2}$$
$$J_{16} = \frac{1}{C_j^2 - C_k^2}, \quad J_{17} = \frac{1 - C_j^2 J_1 - C_k^4 J_3}{C_j^2 C_k^2}, \quad J_{18} = \frac{1}{(C_k^2 - C_j^2)^2}$$

The boundary condition is therefore, in the Laplace domain,

$$\hat{u}_x + \frac{1}{c_\infty} s\hat{u} = \hat{g}$$

where

$$\begin{aligned}
\hat{g} = & \frac{-1}{2\pi(M-L)c_\infty} \sum_{j=1}^{P} \frac{\xi_j}{j^{r-1}} \frac{s^2 \hat{u}(x,s,\omega)}{s^2+C_j^2} \\
& + \sum_{j,k=1}^{P} D_j D_k \Big[E_j E_k (R_1^{j-k} + R_1^{j+k}) + E_j F (R_2^{j+k} - R_2^{j-k}) \Big] \\
& + \sum_{j,k=1}^{P} D_j D_k \Big[E_k F (R_2^{j+k} + R_2^{j-k}) + s^2 F^2 (R_1^{j-k} - R_1^{j+k}) \Big] \hat{u} \\
& - \frac{2}{c_\infty^3} \sum_{j,k=1}^{P} \frac{D_j \xi_k}{\pi^2 k^r} \Big[E_j (R_3^{j+k} - R_3^{j-k}) + F (R_4^{j-k} - R_4^{j+k}) \Big] \hat{u} \\
& \quad + \left(\frac{1}{c_\infty^6} + \frac{1}{c_\infty^4} \right) \sum_{j,k=1}^{P} \frac{\xi_j \xi_k}{\pi^4 j^r k^r} (R_5^{j-k} - R_5^{j+k}) \hat{u}
\end{aligned}$$

Going back to the time domain, we utilize the approach used above in equation (2.1) to deal with the convolution by introducing the auxiliary ODEs

$$\begin{aligned}\frac{du}{dt} &= \frac{d^2\phi_k^1}{dt^2} + C_k^2\phi_k^1\\ u &= \frac{d^2\phi_j^2}{dt^2} + C_j^2\phi_j^2\\ \frac{du}{dt} &= \frac{d^4\phi_j^3}{dt^4} + 2C_j^2\frac{d^2\phi_j^3}{dt^2} + C_j^4\phi_j^3\end{aligned}$$

at the boundary for $k = 1, \ldots, P$, $j = 1, \ldots, 2P$.

Defining

$$\begin{aligned}S_1 &= J_1\phi_j^1 + J_2\phi_k^1 + J_3\phi_{j-k}^1\\ S_2 &= J_4\phi_j^1 + J_5\phi_k^1 + J_6\phi_{j+k}^1\\ S_3 &= J_7\phi_j^1 + J_8\phi_k^1\\ S_4 &= J_9\phi_j^1 + J_{10}\phi_{j-k}^1\\ S_5 &= J_{11}\phi_j^1 + J_{12}\phi_{j+k}^1\\ S_6 &= J_{13}\phi_j^3 + J_{14}\phi_j^1 + J_{15}\phi_k^1\\ S_7 &= J_{16}\phi_k^3 + J_{17}\phi_k^1 + J_{18}\phi_j^1\\ S_8 &= J_{19}\phi_j^3 + J_{20}\phi_j^1 + J_{21}\phi_{2j}^1\\ S_9 &= J_{22}\phi_j^1 + J_{23}\phi_{2j}^1\end{aligned}$$

When $j \neq |j-k|$ and $k \neq |j-k|$, then

$$\begin{aligned}
R_1^{j-k} + R_1^{j+k} &= \frac{c_\infty}{4}\left(2\phi_j^1 - (2C_k^2 + C_{j-k}^2 + C_{j+k}^2)S_3 + C_{j-k}^4 S_1 + C_{j+k}^4 S_2\right) \\
s^2(R_1^{j-k} - R_1^{j+k}) &= \frac{c_\infty}{4}\left((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + (C_{j-k}^4 + C_k^2 C_{j-k}^2 - C_{j+k}^4 - C_k^2 C_{j+k}^2)S_3\right) \\
&\quad - \frac{c_\infty}{4}\left(C_{j-k}^6 S_1 + C_{j+k}^6 S_2\right) \\
R_2^{j+k} - R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2) \\
&\quad - \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_1) \\
R_2^{j+k} + R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2) \\
&\quad + \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_1) \\
R_3^{j+k} - R_3^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - C_{j+k}^2 S_5) - \frac{a(j-k)}{8}(\phi_j^1 - C_{j-k}^2 S_4) \\
R_4^{j-k} - R_4^{j+k} &= \frac{c_\infty}{4}((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + C_{j-k}^4 S_4 - C_{j+k}^4 S_5) \\
R_5^{j-k} - R_5^{j+k} &= \frac{c_\infty}{4}(C_{j+k}^2 \phi_{j+k}^1 - C_{j-k}^2 \phi_{j-k}^1).
\end{aligned}$$

When $j = k$,

$$\begin{aligned}
R_1^{j-k} + R_1^{j+k} &= \frac{c_\infty}{4}\left(2\phi_j^1 - (2C_k^2 + C_{j+k}^2)\phi_j^3 + C_{j+k}^4 S_8\right) \\
s^2(R_1^{j-k} - R_1^{j+k}) &= \frac{c_\infty}{4}\left(C_{j+k}^2 \phi_j^1 - (C_{j+k}^4 + C_k^2 C_{j+k}^2)\phi_j^3 + C_{j+k}^6 S_8\right) \\
R_2^{j+k} - R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)\phi_j^3 + C_{j+k}^4 S_8) \\
R_2^{j+k} + R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)\phi_j^3 + C_{j+k}^4 S_8) \\
R_3^{j+k} - R_3^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - C_{j+k}^2 S_9) \\
R_4^{j-k} - R_4^{j+k} &= \frac{c_\infty}{4}(C_{j+k}^2 \phi_j^1 - C_{j+k}^4 \phi_j^1) \\
R_5^{j-k} - R_5^{j+k} &= \frac{c_\infty}{4} C_{j+k}^2 \phi_{j+k}^1.
\end{aligned}$$

When $j = |j-k|$ then

$$
\begin{aligned}
R_1^{j-k} + R_1^{j+k} &= \frac{c_\infty}{4}\left(2\phi_j^1 - (2C_k^2 + C_{j-k}^2 + C_{j+k}^2)S_3 + C_{j-k}^4 S_6 + C_{j+k}^4 S_2\right) \\
s^2(R_1^{j-k} - R_1^{j+k}) &= \frac{c_\infty}{4}\left((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + (C_{j-k}^4 + C_k^2 C_{j-k}^2 - C_{j+k}^4 - C_k^2 C_{j+k}^2)S_3\right) \\
&\quad + \frac{c_\infty}{4}\left(-C_{j-k}^6 S_6 + C_{j+k}^6 S_2\right) \\
R_2^{j+k} - R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2) \\
&\quad - \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_6) \\
R_2^{j+k} + R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2) \\
&\quad + \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_6) \\
R_3^{j+k} - R_3^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - C_{j+k}^2 S_5) - \frac{a(j-k)}{8}(\phi_j^1 - C_{j-k}^2 \phi_j^3) \\
R_4^{j+k} - R_4^{j-k} &= \frac{c_\infty}{4}((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + C_{j-k}^4 S_4 - C_{j+k}^4 \phi_j^3) \\
R_5^{j-k} - R_5^{j+k} &= \frac{c_\infty}{4}(C_{j+k}^2 \phi_{j+k}^1 - C_{j-k}^2 \phi_{j-k}^1).
\end{aligned}
$$

When $k = |j-k|$, then

$$
\begin{aligned}
R_1^{j-k} + R_1^{j+k} &= \frac{c_\infty}{4}\left(2\phi_j^1 - (2C_k^2 + C_{j-k}^2 + C_{j+k}^2)S_3 + C_{j-k}^4 S_7 + C_{j+k}^4 S_2\right)\\
s^2(R_1^{j-k} - R_1^{j+k}) &= \frac{c_\infty}{4}\left((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + (C_{j-k}^4 + C_k^2 C_{j-k}^2 - C_{j+k}^4 - C_k^2 C_{j+k}^2)S_3\right)\\
&\quad + \frac{c_\infty}{4}\left(-C_{j-k}^6 S_7 + C_{j+k}^6 S_2\right)\\
R_2^{j+k} - R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2)\\
&\quad - \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_7)\\
R_2^{j+k} + R_2^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - (C_k^2 + C_{j+k}^2)S_3 + C_{j+k}^4 S_2)\\
&\quad + \frac{a(j-k)}{8}(\phi_j^1 - (C_k^2 + C_{j-k}^2)S_3 + C_{j-k}^4 S_7)\\
R_3^{j+k} - R_3^{j-k} &= \frac{a(j+k)}{8}(\phi_j^1 - C_{j+k}^2 S_5) - \frac{a(j-k)}{8}(\phi_j^1 - C_{j-k}^2 S_3)\\
R_4^{j+k} - R_4^{j-k} &= \frac{c_\infty}{4}((C_{j+k}^2 - C_{j-k}^2)\phi_j^1 + C_{j-k}^4 S_4 - C_{j+k}^4 S_3)\\
R_5^{j-k} - R_5^{j+k} &= \frac{c_\infty}{4}(C_{j+k}^2 \phi_{j+k}^1 - C_{j-k}^2 \phi_{j-k}^1).
\end{aligned}
$$

We calculate that

$$
\begin{aligned}
g = \frac{-1}{2\pi(M-L)c_\infty}\sum_{j=1}^{P}\frac{\xi_j}{j^{r-1}}(u(L,t,\omega) - \phi_j^2) &+ \sum_{\substack{j,k=1\\|j-k|\neq j\\|j-k|\neq k}}^{P}(A_1^1\phi_j^1 + A_2^1\phi_k^1 + A_3^1\phi_{j-k}^1 + A_4^1\phi_{j+k}^1)\\
+ \sum_{\substack{j,k=1\\|j-k|=j}}^{P}(A_1^2\phi_j^1 + A_2^2\phi_k^1 + A_3^2\phi_{j+k}^1 + A_4^2\phi_j^3) &+ \sum_{\substack{j,k=1\\|j-k|=k}}^{P}(A_1^3\phi_j^1 + A_2^3\phi_k^1 + A_3^3\phi_{j+k}^1 + A_4^3\phi_k^3) \quad (2.17)\\
+ \sum_{\substack{j,k=1\\j=k}}^{P}(A_1^4\phi_j^1 + A_2^4\phi_j^3 + A_3^4\phi_{2j}^1)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^1 = {} & D_1D_2E_1E_2\frac{c_\infty}{4}(2-(2*C_k^2+C_{j-k}^2+C_{j+k}^2)J_7+C_{j-k}^4J_1+C_{j+k}^4J_4) \\
& + D_1D_2E_1F\frac{A_2}{8}(1-(C_k^2+C_{j+k}^2)J_7+C_{j+k}^4J_4 - E_1F\frac{A_1}{8}(1-(C_k^2+C_{j-k}^2)J_7+C_{j-k}^4J_1)) \\
& + D_1D_2E_2F\frac{A_2}{8}(1-(C_k^2+C_{j+k}^2)J_7+C_{j+k}^4J_4) + E_2F\frac{A_1}{8}(1-(C_k^2+C_{j-k}^2)J_7+C_{j-k}^4J_1) \\
& + D_1D_2F^2\frac{c_\infty}{4}((C_{j+k}^2-C_{j-k}^2)+(C_{j-k}^4+C_k^2C_{j-k}^2-C_{j+k}^4-C_k^2C_{j+k}^2)J_7-C_{j-k}^6J_1+C_{j+k}^6J_4) \\
& - 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\left(\frac{A_2}{8}(1-C_{j+k}^2J_{11})-\frac{A_1}{8}(1-C_{j-k}^2J_9)\right) \\
& - 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}((C_{j+k}^2-C_{j-k}^2)-C_{j+k}^4J_{11}+C_{j-k}^4J_9) \\
A_2^1 = {} & D_1D_2E_1E_2\frac{c_\infty}{4}(-(2C_k^2+C_{j-k}^2+C_{j+k}^2)J_8+C_{j-k}^4J_2+C_{j+k}^4J_5) \\
& + D_1D_2E_1F\frac{A_2}{8}(-(C_k^2+C_{j+k}^2)J_8+C_{j+k}^4J_5) - D_1D_2E_1F\frac{A_1}{8}(-(C_k^2+C_{j-k}^2)J_8+C_{j-k}^4J_2) \\
& + D_1D_2E_2F\frac{A_2}{8}(-(C_k^2+C_{j+k}^2)J_8+C_{j+k}^4J_5) + D_1D_2E_2F\frac{A_1}{8}(-(C_k^2+C_{j-k}^2)J_8+C_{j-k}^4J_2) \\
& + D_1D_2F^2\frac{c_\infty}{4}((C_{j-k}^4+C_k^2C_{j-k}^2-C_{j+k}^4-C_k^2C_{j+k}^2)J_8-C_{j-k}^6J_2+C_{j+k}^6J_5) \\
A_3^1 = {} & D_1D_2E_1E_2\frac{c_\infty}{4}C_{j-k}^4J_3 - D_1D_2E_1F\frac{A_1}{8}C_{j-k}^4J_3 + D_1D_2E_2F\frac{A_1}{8}C_{j-k}^4J_3 - D_1D_2F^2\frac{c_\infty}{4}C_{j-k}^6J_3 \\
& - 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_1}{8}C_{j-k}^2J_{10} - 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j-k}^4J_{10} - \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j-k}^2 \\
A_4^1 = {} & D_1D_2E_1E_2\frac{c_\infty}{4}C_{j+k}^4J_6 + D_1D_2E_1F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2E_2F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2F^2\frac{c_\infty}{4}C_{j+k}^6J_6 \\
& + 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_2}{8}C_{j+k}^2J_{12} + 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j+k}^4J_{12} + \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j+k}^2 \\
A_1^2 = {} & D_1D_2E_1E_2\frac{c_\infty}{4}(2-(2*C_k^2+C_{j-k}^2+C_{j+k}^2)J_7+C_{j-k}^4J_{14}+C_{j+k}^4J_4) \\
& + D_1D_2E_1F\frac{A_2}{8}(1-(C_k^2+C_{j+k}^2)J_7+C_{j+k}^4J_4) - E_1F\frac{A_1}{8}(1-(C_k^2+C_{j-k}^2)J_7+C_{j-k}^4J_{14}) \\
& + D_1D_2E_2F\frac{A_2}{8}(1-(C_k^2+C_{j+k}^2)J_7+C_{j+k}^4J_4) + E_2F\frac{A_1}{8}(1-(C_k^2+C_{j-k}^2)J_7+C_{j-k}^4J_{14}) \\
& + D_1D_2F^2\frac{c_\infty}{4}((C_{j+k}^2-C_{j-k}^2)+(C_{j-k}^4+C_k^2C_{j-k}^2-C_{j+k}^4-C_k^2C_{j+k}^2)J_7-C_{j-k}^6J_{14}+C_{j+k}^6J_4) \\
& - 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\left(\frac{A_2}{8}(1-C_{j+k}^2J_{11})-\frac{A_1}{8}\right) \\
& - 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}((C_{j+k}^2-C_{j-k}^2)-C_{j+k}^4J_{11}) - \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j-k}^2
\end{aligned}
$$

$$
\begin{aligned}
A_2^2 &= D_1D_2E_1E_2\frac{c_\infty}{4}(-(2C_k^2 + C_{j-k}^2 + C_{j+k}^2)J_8 + C_{j-k}^4J_{15} + C_{j+k}^4J_5) \\
&+ D_1D_2E_1F\frac{A_2}{8}(-(C_k^2 + C_{j+k}^2)J_8 + C_{j+k}^4J_5) - D_1D_2E_1F\frac{A_1}{8}(-(C_k^2 + C_{j-k}^2)J_8 + C_{j-k}^4J_{15}) \\
&+ D_1D_2E_2F\frac{A_2}{8}(-(C_k^2 + C_{j+k}^2)J_8 + C_{j+k}^4J_5) + D_1D_2E_2F\frac{A_1}{8}(-(C_k^2 + C_{j-k}^2)J_8 + C_{j-k}^4J_{15}) \\
&+ D_1D_2F^2\frac{c_\infty}{4}((C_{j-k}^4 + C_k^2C_{j-k}^2 - C_{j+k}^4 - C_k^2C_{j+k}^2)J_8 - C_{j-k}^6J_{15} + C_{j+k}^6J_5) \\
A_3^2 &= D_1D_2E_1E_2\frac{c_\infty}{4}C_{j+k}^4J_6 + D_1D_2E_1F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2E_2F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2F^2\frac{c_\infty}{4}C_{j+k}^6J_6 \\
&+ 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_2}{8}C_{j+k}^2J_{12} + 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j+k}^4J_{12} + \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j+k}^2 \\
A_4^2 &= D_1D_2E_1E_2\frac{c_\infty}{4}C_{j-k}^4J_{13} - D_1D_2E_1F\frac{A_1}{8}C_{j-k}^4J_{13} + D_1D_2E_2F\frac{A_1}{8}C_{j-k}^4J_{13} - D_1D_2F^2\frac{c_\infty}{4}C_{j-k}^6J_6 \\
&+ 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_1}{8}C_{j-k}^2 + 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j-k}^4 \\
A_1^3 &= D_1D_2E_1E_2\frac{c_\infty}{4}(2 - (2 * C_k^2 + C_{j-k}^2 + C_{j+k}^2)J_7 + C_{j-k}^4J_{18} + C_{j+k}^4J_4) \\
&+ D_1D_2E_1F\frac{A_2}{8}(1 - (C_k^2 + C_{j+k}^2)J_7 + C_{j+k}^4J_4 - E_1F\frac{A_1}{8}(1 - (C_k^2 + C_{j-k}^2)J_7 + C_{j-k}^4J_{18})) \\
&+ D_1D_2E_2F\frac{A_2}{8}(1 - (C_k^2 + C_{j+k}^2)J_7 + C_{j+k}^4J_4) + E_2F\frac{A_1}{8}(1 - (C_k^2 + C_{j-k}^2)J_7 + C_{j-k}^4J_{18}) \\
&+ D_1D_2F^2\frac{c_\infty}{4}((C_{j+k}^2 - C_{j-k}^2) + (C_{j-k}^4 + C_k^2C_{j-k}^2 - C_{j+k}^4 - C_k^2C_{j+k}^2)J_7 - C_{j-k}^6J_{18} + C_{j+k}^6J_4) \\
&- 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\left(\frac{A_2}{8}(1 - C_{j+k}^2J_{11}) - \frac{A_1}{8}(1 - C_{j-k}^2J_7)\right) \\
&- 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}((C_{j+k}^2 - C_{j-k}^2) - C_{j+k}^4J_{11} + C_{j-k}^4J_7) \\
A_2^3 &= D_1D_2E_1E_2\frac{c_\infty}{4}(-(2C_k^2 + C_{j-k}^2 + C_{j+k}^2)J_8 + C_{j-k}^4J_{17} + C_{j+k}^4J_5) \\
&+ D_1D_2E_1F\frac{A_2}{8}(-(C_k^2 + C_{j+k}^2)J_8 + C_{j+k}^4J_5) - D_1D_2E_1F\frac{A_1}{8}(-(C_k^2 + C_{j-k}^2)J_8 + C_{j-k}^4J_{17}) \\
&+ D_1D_2E_2F\frac{A_2}{8}(-(C_k^2 + C_{j+k}^2)J_8 + C_{j+k}^4J_5) + D_1D_2E_2F\frac{A_1}{8}(-(C_k^2 + C_{j-k}^2)J_8 + C_{j-k}^4J_{17}) \\
&+ D_1D_2F^2\frac{c_\infty}{4}((C_{j-k}^4 + C_k^2C_{j-k}^2 - C_{j+k}^4 - C_k^2C_{j+k}^2)J_8 - C_{j-k}^6J_{17} + C_{j+k}^6J_5) \\
&- 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_1}{8}C_{j-k}^2J_8 - 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j-k}^4J_8 - \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j-k}^2 \\
A_3^3 &= D_1D_2E_1E_2\frac{c_\infty}{4}C_{j+k}^4J_6 + D_1D_2E_1F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2E_2F\frac{A_2}{8}C_{j+k}^4J_6 + D_1D_2F^2\frac{c_\infty}{4}C_{j+k}^6J_6 \\
&+ 2D_1E_1\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{A_2}{8}C_{j+k}^2J_{12} + 2D_1F\frac{\xi_k}{\pi^2k^rc_\infty^3}\frac{c_\infty}{4}C_{j+k}^4J_{12} + \frac{\xi_j\xi_k}{c_\infty^6\pi^4j^rk^r}\frac{c_\infty}{4}C_{j+k}^2
\end{aligned}
$$

$$A_4^3 = D_1D_2E_1E_2\frac{c_\infty}{4}C_{j-k}^4J_{16} - D_1D_2E_1F\frac{A_1}{8}C_{j-k}^4J_{16} + E_2F\frac{A_1}{8}C_{j-k}^4J_{16} - D_1D_2F^2\frac{c_\infty}{4}C_{j-k}^6J_{16}$$

$$\begin{aligned}
A_1^4 &= D_1^2E_1^2\frac{c_\infty}{4}(2 + C_{2j}^2J_{20}) + 2D_1^2E_1F\frac{A_3}{8}(1 + C_{2j}^4J_{20}) + D_1^2F^2\frac{c_\infty}{4}(C_{2j}^2 + C_{2j}^6J_{20}) \\
&\quad - 2D_1E_1\frac{\xi_j}{\pi^2j^rc_\infty^3}\frac{A_3}{8}(1 - C_{2j}^2J_{22}) - 2D_1F\frac{\xi_j}{\pi^2j^rc_\infty^3}\frac{c_\infty}{4}(C_{2j}^2 - C_{2j}^4J_{22}) \\
A_2^4 &= D_1^2E_1^2\frac{c_\infty}{4}(-(2C_j^2 + C_{2j}^2) + C_{2j}^4J_{19}) + 2D_1^2E_1F\frac{A_3}{8}(-(C_j^2 + C_{2j}^2) + C_{2j}^4J_{19}) \\
&\quad + D_1^2F^2\frac{c_\infty}{4}((-C_{2j}^4 - C_j^2C_{2j}^2) + C_{2j}^6J_{19}) \\
A_3^4 &= D_1^2E_1^2\frac{c_\infty}{4}C_{2j}^4J_{21} + 2D_1^2E_1F\frac{A_3}{8}C_{2j}^4J_{21} + D_1^2F^2\frac{c_\infty}{4}C_{2j}^6J_{21} \\
&\quad + 2D_1E_1\frac{\xi_j}{\pi^2j^rc_\infty^3}\frac{A_3}{8}C_{2j}^2J_{23} + 2D_1F\frac{\xi_j}{\pi^2j^rc_\infty^3}\frac{c_\infty}{4}C_{2j}^4J_{23} + \frac{\xi_j^2}{c_\infty^6\pi^4j^{2r}}\frac{c_\infty}{4}C_{2j}^2
\end{aligned}$$

2.3. Numerical Results

Numerical experiments were performed to compare the stochastic boundary condition obtained with a solution on the extended domain. The system corresponding to the linear approximation obtained in section 2.1 is

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t,\omega) &= \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}(x,t,\omega)\right) + f(x,t), \qquad x \in [-L, L] \\
u(x,0,\omega) &= u_0(x), \qquad \frac{\partial u}{\partial t}(x,0,\omega) = v_0(x), \\
\frac{\partial u}{\partial x}(L,t,\omega) + \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t,\omega) &= g_1(L,t,\omega), \\
\frac{\partial u}{\partial x}(-L,t,\omega) - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(-L,t,\omega) &= 0, \\
\frac{d^2\phi_j}{dt^2}(t,\omega) + B_j^2\phi_j(t,\omega) &= B_j^2 u(L,t,\omega).
\end{aligned} \tag{2.18}$$

where $g_1(x,t,\omega) = \frac{1}{4c_\infty^2\pi}\sum_{j=1}^{P}\frac{1}{j^{k-1}}\frac{u(L,t,\omega)-\phi_j(t,\omega)}{M-L}\xi_j(\omega)$ and we assume that $c = c_\infty$ for $x < -L$ so that we use the exact boundary condition there. The system corresponding to the quadratic approximation obtained in Section 2.2 is

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t,\omega) &= \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}(x,t,\omega)\right) + f(x,t), \qquad x \in [-L, L] \\
u(x,0,\omega) &= u_0(x), \qquad \frac{\partial u}{\partial t}(x,0,\omega) = v_0(x), \\
\frac{\partial u}{\partial x}(L,t,\omega) + \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t,\omega) &= g_2(L,t,\omega), \\
\frac{\partial u}{\partial x}(-L,t,\omega) - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(-L,t,\omega) &= 0, \\
\frac{du}{dt}(L,t,\omega) &= \frac{d^2\phi_k^1}{dt^2}(t,\omega) + C_k^2\phi_k^1(t,\omega) \\
u(L,t,\omega) &= \frac{d^2\phi_j^2}{dt^2}(t,\omega) + C_j^2\phi_j^2(t,\omega) \\
\frac{du}{dt}(L,t,\omega) &= \frac{d^4\phi_j^3}{dt^4}(t,\omega) + 2C_j^2\frac{d^2\phi_j^3}{dt^2}(t,\omega) + C_j^4\phi_j(t,\omega),
\end{aligned} \tag{2.19}$$

where $g_2(L,t,\omega)$ is given by (2.2). Writing the system (2.3) or (2.3) in weak form, we obtain

$$\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dx = -\int_{-L}^{L} \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}(x,t)\right)v(x,t)dx + \int_{-L}^{L} f(x,t)v(x,t)dx$$

for test functions $v \in C^\infty(-L,L)$. Using integration by parts,

$$\begin{aligned}\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c^2(x)\frac{\partial u}{\partial x}(x,t)v(x,t)\mid_{-L}^{L} \\ &\quad + \int_{-L}^{L} f(x,t)v(x,t)dx \\ \int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\ &\quad - c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t) + \int_{-L}^{L} f(x,t)v(x,t)dx\end{aligned}$$

We use the Galerkin approximation

$$u(x,t,\omega) \approx \sum_{k=0}^{n_x} u_k(t,\omega)\psi_k(x) \tag{2.20}$$

where ψ_k are the Galerkin difference basis functions as in [3]. These are piecewise polynomials defined by values on a uniform grid whose restriction to any interval bounded by grid points is the Lagrange interpolant of the nodal data. Near boundaries we simply take the values of the solution at external ghost points to be free, called the ghost basis method in [3]. Here we take the local polynomial degrees to be 3 and the grid spacing to be $\Delta x = 1/100$. This seems sufficient to resolve the waves to the accuracy provided by the linearized approximate boundary condition, but some discretization errors are noticeable for the more accurate quadratic approximation when the amplitude of the perturbation is very small.

$$\begin{aligned}\sum_{k=1}^{n_x} M_{jk}\frac{d^2 u_k}{dt^2} = \sum_{k=1}^{n_x}\left(-S_{jk}\frac{du_k}{dt} + M_{jk}f_k\right) + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\ -c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t)\end{aligned} \tag{2.21}$$

where

$$M_{jk} = \int_{-L}^{L} \psi_j(x)\psi_k(x)dx, \qquad S_{jk} = \int_{-L}^{L} c^2(x)\frac{d\psi_j}{dx}(x)\frac{d\psi_k}{dx}(x)dx$$

The standard 4th-order Runge-Kutta method is used to discretize the time-variable with $\Delta t = 1/10000$. We take the source term $f(x,t) = 0$, and the initial conditions to represent a pulse $u_0(x) = e^{-x^2} I_{[-5,5]}$, $v_0(x) = 0$. The wave speed on the domain $[-L, L]$ is $c(x) = c_\infty = 10$.

2.3.1. Consistency with Extended Domain

First we will test the accuracy of the method by comparing the solution obtained with the random boundary condition to the solution obtained by solving the problem on the extended domain. Let $u_1(x,t)$ denote the solution to equation (2.3) and $u_2(x,t)$ the solution of

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t,\omega) &= \frac{\partial}{\partial x}\left(c^2(x,\omega)\frac{\partial u}{\partial x}(x,t,\omega)\right) + f(x,t), \qquad x \in [-L, M] \\
u(x,0) &= u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x), \\
\frac{\partial u}{\partial x}(M,t,\omega) &+ \frac{1}{c_\infty}\frac{\partial u}{\partial t}(M,t,\omega) = 0 \\
\frac{\partial u}{\partial x}(-L,t,\omega) &- \frac{1}{c_\infty}\frac{\partial u}{\partial t}(-L,t,\omega) = 0
\end{aligned} \tag{2.22}$$

with discretization analogous to equation (2.3). We choose a single sample $\tilde{c}$ scaled by various amplitudes A so that the solutions are deterministic. We measure the difference between u_1 and u_2 at a time $t = T$ by using the maximum norm

$$err = \max_{x=1}^{n_x} |u_1(x,T) - u_2(x,T)|.$$

In Figure (2.3) we plot err vs A, for the single sample $A\tilde{c}$. The parameters $r = 6$, $P = 10$, $L = 10$, $M = 20$, and $T = 2$. As A increases, the error of the method increases at second-order rate, as expected.

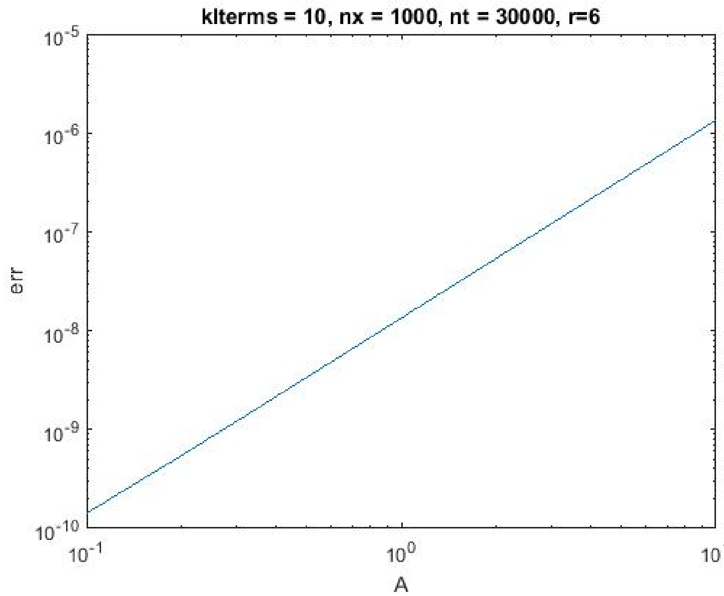

Figure 2.3. Error for Linear Approximation

In Figure (2.4) we plot err vs A, for the single sample $A\tilde{c}$. As A increases, the error of the method increases at third-order rate for sufficiently large A, as expected. There is not 3rd order accuracy at small fluctuation values because the error in discretization is larger than the error in the linearization of the Ricatti equation for these values of A. However, it is observed that the slope of a sufficiently short tail of the graph is 3.

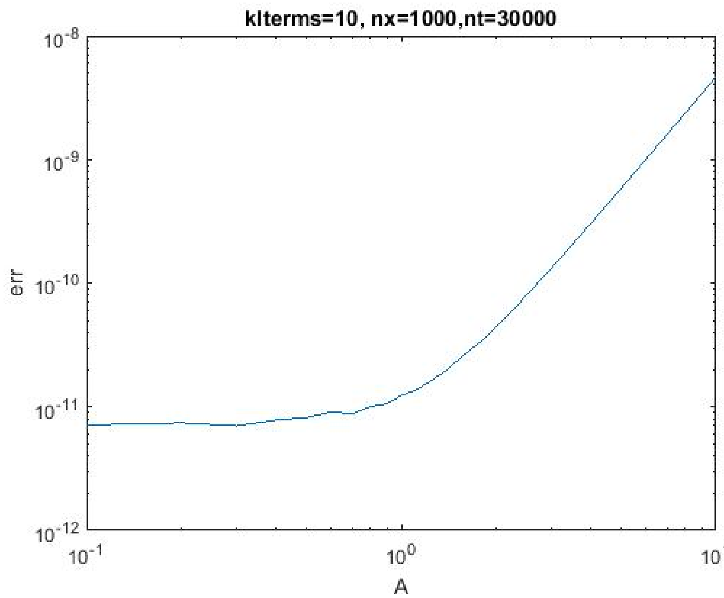

Figure 2.4. Error for Quadratic Approximation

2.3.2. Monte-Carlo

We approximate the mean and variance of the reflected wave at the boundary by using the Monte-Carlo method. The system (2.3) is solved N times to approximate the statistics of the solution. Let $X_i(x,t) = f(u_1^i(x,t))$ be a quantity of interest, where $u_1^i(x,t)$ is the solution using the ith sample. The sample mean $m(x,t)$ and sample variance $\sigma^2(x,t)$ are given by

$$m(x,t) = \frac{1}{N}\sum_{i=1}^{N} X_i(x,t)$$

$$\sigma^2(x,t) = \frac{1}{N-1}\sum_{i=1}^{N}(X_i(x,t) - m(x,t))^2$$

where N is the number of samples.

The first quantity computed is the maximum value of the reflected wave at $t = 2$, given by $X_i(x,t) = \max_x(|u_1^i(x,t)|)$. The value $t = 2$ is selected because this is the value of time when the wave has passed completely through the random medium. The same parameters were chosen as in the previous experiment. The mean m and variance σ^2 is given below in Figures (2.5) and (2.6). The small variance is due to small perturbations in the random medium. We do not expect these quantities to be zero because the small perturbations in the wave speed should generate reflected wave energy.

The second quantity computed is the mean value of the reflected wave at $t = 2$, given by $X_i(x,t) = \frac{1}{n}\sum_j (u_1^i(x_j,t))$. The mean m and variance σ^2 is given below in Figures (2.7) and (2.8). In these preliminary uncertainty quantification experiments we expect the sample mean and sample variance to converge to their true values as $N \to \infty$. A simulation with several orders of magnitude increase in the number of samples used here is expected to be necessary to observe convergence.

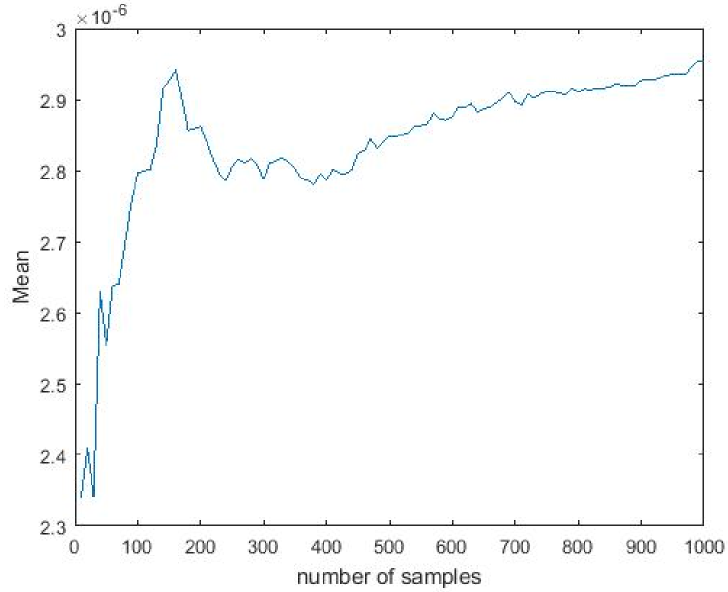

Figure 2.5. Mean Maximum of Reflected Wave at Boundary

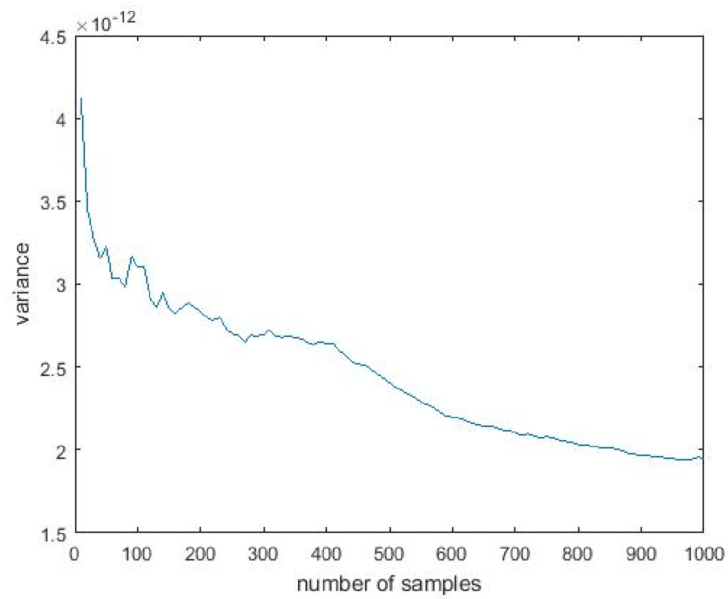

Figure 2.6. Variance of Maximum of Reflected Wave at Boundary

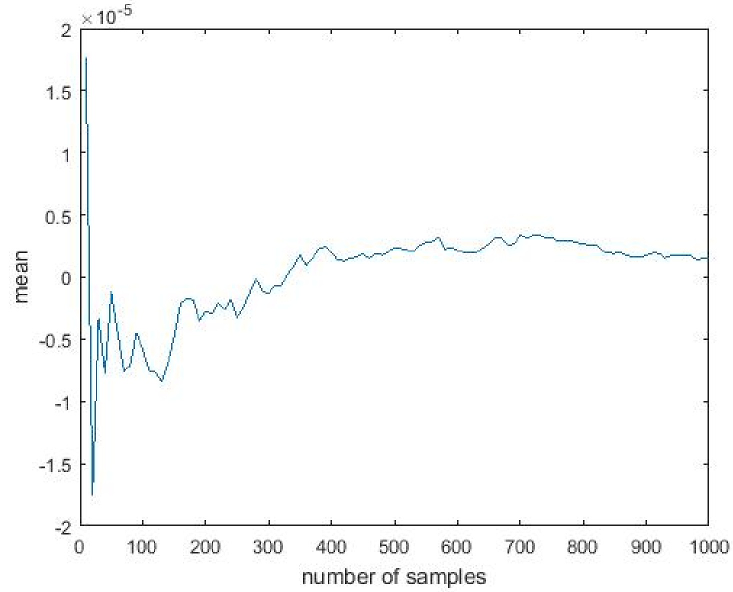

Figure 2.7. Mean of Average Reflected Wave at Boundary

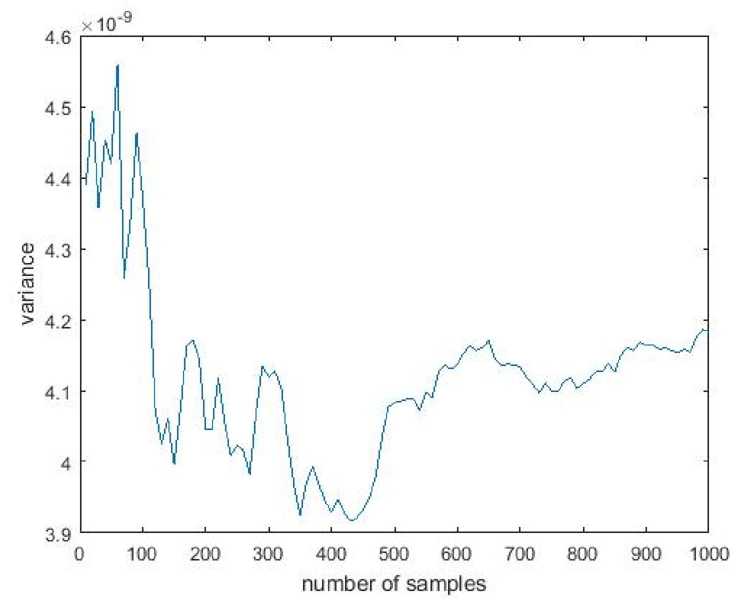

Figure 2.8. Variance of Average of Reflected Wave at Boundary

Chapter 3

EXTENSIONS TO A STATIONARY PROCESS

3.1. DtN Map for Whittle-Matérn Covariance Process

In this chapter we will repeat the procedure carried out in Chapter 2 for a stationary Gaussian process. This will prepare us for Chapter 4 when we will compare the reflection of the wave at the boundary caused by the random boundary condition obtained here to an asymptotic calculation carried out in [8]. A class of stationary covariance functions are given by the Matérn class

$$C_v(d) = \frac{2^{1-v}}{\Gamma(v)} \left(\sqrt{2v}\frac{d}{\rho}\right)^v K_v\left(\sqrt{2\nu}\frac{d}{\rho}\right), \tag{3.1}$$

where Γ is the gamma function, K_v is the modified Bessel function of the second kind, $d = |x-y|$ is the distance between points, and ρ, v are positive parameters. It is shown in [9] that a Gaussian process with Matérn covariance function is $\lceil v \rceil -$-times differentiable. The selection of the Whittle-Matérn covariance function was motivated by the desire to find an orthogonal expansion which has closed-form eigenfunctions. In the future it would be interesting to choose a process which is motivated by a physical example.

In the case $v = p + 1/2$ equation (3.1) simplifies to

$$C_{p+1/2}(d) = \exp\left(-\frac{\sqrt{2p+1}d}{p}\right) \frac{p!}{(2p)!} \sum_{i=0}^{p} \frac{(p+i)!}{i!\,(p-i)!} \left(\frac{2\sqrt{2p+1}d}{\rho}\right)^{p-i}$$

For $p = 0$ we have the exponential covariance,

$$C_{1/2}(d) = \exp(-d),$$

where we chose $\rho = 1$ for simplicity. Here the sample paths are continuous but not differentiable. We will find the eigenfunction expansion for the $p = 1$ case

$$C_{3/2}(d) = (1 + \sqrt{3}d)\exp(-\sqrt{3}d).$$

which is continuous and differentiable. The eigenfunction expansion is given by

$$\tilde{c}(x,\omega) = c_\infty + \sum_{j=1}^{\infty} \sqrt{\nu_j}\phi_j(x)\xi_j(\omega), \qquad \xi_j(\omega) \sim N(0,1)$$

where, taking $d = |x-y|$, ϕ and ν are the eigenfunctions and eigenvalues of the covariance operator satisfying

$$\int_{-a}^{a} (1 + \sqrt{3}|x-y|)e^{-\sqrt{3}|x-y|}\phi(y)dy = \nu\phi(x), \qquad x \in [-a,a]$$

Differentiating under the integral sign gives

$$\begin{aligned}
\nu\phi'(x) &= -3\int_{-a}^{x}(x-y)e^{-\sqrt{3}(x-y)}\phi(y)dy + 3\int_{x}^{a}(y-x)e^{-\sqrt{3}(y-x)}\phi(y)dy \\
\nu\phi''(x) &= \int_{-a}^{x}(3\sqrt{3}(x-y)-3)e^{-\sqrt{3}(x-y)}\phi(y)dy + \int_{x}^{a}(3\sqrt{3}(y-x)-3)e^{-\sqrt{3}(y-x)}\phi(y)dy \\
\nu\phi'''(x) &= \int_{-a}^{x}(-9(x-y)+6\sqrt{3})e^{-\sqrt{3}(x-y)}\phi(y)dy + \int_{x}^{a}(9(y-x)-6\sqrt{3})e^{-\sqrt{3}(y-x)}\phi(y)dy \\
\nu\phi''''(x) &= \int_{-a}^{x}(9\sqrt{3}(x-y)-27)e^{-\sqrt{3}(x-y)}\phi(y)dy + \int_{x}^{a}(9\sqrt{3}(y-x)-27)e^{-\sqrt{3}(y-x)}\phi(y)dy \\
&\quad + 12\sqrt{3}\phi(x)
\end{aligned}$$

Thus we see that the integral equation is equivalent to the linear 4th order ODE

$$\begin{aligned}
\phi''''(x) - 6\phi''(x) + \frac{9\nu - 12\sqrt{3}}{\nu}\phi(x) &= 0 \\
\phi''(-a) - 2\sqrt{3}\phi'(-a) + 3\phi(-a) = 0, \quad \phi'''(-a) - 2\sqrt{3}\phi''(-a) + 3\phi'(-a) &= 0 \\
\phi''(a) + 2\sqrt{3}\phi'(a) + 3\phi(a) = 0, \quad \phi'''(a) + 2\sqrt{3}\phi''(a) + 3\phi'(a) &= 0
\end{aligned}$$

Solving the ODE gives the characteristic polynomial $r^4 - 6r^2 + (9 - 12\sqrt{3}/\nu) = 0$, which has roots

$$r^2 = 3 \pm \sqrt{\frac{12\sqrt{3}}{\nu}}.$$

The eigenvalues satisfy $\nu_1 \geq \nu_2 \geq \ldots \geq 0$, so we expect two real and two imaginary roots, giving the general solution

$$\phi(x) = A\exp(-r_1(x+a)) + B\exp(r_1(x-a)) + C\cos(r_2(x+a)) + D\sin(r_2(x+a))$$

with $r_1 = \sqrt{3 + \sqrt{12\sqrt{3}/\nu}}$, $r_2 = \sqrt{\sqrt{12\sqrt{3}/\nu} - 3}$.

Applying the boundary conditions leads to

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.2}$$

where

$$A_{11} = r_1^2 + 2\sqrt{3}r_1 + 3, \quad A_{12} = (r_1^2 - 2\sqrt{3}r_1 + 3)\exp(-2ar_1)$$
$$A_{13} = 3 - r_2^2, \quad A_{14} = -2\sqrt{3}r_2, \quad A_{21} = (-r_1^3 - 2\sqrt{3}r_1^2 - 3r_1)$$
$$A_{22} = (r_1^3 - 2\sqrt{3}r_1^2 + 3r_1)\exp(-2ar_1), \quad A_{23} = 2\sqrt{3}r_2^2, \quad A_{24} = (3r_2 - r_2^3)$$
$$A_{31} = (r_1^2 - 2\sqrt{3}r_1 + 3)\exp(-2ar_1), \quad A_{32} = r_1^2 + 2\sqrt{3}r_1 + 3$$
$$A_{33} = (3 - r_2^2)C - 2\sqrt{3}r_2 S, \quad A_{34} = 2\sqrt{3}r_2 C + (3 - r_2^2)S$$
$$A_{41} = (-r_1^3 + 2\sqrt{3}r_1^2 - 3r_1)\exp(-2ar_1), \quad A_{42} = (r_1^3 + 2\sqrt{3}r_1^2 + 3r_1)$$
$$A_{43} = (r_2^3 - 3r_2)S - 2\sqrt{3}r_2^2 C, \quad A_{44} = (3r_2 - r_2^3)C - 2\sqrt{3}r_2^2 S$$

where $S = \sin(2ar_2)$, $C = \cos(2ar_2)$.

The eigenvalues ν and corresponding eigenvectors ϕ are found numerically by applying the power method to the eigenvalue problem $A\phi = \nu\phi$ where A is given in (3.1). The eigenvectors ϕ are then normalized by enforcing $\|\phi\|_2 = 1$. Sample paths are given in Figure (3.1), where the process has been truncated to $P = 100$ terms.

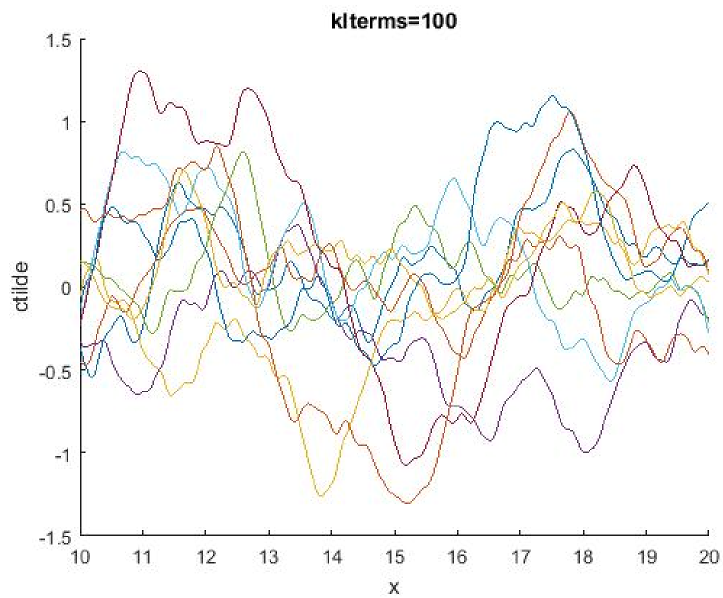

Figure 3.1. Sample Paths for Stationary Process

Starting from equation (2.1), we follow the same procedure to obtain the DtN map with the added complication that there will generally be a jump at the boundary $x = L$

$$\begin{aligned}
\tilde{\sigma}(L, s, \omega) &= \frac{2s^2}{c_\infty^3} \int_L^M e^{-\frac{2s}{c_\infty}(z-L)} \tilde{c}(z,\omega) dz - \frac{2s}{c_\infty^2} \tilde{c}(L,\omega) \\
&= \frac{2s^2}{c_\infty^3} \sum_{i=1}^{\infty} \xi_i(\omega) \sqrt{\nu_i} \left(A_i \int_L^M e^{-\frac{2s}{c_\infty}(z-L)} e^{-r_1(z-L)} dz + B_i \int_L^M e^{-\frac{2s}{c_\infty}(z-L)} e^{r_1(z-M)} dz \right) \\
&\quad + \frac{2s^2}{c_\infty^3} \sum_{i=1}^{\infty} \xi_i(\omega) \sqrt{\nu_i} C_i \int_L^M e^{-\frac{2s}{c_\infty}(z-L)} \cos(r_2(z-L)) dz \\
&\quad + \frac{2s^2}{c_\infty^3} \sum_{i=1}^{\infty} \xi_i(\omega) \sqrt{\nu_i} D_i \int_L^M e^{-\frac{2s}{c_\infty}(z-L)} \sin(r_2(z-L)) dz - \frac{2s}{c_\infty^2} \tilde{c}(L,\omega).
\end{aligned}$$

Using the following integrals

$$\int_L^M e^{-\frac{2s}{c_\infty}(z-L)}e^{-r_1(z-L)}dz = \frac{c_\infty/2}{s+c_\infty r_1/2}$$
$$\int_L^M e^{-2s(z-L)/c_\infty}e^{r_1(z-M)}dz = \frac{c_\infty e^{-(M-L)r_1}/2}{s-c_\infty r_1/2}$$
$$\int_L^M e^{-2s(z-L)/c_\infty}\cos(r_2(z-L))dz = \frac{c_\infty s/2}{s^2+c_\infty^2 r_2^2/4}$$
$$\int_L^M e^{-2s(z-L)/c_\infty}\sin(r_2(z-L))dz = \frac{c_\infty^2 r_2/4}{s^2+c_\infty^2 r_2^2/4}$$

the DtN map in the Laplace domain is then, in the Laplace domain

$$\begin{aligned}
\tilde{\sigma}(L,s,\omega) &= \frac{2s^2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty/2}{s+c_\infty r_1/2}+B_i\frac{c_\infty e^{-(M-L)r_1}/2}{s-c_\infty r_1/2}\right)\\
&+\frac{2s^2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(C_i\frac{c_\infty s/2}{s^2+c_\infty^2r_2^2/4}+D_i\frac{c_\infty^2 r_2/4}{s^2+c_\infty^2r_2^2/4}\right)-\frac{2s}{c_\infty^2}\tilde{c}(L,\omega).\\
&=\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty}{2}\left(s-\frac{sc_\infty r_1/2}{s+c_\infty r_1/2}\right)+B_i\frac{c_\infty e^{-(M-L)r_1}}{2}\left(s+\frac{sc_\infty r_1/2}{s-c_\infty r_1/2}\right)\right)\\
&+\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(C_i\frac{c_\infty}{2}\left(s-\frac{sc_\infty^2r_2^2/4}{s^2+c_\infty^2r_2^2/4}\right)+D_i\frac{c_\infty^2 r_2}{4}\left(1-\frac{c_\infty^2r_2^2/4}{s^2+c_\infty^2r_2^2/4}\right)\right)\\
&-\frac{2s}{c_\infty^2}\tilde{c}(L,\omega).\\
&=\frac{2s}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty}{2}+B_i\frac{c_\infty e^{-(M-L)r_1}}{2}+C_i\frac{c_\infty}{2}\right)\\
&+\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(-A_i\frac{c_\infty}{2}\frac{c_\infty r_1}{2}\left(1-\frac{c_\infty r_1/2}{s+c_\infty r_1/2}\right)\right)\\
&+\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(B_i\frac{c_\infty e^{-(M-L)r_1}}{2}\frac{c_\infty r_1}{2}\left(1+\frac{c_\infty r_1/2}{s-c_\infty r_1/2}\right)\right)\\
&+\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(-C_i\frac{c_\infty}{2}\frac{c_\infty^2r_2^2}{4}\left(\frac{s}{s^2+c_\infty^2r_2^2/4}\right)+D_i\frac{c_\infty^2 r_2}{4}\left(1-\frac{c_\infty^2r_2^2/4}{s^2+c_\infty^2r_2^2/4}\right)\right)\\
&-\frac{2s}{c_\infty^2}\tilde{c}(L,\omega).
\end{aligned}$$

Define

$$
\begin{aligned}
h &= \frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty}{2}+B_i\frac{c_\infty e^{-(M-L)r_1}}{2}+C_i\frac{c_\infty}{2}\right)\\
\hat{g} &= \frac{2\hat{u}}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty}{2}\frac{c_\infty r_1}{2}\left(1-\frac{c_\infty r_1/2}{s+c_\infty r_1/2}\right)\right)\\
&\quad-\frac{2\hat{u}}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(B_i\frac{c_\infty e^{-(M-L)r_1}}{2}\frac{c_\infty r_1}{2}\left(1+\frac{c_\infty r_1/2}{s-c_\infty r_1/2}\right)\right)\\
&\quad-\frac{2\hat{u}}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(-C_i\frac{c_\infty}{2}\frac{c_\infty^2 r_2^2 s/4}{s^2+c_\infty^2 r_2^2/4}+D_i\frac{c_\infty^2 r_2}{4}\left(1-\frac{c_\infty^2 r_2^2/4}{s^2+c_\infty^2 r_2^2/4}\right)\right)
\end{aligned}
$$

The boundary condition is then

$$
\frac{d\hat{u}}{dx}+s\left(\frac{1}{c_\infty}-\frac{2}{c_\infty^2}\tilde{c}(L)+\hat{h}\right)\hat{u}=\hat{g}
$$

In the time domain,

$$
\frac{du}{dx}+\left(\frac{1}{c_\infty}-\frac{2}{c_\infty^2}\tilde{c}(L,y)+h\right)\frac{du}{dt}=g
$$

Where

$$
\begin{aligned}
g &= \frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(A_i\frac{c_\infty}{2}\frac{c_\infty r_1}{2}\left(u-c_\infty r_1\phi_1/2\right)-B_i\frac{c_\infty e^{-(M-L)r_1}}{2}\frac{c_\infty r_1}{2}\left(u+c_\infty r_1\phi_2/2\right)\right)\\
&\quad-\frac{2}{c_\infty^3}\sum_{i=1}^{\infty}\xi_i\sqrt{\nu_i}\left(-C_i\frac{c_\infty}{2}c_\infty^2 r_2^2\phi_3/4+D_i\frac{c_\infty^2 r_2}{4}\left(u-c_\infty^2 r_2^2\phi_4/4\right)\right)
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
u(t,\omega) &= \frac{d\phi_1}{dt} + c_\infty r_1 \phi_1/2 \\
u(t,\omega) &= \frac{d\phi_2}{dt} - c_\infty r_1 \phi_2/2 \\
\frac{du}{dt}(t,\omega) &= \frac{d^2\phi_3}{dt^2} + c_\infty^2 r_2^2 \phi_3/4 \\
u(t,\omega) &= \frac{d^2\phi_4}{dt^2} + c_\infty^2 r_2^2 \phi_4/4
\end{aligned}
$$

with zero initial conditions for ϕ_j, $j = 1, 2, 3, 4$.

3.2. Numerical Results

Numerical experiments were performed to compare the solution computed with the stochastic boundary condition to a solution computed on the extended domain. The system is

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) + f(x,t), \qquad x \in [-L, L] \\
&u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x), \\
&\frac{\partial u}{\partial x} + \frac{1}{c_\infty}\frac{\partial u}{\partial t} = g(x,t,\omega), \quad x = L \\
&\frac{\partial u}{\partial x} - \frac{1}{c_\infty}\frac{\partial u}{\partial t} = 0, \quad x = -L \\
&\frac{d^2\phi_j}{dt^2} + B_j^2 \phi_j = B_j^2 u.
\end{aligned}
\tag{3.4}
$$

where $g(x,t,\omega)$ is given by (3.1). Writing the system (3.2) in weak form, we obtain

$$
\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dx = -\int_{-L}^{L} \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}(x,t)\right)v(x,t)dx + \int_{-L}^{L} f(x,t)v(x,t)dx
$$

for test functions $v \in C^\infty(-L,L)$. Using integration by parts,

$$\begin{aligned}
\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c^2(x)\frac{\partial u}{\partial x}(x,t)v(x,t)\,|_{-L}^{L} \\
&\quad + \int_{-L}^{L} f(x,t)v(x,t)dx \\
\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\
&\quad - c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t) + \int_{-L}^{L} f(x,t)v(x,t)dx
\end{aligned}$$

We use the Galerkin approximation

$$u(x,t,\omega) \approx \sum_{k=0}^{n_x} u_k(t,\omega)\psi_k(x) \tag{3.5}$$

where ψ_k are the Galerkin difference basis functions as in [3]. These are piecewise polynomials defined by values on a uniform grid whose restriction to any interval bounded by grid points is the Lagrange interpolant of the nodal data. Near boundaries we simply take the values of the solution at external ghost points to be free, called the ghost basis method in [3]. Here we take the local polynomial degrees to be 3 and the grid spacing to be $\Delta x = 1/100$. This seems sufficient to resolve the waves to the accuracy provided by the linearized approximate boundary condition, but some discretization errors are noticeable for the more accurate quadratic approximation when the amplitude of the perturbation is very small.

$$\begin{aligned}
\sum_{k=1}^{n_x} M_{jk}\frac{d^2 u_k}{dt^2} = \sum_{k=1}^{n_x}\left(-S_{jk}\frac{du_k}{dt} + M_{jk}f_k\right) + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\
-c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t)
\end{aligned} \tag{3.6}$$

where

$$M_{jk} = \int_{-L}^{L} \psi_j(x)\psi_k(x)dx, \qquad S_{jk} = \int_{-L}^{L} c^2(x)\frac{d\psi_j}{dx}(x)\frac{d\psi_k}{dx}(x)dx$$

The standard 4th-order Runge-Kutta method is used to discretize the time-variable with $\Delta t = 1/10000$. We take the source term $f(x,t) = 0$, and the initial conditions to represent a pulse $u_0(x) = e^{-x^2} I_{[-5,5]}$, $v_0(x) = 0$. The wave speed on the domain $[-L, L]$ is $c(x) = c_\infty = 10$.

3.2.1. Consistency with Extended Domain

First we will test the accuracy of the method by comparing the solution obtained with the random boundary condition to the solution obtained by solving the problem on the extended domain. Let $u_1(x,t)$ denote the solution to equation (2.3) and $u_2(x,t)$ the solution of

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t,\omega) &= \frac{\partial}{\partial x}\left(c^2(x,\omega)\frac{\partial u}{\partial x}(x,t,\omega)\right) + f(x,t), \qquad x \in [-L, M] \\
u(x,0) &= u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x), \\
\frac{\partial u}{\partial x}(M,t,\omega) &+ \frac{1}{c_\infty}\frac{\partial u}{\partial t}(M,t,\omega) = 0 \\
\frac{\partial u}{\partial x}(-L,t,\omega) &- \frac{1}{c_\infty}\frac{\partial u}{\partial t}(-L,t,\omega) = 0
\end{aligned}
\tag{3.7}
$$

with discretization analogous to equation (4.5). We choose a single sample $\tilde{c}$ scaled by various amplitudes A so that the solutions are deterministic. We measure the difference between u_1 and u_2 at a time $t = T$ by using the maximum norm

$$
err = \max_{x=1}^{n_x} |u_1(x,T) - u_2(x,T)|.
$$

In Figure (3.2) we plot err vs A, for the single sample $A\tilde{c}$. The parameters are $r = 6$, $P = 100$, $L = 10$, $M = 20$, and $T = 2$. As A increases, the error of the method increases at second-order rate, as expected.

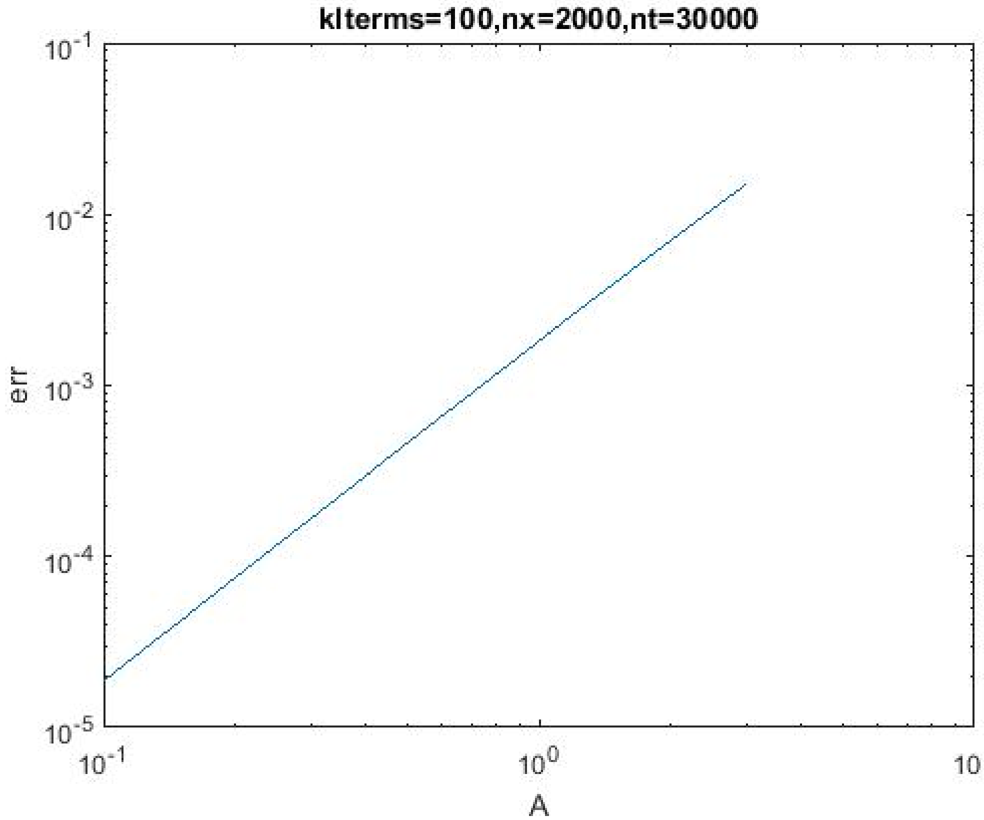

Figure 3.2. Error for Linear Approximation

Chapter 4

COMPARISON WITH ASYMPTOTIC RESULTS

In this section we will summarize an alternative approach developed in [8] to study the transmission and reflection of waves through a random medium. In the final section we will compare the method obtained in previous chapters to an asymptotic one developed here. Preliminary results are obtained.

4.1. Scaling Regimes of Wave Propagation through Random Medium

Three parameters of interest in the random layer wave propagation problem are the random layer size l, the typical wavelength of the propagating pulse λ_0, and the propagation distance L. See Figure (4.1). The relative magnitude of these parameters determines the qualitative behavior of the wave as it passes through the random slab. First we would like to nondimensionalize the problem in order to introduce our scaling parameters. We start with the 1D wave equation written in the form

$$\begin{aligned} \rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} &= F(t,z), \\ \frac{1}{K(z)}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial z} &= 0, \end{aligned} \tag{4.1}$$

where ρ and K are the density and permissibility of the medium. Upon differentation and substitution this system is equivalent to

$$\rho(z)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left(K(z)\frac{\partial u}{\partial z}\right) + \frac{\partial F}{\partial t}(z,t) \tag{4.2}$$

with $c^2(z) = K(z)/\rho(z)$. We will go through the asymptotic analysis using the first order system (4.1), keeping equation (4.1) in mind when we do the comparison in Section 4.5. The

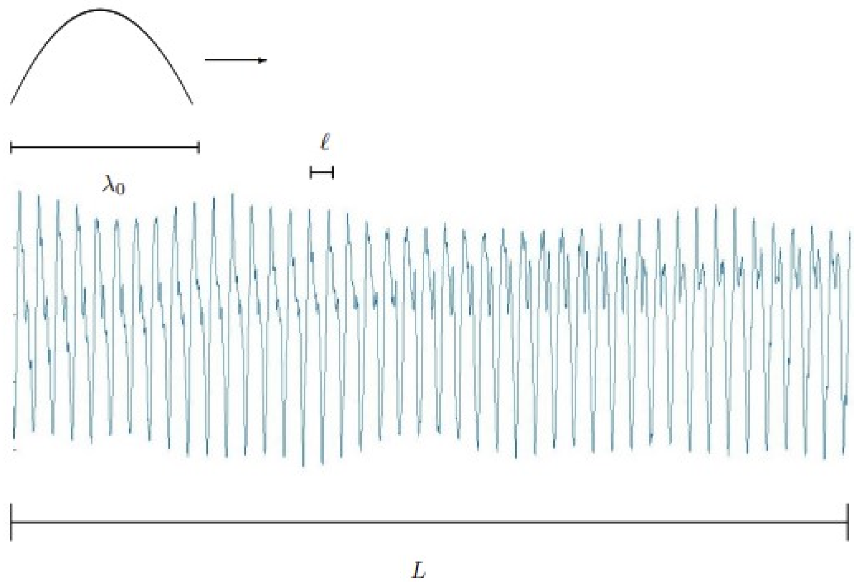

Figure 4.1. Scaling regime of asymptotic approach

random fluctuations are modeled in the form

$$\frac{1}{K(z)} = \begin{cases} \frac{1}{\bar{K}}(1+\nu_K(z,\omega)) & \text{for } z \in [0,L], \\ \frac{1}{\bar{K}} & \text{for } z \in (-\infty,0) \cup (L,\infty), \end{cases}$$

$$\rho(z) = \overline{\rho} \qquad \text{for all } z.$$

This corresponds to

$$\frac{1}{c^2(z,\omega)} = \frac{\bar{\rho}}{\bar{K}}(1+\nu_K(z,\omega))$$

We therefore assume for simplification that the properties of the medium on either side of the random slab are the same, so that in the absence of random perturbations there is no reflection.

The randomness is therefore contained in the zero-mean stationary process $\nu_K(z,\omega)$. A process being stationary means that the (transition) probability of $\nu_K(z_0+z,\omega) = y$ given

$\nu_K(z_0, \omega) = x$, $z \geq 0$ does not depend on z_0. We write the process $\nu_K(z, \omega)$ in scaled form as

$$\nu_K(z, \omega) = \sigma\nu(z/l, \omega).$$

The source is $F(t, z)$ is a point source given by

$$F(t, z) = \bar{\zeta}^{1/2} g(t)\delta(z - z_0),$$

where $\bar{\zeta} = \sqrt{\bar{K}\bar{\rho}}$ is the impedance, so that the right-going wave that travels to the random slab has the form

$$A(t, z) = g\left(t - \frac{z - z_0}{\bar{c}}\right), \quad z < 0.$$

$\bar{c}$ is the wave speed given by $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$.

One can define a pulse width T_0 by root mean square

$$T_0^2 = \frac{\int_{-\infty}^{\infty}(t - \bar{T})^2 g^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt}, \quad \text{where} \quad \bar{T} = \frac{\int_{\infty}^{\infty} tg^2(t)dt}{\int_{-\infty}^{\infty} g^2(t)dt},$$

so that the typical frequency is $\omega_0 = 2\pi/T_0$ and the typical wavelength is $\lambda_0 = 2\pi\bar{c}/\omega_0$. We can write the source term in terms of these variables as

$$F(t, z) = \bar{\zeta}^{1/2} f(\omega_0 t)\delta(z - z_0).$$

Now we define the dimensionless space and time variables

$$\tilde{z} = \frac{z}{L_0}, \quad \tilde{t} = \frac{c_0 t}{L_0},$$

where L_0 is a typical propagation distance and c_0 is a reference speed of propagation. We introduce a reference impedance ζ_0 and take the normalized pressure and velocity fields to be

$$\tilde{p}(\tilde{t}, \tilde{z}) = \zeta_0^{-1/2} p\left(\tilde{t}\frac{L_0}{c_0}, \tilde{z}L_0\right), \qquad \tilde{u}(\tilde{t}, \tilde{z}) = \zeta_0^{1/2} u\left(\tilde{t}\frac{L_0}{c_0}, \tilde{z}L_0\right) \tag{4.3}$$

and the normalized source and fluctuation terms as

$$\tilde{F}(\tilde{t}, \tilde{z}) = L_0 \zeta_0^{-1/2} F\left(\tilde{t}\frac{L_0}{c_0}, \tilde{z}L_0\right), \qquad \tilde{\nu}(\tilde{z}) = \nu(\tilde{z}L_0).$$

Then the wave equation (4.1) is given by

$$\begin{aligned} \tilde{\rho}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{z}} &= \tilde{F}(\tilde{t}, \tilde{z}), \\ \frac{1}{\tilde{K}}\left(1 + \sigma\tilde{\nu}\left(\tilde{z}\frac{L_0}{l}\right)\right)\frac{\partial \tilde{p}}{\partial \tilde{t}} + \frac{\partial \tilde{u}}{\partial \tilde{z}} &= 0, \end{aligned}$$

with $\tilde{p} = c_0(\bar{\rho}/\zeta_0)$ and $\tilde{K} = \bar{K}/(c_0\zeta_0)$. The source is of the form

$$\tilde{F}(\tilde{t}, \tilde{z}) = \tilde{\zeta}^{1/2} f\left(\tilde{t}\frac{\omega_0 L_0}{c_0}\right)\delta(z - z_0),$$

where

$$\tilde{\zeta} = \sqrt{\tilde{K}\tilde{\rho}} = \bar{\zeta}/\zeta_0, \qquad \tilde{z}_0 = z_0/L_0.$$

Now we introduce our scaling parameters ϵ and δ as

$$\frac{L_0}{l} = \frac{1}{\epsilon^2}, \qquad \frac{\omega_0 L_0}{c_0} = \frac{\theta}{\epsilon}.$$

The ratio δ/ϵ is thus the propagation distance measured in terms of the wavelength. Inverting, we have

$$\epsilon = \sqrt{\frac{l}{L_0}}, \qquad \theta = \frac{\omega_0}{c_0}\sqrt{lL_0}.$$

$\epsilon \ll 1$ obviously in all cases of interest. When $\theta \sim \epsilon$, we are in the effective medium regime. Here, there is not enough wave interaction with the medium to cause much random scattering, and homogenization can be used to find effective medium parameters. When $\theta \sim \epsilon^{-1}$ and $\sigma \sim \epsilon$, we are in the weakly heterogeneous regime. It is weak because the variations in the random medium are small, but the propagation distance is large enough to experience significant scattering nonetheless. When $\theta \sim 1$, $\sigma \sim 1$, we are in the strongly heterogeneous regime.

The scaled and dimensionless wave equation in terms of these parameters has the form

$$\begin{aligned}
\bar{\rho}\frac{\partial u^\epsilon}{\partial t} + \frac{\partial p^\epsilon}{\partial z} &= \bar{\zeta}^{1/2} f\left(\frac{\theta t}{\epsilon}\right)\delta(z - z_0), \\
\frac{1}{\overline{\overline{K}}}\left(1 + \sigma\nu\left(\frac{z}{\epsilon^2}\right)\right)\frac{\partial p^\epsilon}{\partial t} + \frac{\partial u^\epsilon}{\partial z} &= 0.
\end{aligned}$$

Decomposing the wave into right- and left-going modes via the transformation

$$A^\epsilon = \frac{p^\epsilon}{\bar{\zeta}^{1/2}} + \bar{\zeta}^{1/2} u^\epsilon, \quad B^\epsilon = -\frac{p^\epsilon}{\bar{\zeta}^{1/2}} + \bar{\zeta}^{1/2} u^\epsilon,$$

The boundary conditions correspond to a right-going wave

$$A^\epsilon(t, z) = f\left(\frac{\theta}{\epsilon}\left(t - \frac{z}{\bar{c}}\right)\right), \quad z < 0,$$

and $B^\epsilon(t, z) = 0$ coming from the left $(z > L)$. Transform the waves along the characteristics

$$\begin{aligned}
a^\epsilon(s, z) &= A^\epsilon\left(\frac{\epsilon}{\theta}s + \frac{z}{\bar{c}}, z\right), \\
b^\epsilon(s, z) &= B^\epsilon\left(\frac{\epsilon}{\theta}s - \frac{z}{\bar{c}}, z\right),
\end{aligned}$$

and taking the Fourier transform with respect to the time variable s,

$$\hat{a}^\epsilon(\omega, z) = \int e^{i\omega s} a^\epsilon(s, z) ds, \qquad \hat{b}^\epsilon(\omega, z) = \int e^{i\omega s} b^\epsilon(s, z) ds,$$

we arrive at the system

$$\frac{d}{dz}\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} = \frac{i\theta\omega\sigma}{2\bar{c}\epsilon}\nu\left(\frac{z}{\epsilon^2}\right)\begin{bmatrix} 1 & -e^{-2i\theta\omega z/(\epsilon\bar{c})} \\ e^{+2i\theta\omega z/(\epsilon\bar{c})} & -1 \end{bmatrix}\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix}$$

with boundary conditions given by

$$\hat{a}^\epsilon(\omega,0) = \int e^{i\omega s} A^\epsilon\left(\frac{\epsilon}{\delta}s, 0\right) ds = \int e^{i\omega s} f(s) ds = \hat{f}(\omega),$$
$$\hat{b}^\epsilon(\omega,L) = 0.$$

4.2. Reflection of Monochromatic Waves

Now we are ready to derive the reflection of monochromatic waves through a random slab on $[0,\hat{L}]$ in the weakly heterogeneous regime, where the frequency of the waves is ω/ϵ^2, the fluctuations in the random medium are of order ϵ^2, and the size of the slab is order 1. We will take $\hat{L} = M - L$ so that the width of the slab is consistent with other chapters.

The right-going and left-going modes $\hat{a}^\epsilon$ and $\hat{b}^\epsilon$ satisfy the BVP

$$\frac{d}{dz}\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} = \frac{1}{\epsilon}\mathbf{H}_\omega\left(\frac{z}{\epsilon^2}, \nu\left(\frac{z}{\epsilon^2}\right)\right)\begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix}, \tag{4.4}$$

$$\mathbf{H}_\omega(z,\nu) = \frac{i\omega}{2\bar{c}}\nu\begin{bmatrix} 1 & -e^{-2i\omega z/\bar{c}} \\ e^{2i\omega z/\bar{c}} & -1 \end{bmatrix}$$

with boundary conditions

$$\hat{a}^\epsilon(\omega,0) = 1, \qquad \hat{b}^\epsilon(\omega,\hat{L}) = 0. \tag{4.5}$$

The reflection and transmission coefficients are given by

$$R_\omega^\epsilon(0,\hat{L}) = \hat{b}^\epsilon(\omega,0), \qquad T_\omega^\epsilon(0,\hat{L}) = \hat{a}^\epsilon(\omega,\hat{L}).$$

First, we convert the above BVP into an IVP by using the propagator $\mathbf{P}_\omega$, a 2×2 complex matrix function satisfying

$$\frac{d}{dz}\mathbf{P}_\omega(0,z) = \mathbf{H}_\omega(z, z/l)\mathbf{P}_\omega(0,z), \qquad \mathbf{P}_\omega(0,0) = \mathbf{I}.$$

The matrix $\mathbf{P}_\omega(0,z)$ is of the form

$$\mathbf{P}_\omega = \begin{bmatrix} \alpha_\omega & \overline{\beta_\omega} \\ \beta\omega & \overline{\alpha_\omega} \end{bmatrix}. \tag{4.6}$$

This is seen by using Jacobi's formula for the derivative of a determinant

$$\frac{d\det(\mathbf{P}_\omega)}{dz} = \mathrm{tr}\left(\mathrm{Adj}(\mathbf{P}_\omega)\frac{d\mathbf{P}_\omega}{dz}\right)$$

where the adjugate matrix $\mathbf{P}_\omega\mathrm{Adj}(\mathbf{P}_\omega) = \det(\mathbf{P}_\omega)\mathbf{I}$, so that

$$\begin{aligned}\frac{d\det(\mathbf{P}_\omega)}{dz} &= \mathrm{Tr}(\mathrm{Adj}(\mathbf{P}_\omega)\mathbf{H}_\omega\mathbf{P}_\omega) = \mathrm{Tr}(\mathbf{H}_\omega\mathbf{P}_\omega\mathrm{Adj}(\mathbf{P}_\omega)) \\ &= \mathrm{Tr}(\mathbf{H}_\omega)\det(\mathbf{P}_\omega) = 0.\end{aligned}$$

Thus

$$\det(\mathbf{P}_\omega) = \det(\mathbf{H}_\omega) = 1.$$

If $(\alpha_\omega, \beta\omega)^T$ satisfies equation (4.2) with initial condition $(1,0)^T$, then $(\overline{\beta_\omega}, \overline{\alpha_\omega})^T$ satisfies the same equation with initial condition $(0,1)^T$. Thus, we get equation (4.2) with

$$|\alpha_\omega|^2 - |\beta_\omega|^2 = 1. \tag{4.7}$$

Using the identity

$$\begin{bmatrix} \hat{a}^\epsilon(\omega, z) \\ \\ \hat{b}^\epsilon(\omega, z) \end{bmatrix} = \mathbf{P}^\epsilon_\omega(0, z) \begin{bmatrix} \hat{a}^\epsilon(\omega, 0) \\ \\ \hat{b}^\epsilon(\omega, 0) \end{bmatrix}$$

applied at $z = \hat{L}$ and the boundary condition (4.2), we find that

$$R^\epsilon_\omega(0, \hat{L}) = -\frac{\beta^\epsilon_\omega(0, \hat{L})}{\overline{\overline{\alpha^\epsilon_\omega(0, \hat{L})}}}.$$

By using Euler's identity $e^{ix} = \cos(x) + i\sin(x)$, we get the ODE

$$\begin{aligned} \frac{d}{dz}\mathbf{P}^\epsilon_\omega(0, z) = \frac{i\omega}{2\epsilon\bar{c}}\nu\left(\frac{z}{\epsilon^2}\right) & \begin{bmatrix} 1 & 0 \\ \\ 0 & -1 \end{bmatrix} \mathbf{P}^\epsilon_\omega(0, z) \\ & - \frac{\omega}{2\bar{c}\epsilon}\nu\left(\frac{z}{\epsilon^2}\right)\sin\left(\frac{2\omega z}{\bar{c}\epsilon^2}\right) \begin{bmatrix} 0 & 1 \\ \\ 1 & 0 \end{bmatrix} \mathbf{P}^\epsilon_\omega(0, z) \\ & - \frac{i\omega}{2\bar{c}\epsilon}\nu\left(\frac{z}{\epsilon^2}\right)\cos\left(\frac{2\omega z}{\bar{c}\epsilon^2}\right) \begin{bmatrix} 0 & 1 \\ \\ -1 & 0 \end{bmatrix} \mathbf{P}^\epsilon_\omega(0, z) \end{aligned}$$

The ODE is of the form

$$\frac{d}{dz}\mathbf{P}^\epsilon_\omega(0, z) = \frac{1}{\epsilon}\mathbf{F}\left(\mathbf{P}^\epsilon_\omega(0, z), \nu\left(\frac{z}{\epsilon^2}\right), \frac{z}{\epsilon^2}\right) \tag{4.8}$$

where

$$\mathbf{F}(\mathbf{P}, \nu, \tau) = \frac{\omega}{2\bar{c}} \sum_{p=0}^{2} g^{(p)}(\nu, \tau)\mathbf{h}_p\mathbf{P}$$

$$\mathbf{h}_0 = i\text{œ}_3, \quad \mathbf{h}_1 = -\text{œ}_1, \quad \mathbf{h}_2 = \text{œ}_2,$$

$$\text{œ}_1 = \begin{bmatrix} 0 & 1 \\ \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -i \\ \\ i & 0 \end{bmatrix}, \qquad \text{œ}_3 = \begin{bmatrix} 1 & 0 \\ \\ 0 & -1 \end{bmatrix}$$

$$g^{(0)}(\nu,\tau)=\nu, \qquad g^{(1)}(\nu,\tau)=\nu\sin\left(\frac{2\omega z}{\bar{c}}\right), \qquad g^{(2)}(\nu,\tau)=\nu\cos\left(\frac{2\omega z}{\bar{c}}\right)$$

We are interested in the limiting stochastic process as $\epsilon \to 0$. We will use the following Theorem from [8], page 140:

Theorem 4.1 *Let the process $X^\epsilon(z)$ be defined by the system of random ordinary differential equations*

$$\frac{dX^\epsilon}{dz}(z)=\frac{1}{\epsilon}F\left(X^\epsilon(z),Y\left(\frac{z}{\epsilon^2}\right),\frac{z}{\epsilon^2}\right)$$

starting from $X^\epsilon(0)=x_0\in\mathbb{R}^d$. Assume that $Y(z)$ is a z-homogeneous Markov process on S with generator $\mathcal{L}_Y$ satisfying the Fredholm alternative, and the $\mathbb{R}^d$-valued function F satisfies the centering condition $\mathbb{E}[F(x,Y(0))]=0$, where $\mathbb{E}[\cdot]$ denotes expectation with respect to the invariant probability distribution of $Y(z)$. Assume also that $F(x,y,\tau)$ and $G(x,y,\tau)$ are at most linearly growing and smooth in x and that $F(x,y,\tau)$ and $G(x,y,\tau)$ are periodic with respect to τ with period Z_0 where F satisfies the centering condition

$$\int_0^{Z_0}\mathbb{E}[F(x,Y(0),\tau)]d\tau=0$$

for all x. Then the random processes $X^\epsilon(z)$ converge in distribution to the diffusion process $X(z)$ with generator

$$\mathcal{L}\phi(x)=\frac{1}{Z_0}\int_0^{Z_0}\int_0^\infty\mathbb{E}[F(x,Y(0),\tau)\cdot\nabla_x(F(x,Y(z),\tau+z)\cdot\nabla_x\phi(x))]dzd\tau. \tag{4.9}$$

Inserting the F below

$$\mathbf{F}(\mathbf{P},\nu,\tau)=\frac{\omega}{2\bar{c}}\sum_{p=0}^{2}g^{(p)}(\nu,\tau)\mathbf{h}_p\mathbf{P}$$

into the expression for the generator gives

$$\mathcal{L}\phi(\mathbf{P})=\frac{1}{2\pi}\sum_{i,j=0}^{2}\int_0^{2\pi}\int_0^\infty\mathbb{E}[g^{(i)}(\nu(0),\tau)g^{(j)}(\nu(z),\tau+z)]dzd\tau$$
$$\times\,\mathbf{h}_i\mathbf{P}\cdot\nabla_{\mathbf{P}}(\mathbf{h}_j\mathbf{P}\cdot\nabla_{\mathbf{P}}\phi(\mathbf{P}))$$

where $\mathbf{P} \cdot \mathbf{Q} = \sum_{i,j} P_{ij} Q_{ij}$. Define the correlation integrals C_{ij} by

$$C_{ij} = 2\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \mathbb{E}[g^{(i)}(\nu(0),\tau)g^{(j)}(\nu(z),\tau+z)]dzd\tau, \quad p,q=0,1,2$$

and expanding out the dot product terms:

$$\begin{aligned}
\mathbf{h}_i\mathbf{P} \cdot \nabla_{\mathbf{P}}(\mathbf{h}_j\mathbf{P} \cdot \nabla_{\mathbf{P}}\phi(\mathbf{P})) &= \sum_{a,b=1}^{2} (\mathbf{h}_i\mathbf{P})_{ab} \frac{\partial}{\partial P_{ab}} \left(\sum_{m,n=1}^{2} (\mathbf{h}_j\mathbf{P})_{mn} \frac{\partial \phi(\mathbf{P})}{\partial P_{mn}} \right) \\
&= \sum_{a,b=1}^{2} \sum_{m,n=1}^{2} (\mathbf{h}_i\mathbf{P})_{ab} \frac{\partial (\mathbf{h}_j\mathbf{P})_{mn}}{\partial P_{ab}} \frac{\partial \phi(\mathbf{P})}{\partial P_{mn}} \\
&+ \sum_{a,b=1}^{2} \sum_{m,n=1}^{2} (\mathbf{h}_i\mathbf{P})_{ab} (\mathbf{h}_j\mathbf{P})_{mn} \frac{\partial^2 \phi(\mathbf{P})}{\partial P_{ab} \partial P_{mn}}
\end{aligned}$$

Then, the infinitesimal generator is

$$\begin{aligned}
\mathcal{L}\phi(x) = \frac{1}{2} \sum_{i,j=0}^{2} \sum_{a,b,m,n=1}^{2} C_{ij} (\mathbf{h}_i\mathbf{P})_{ab} (\mathbf{h}_j\mathbf{P})_{mn} \frac{\partial^2 \phi(\mathbf{P})}{\partial P_{ab} \partial P_{mn}} \\
+ \frac{1}{2} \sum_{i,j=0}^{2} \sum_{a,b,m,n=1}^{2} C_{ij} (\mathbf{h}_i\mathbf{P})_{ab} \frac{\partial (\mathbf{h}_j\mathbf{P})_{mn}}{\partial P_{ab}} \frac{\partial \phi(\mathbf{P})}{\partial P_{mn}}
\end{aligned} \tag{4.10}$$

Recall that a general diffusion process with diffusion matrix $a_{ij}(x)$ and drift vector $b_i(x)$ has infinitesimal generator given by

$$\mathcal{L}\phi = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial \phi}{\partial x_i}. \tag{4.11}$$

Moreover, if the diffusion coefficients $a_{ij}(x)$ can be factored as

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x) \tag{4.12}$$

then the diffusion process $X(z)$ is the solution of the SDE

$$dX_i(z) = b_i(X(z))dz + \sum_{j=1}^{d} \sigma_{ij}(X(z))dW_j(z), \quad i = 1, \dots, d,$$

where the W_j are independent Brownian motion processes. Comparing equation (4.2) with equation (4.2), we see that for our case the diffusion and drift coefficients are given by

$$\begin{aligned} a_{abmn}(\mathbf{P}) &= \sum_{i,j=0}^{2} C_{ij}(\mathbf{h}_i\mathbf{P})_{ab}(\mathbf{h}_j\mathbf{P})_{mn} \\ b_{mn}(x) &= \frac{1}{2}\sum_{i,j=0}^{2}\sum_{a,b=1}^{2} C_{ij}(\mathbf{h}_i\mathbf{P})_{ab}\frac{\partial(\mathbf{h}_j\mathbf{P})_{mn}}{\partial P_{ab}} \end{aligned}$$

Now we just need to factor $a_{cd}(x)$ as in equation (4.2) to derive our SDE. First, we shall compute the components of $\mathbf{C}$:

$$\begin{aligned} C_{00} &= 2\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]dz, \\ C_{11} &= 2\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]dz\sin(x)\sin\left(x + \frac{2\omega z}{\bar{c}}\right)dx \\ &= \frac{1}{\pi}\int_0^{2\pi}\sin^2(x)dx\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]\cos\left(\frac{2\omega z}{\bar{c}}\right)dz \\ &\quad + \frac{1}{\pi}\int_0^{2\pi}\sin(x)\cos(x)dx\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]\sin\left(\frac{2\omega z}{\bar{c}}\right)dz \\ &= \int_0^\infty \mathbb{E}[\nu(0)\nu(z)]\cos\left(\frac{2\omega z}{\bar{c}}\right)dz. \end{aligned}$$

The other entries are computed similarly and we have

$$\mathbf{C} = \begin{bmatrix} \gamma(0) & 0 & 0 \\ 0 & \frac{1}{2}\gamma(\omega) & -\frac{1}{2}\gamma^{(s)}(\omega) \\ 0 & \frac{1}{2}\gamma^{(s)}(\omega) & \frac{1}{2}\gamma(\omega) \end{bmatrix}$$

where

$$\gamma(\omega) = 2\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]\cos\left(\frac{2\omega z}{\bar{c}}\right)dz,$$
$$\gamma^{(s)}(\omega) = 2\int_0^\infty \mathbb{E}[\nu(0)\nu(z)]\sin\left(\frac{2\omega z}{\bar{c}}\right)dz.$$

We decompose $\mathbf{C}$ into its symmetric and antisymmetric parts

$$\mathbf{C}^{(S)} = \begin{bmatrix} \gamma(0) & 0 & 0 \\ 0 & \frac{1}{2}\gamma(\omega) & 0 \\ 0 & 0 & \frac{1}{2}\gamma(\omega) \end{bmatrix}, \qquad \mathbf{C}^{(A)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\gamma^{(s)}(\omega) \\ 0 & \frac{1}{2}\gamma^{(s)}(\omega) & 0 \end{bmatrix}$$

Now define $\tilde{\text{œ}}$ as the square root of $\mathbf{C}^{(S)}$

$$\tilde{\text{œ}} = \begin{bmatrix} \sqrt{\gamma(0)} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\sqrt{\gamma(\omega)} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}\sqrt{\gamma(\omega)} \end{bmatrix}$$

Then, taking

$$\sigma_{ijl}(\mathbf{P}) = \sum_{p=0}^{2} \tilde{\sigma}_{lp}(\mathbf{h}_p\mathbf{P})_{ij}$$

$\sigma_{ijl}(x)$ satisfies equation (4.2) so $\mathbf{P}_\omega(0,z)$ satisfies the (Itô) SDE

$$d\mathbf{P}_{ij\omega}(0,z) = \sum_{l=0}^{2} \sigma_{ijl}(\mathbf{P}_\omega(0,z))dW_l(z) + b_{ij}(\mathbf{P}_\omega(0,z))dz, \quad i,j = 1,2. \tag{4.13}$$

In the Itô Calculus, the ordinary chain rule does not hold. One must use Itô's Lemma. However, if we convert this SDE to Stratonovich form, the ordinary chain rule applies. The relationship between the Itô and Stratonovich integral above is

$$\int_0^z \sigma_{ijl}(\mathbf{P})dW_l(s) = \int_0^z \sigma_{ijl}(\mathbf{P}) \circ dW_l(s) - \frac{1}{2}\sum_{a,b=1}^{2}\int_0^z \sigma_{abl}(\mathbf{P})\frac{\partial \sigma_{ijl}}{\partial P_{ab}}(\mathbf{P})ds. \tag{4.14}$$

where $\circ\, dW_l(s)$ denotes Stratonovich integration. Using the fact that

$$\frac{1}{2}\sum_{p,q=0}^{2}\sum_{i,j=1}^{2} C_{pq}^{(S)}(\mathbf{h}_p\mathbf{P})_{ij}\frac{\partial(\mathbf{h}_q\mathbf{P})_{ab}}{\partial P_{ij}} = \frac{1}{2}\sum_{l=0}^{2}\sum_{i,j=1}^{2}\sigma_{ijl}\frac{\partial \sigma_{abl}}{\partial P_{ij}}$$

and equation (4.2) we can convert equation (4.2) to Stratonovich form:

$$\begin{aligned}
dP_{ij\omega} &= \sum_{l=0}^{2}\sigma_{ijl}\circ dW_l(z) - \frac{1}{2}\sum_{l=0}^{2}\sum_{a,b=1}^{2}\sigma_{abl}\frac{\partial \sigma_{ijl}}{\partial P_{ab}}dz + b_{ij}dz \\
&= \sum_{l=0}^{2}\sum_{p=0}^{2}\tilde{\sigma}_{lp}(\mathbf{h}_p\mathbf{P})_{ij}\circ dW_l(z) - \frac{1}{2}\sum_{p,q=0}^{2}\sum_{a,b=0}^{2}C_{pq}^{(S)}(\mathbf{h}_p\mathbf{P})_{ab}\frac{\partial(\mathbf{h}_p\mathbf{P})_{ij}}{\partial P_{ab}} + b_{ij}dz \\
&= \sum_{l=0}^{2}\sum_{p=0}^{2}\tilde{\sigma}_{lp}(\mathbf{h}_p\mathbf{P})_{ij}\circ dW_l(z) + \frac{1}{2}\sum_{p,q=0}^{2}C_{pq}^{(A)}(\mathbf{h}_q\mathbf{h}_p\mathbf{P})_{ij}dz.
\end{aligned}$$

In matrix form, we have

$$d\mathbf{P}_\omega(0,z) = \sum_{l=0}^{2}\tilde{\sigma}_{ll}\mathbf{h}_p\mathbf{P}\circ dW_l(z) + \frac{1}{2}\sum_{p,q=0}^{2}C_{pq}^{(A)}\mathbf{h}_q\mathbf{h}_p\mathbf{P}dz$$

which upon expansion and using the fact that $\mathbf{h}_2\mathbf{h}_1 = \mathbf{h}_0$ in the dz term gives

$$\begin{aligned}
d\mathbf{P}_\omega(0,z) = {} & \frac{i\omega\sqrt{\gamma(0)}}{2\bar{c}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{P}_\omega(0,z) \circ dW_0(z) \\
& - \frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\bar{c}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{P}_\omega(0,z) \circ dW_1(z) \\
& - \frac{i\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\bar{c}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{P}_\omega(0,z) \circ dW_2(z) \\
& - \frac{i\omega^2\gamma^{(s)}(\omega)}{8\bar{c}^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{P}_\omega(0,z) dz.
\end{aligned} \tag{4.15}$$

We saw earlier in equation (4.2) that $\mathbf{P}_\omega(0,z)$ is of the form

$$\mathbf{P}_\omega(0,z) = \begin{bmatrix} \alpha_\omega(0,z) & \overline{\beta_\omega(0,z)} \\ \beta_\omega(0,z) & \overline{\alpha_\omega(0,z)}. \end{bmatrix}$$

It follows that $(\alpha_\omega, \beta_\omega)$ satisfies the system

$$\begin{aligned}
d\alpha_\omega = {} & \frac{\omega}{2\bar{c}} \left(i\sqrt{\gamma(0)}\alpha_\omega \circ dW_0(z) - \frac{\sqrt{\gamma(\omega)}}{\sqrt{2}}\beta_\omega \circ (dW_1(z) + idW_2(z)) \right) \\
& - \frac{i\omega^2\gamma^{(s)}(\omega)}{8\bar{c}^2}\alpha_\omega dz, \\
d\beta_\omega = {} & \frac{\omega}{2\bar{c}} \left(-i\sqrt{\gamma(0)}\beta_\omega \circ dW_0(z) - \frac{\sqrt{\gamma(\omega)}}{\sqrt{2}}\alpha_\omega \circ (dW_1(z) - idW_2(z)) \right) \\
& + \frac{i\omega^2\gamma^{(s)}(\omega)}{8\bar{c}^2}\beta_\omega dz
\end{aligned}$$

and satisfies $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$. Also each matrix in equation (4.2) has trace equal to zero, so we can parameterize $(\alpha_\omega, \beta_\omega)$ as

$$\alpha_\omega(0,z) = \cosh\left(\frac{\theta_\omega(z)}{2}\right) e^{i\phi_\omega(z)},$$
$$\beta_\omega(0,z) = \sinh\left(\frac{\theta_\omega(z)}{2}\right) e^{i(\psi_\omega(z)+\phi_\omega(z))},$$

where $\theta_\omega(z) \in [0,\infty)$, $\psi_\omega(z)$, $\phi_\omega(z) \in \mathbb{R}$. Since we are in the Stratonovich framework, we can use the ordinary chain rule of calculus and select a branch $\theta_\omega(z) \in [0,\infty)$ to obtain the system

$$\begin{aligned}
d\phi_\omega &= -\frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\bar{c}} \tanh\left(\frac{\theta_\omega}{2}\right)(\sin(\psi_\omega) \circ dW_1(z) + \cos(\psi_\omega) \circ dW_2(z)) \\
&\quad + \frac{\omega\sqrt{\gamma(0)}}{2\bar{c}} \circ dW_0(z) - \frac{\omega^2\gamma^{(s)}(\omega)}{8\bar{c}^2} dz, \\
d\psi_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}\tanh(\theta_\omega)}(\sin(\psi_\omega) \circ dW_1(z) + \cos(\psi_\omega) \circ dW_2(z)) \\
&\quad - \frac{\omega\sqrt{\gamma(0)}}{\bar{c}} \circ dW_0(z) + \frac{\omega^2\gamma^{(s)}(\omega)}{4\bar{c}^2} dz, \\
d\theta_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}}(-\cos(\psi_\omega) \circ dW_1(z) + \sin(\psi_\omega) \circ dW_2(z)).
\end{aligned}$$

Next we wish to convert the above system back to Itô form. To do this, we need to compute the correction terms in the Itô to Stratonovich formula (4.2) given by

$$\int_0^z \sigma_{ip}(\mathbf{x}(s)) \circ dW_p(s) = \int_0^z \sigma_{ip}(\mathbf{x}(s)) dW_p(s) + \frac{1}{2}\sum_{j=1}^{3} \int_0^z \sigma_{jp}(\mathbf{x}(s)) \frac{\partial \sigma_{ip}(\mathbf{x}(s))}{\partial x_j} ds$$

where $\mathbf{x} = (\phi_\omega, \psi_\omega, \theta_\omega)^T$ and σ_{ij}, $i,j = 0,1,2$ is the matrix of coefficients in front of the

$dW_j(z)$ terms. For instance, we can find the correction in the $d\phi_\omega$ equation by calculating

$$\begin{aligned}
\sum_{j=1}^{3} \sigma_{j0} \frac{\partial \sigma_{00}}{\partial x_j} &= 0, \\
\sum_{j=1}^{3} \sigma_{j1} \frac{\partial \sigma_{01}}{\partial x_j} &= \frac{\omega^2 \gamma(\omega)}{8\bar{c}^2} \sin(\psi)\cos(\psi) \left(\frac{2\tanh(\theta/2)}{\tanh(\theta)} - \frac{1}{\cosh^2(\theta/2)} \right), \\
\sum_{j=1}^{3} \sigma_{j2} \frac{\partial \sigma_{02}}{\partial x_j} &= -\frac{\omega^2 \gamma(\omega)}{8\bar{c}^2} \sin(\psi)\cos(\psi) \left(\frac{2\tanh(\theta/2)}{\tanh(\theta)} - \frac{1}{\cosh^2(\theta/2)} \right).
\end{aligned}$$

Thus, the correction terms for the dW_1 and dw_2 terms cancel, and therefore the $d\phi_\omega$ line is unaffected. After calculating the $d\psi_\omega$ and $d\theta_\omega$ corrections, we get

$$\begin{aligned}
d\phi_\omega &= -\frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\bar{c}} \tanh\left(\frac{\theta_\omega}{2}\right) \left(\sin(\psi_\omega) dW_1(z) + \cos(\psi_\omega) dW_2(z)\right) \\
&\quad + \frac{\omega\sqrt{\gamma(0)}}{2\bar{c}} dW_0(z) - \frac{\omega^2 \gamma^{(s)}(\omega)}{8\bar{c}^2} dz, \\
d\psi_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}\tanh(\theta_\omega)} \left(\sin(\psi_\omega) dW_1(z) + \cos(\psi_\omega) dW_2(z)\right) \\
&\quad - \frac{\omega\sqrt{\gamma(0)}}{\bar{c}} dW_0(z) + \frac{\omega^2 \gamma^{(s)}(\omega)}{4\bar{c}^2} dz, \\
d\theta_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}} \left(-\cos(\psi_\omega) dW_1(z) + \sin(\psi_\omega) dW_2(z)\right) \\
&\quad + \frac{\omega^2 \gamma(\omega)}{4\bar{c}^2 \tanh(\theta_\omega)} dz.
\end{aligned}$$

Finally, we can introduce a pair of new processes (W_1^*, W_2^*) by the orthogonal transformation

$$\begin{bmatrix} W_1^*(z) \\ W_2^*(z) \end{bmatrix} = \int_0^z \begin{bmatrix} \sin(\psi_\omega) & \cos(\psi_\omega) \\ -\cos(\psi_\omega) & \sin(\psi_\omega) \end{bmatrix} d \begin{bmatrix} W_1(z) \\ W_2(z) \end{bmatrix}.$$

(W_1^*, W_2^*) by the orthogonal transformation remain independent standard Brownian motions.

With these new processes we have the simplified system

$$
\begin{aligned}
d\phi_\omega &= -\frac{\omega\sqrt{\gamma(\omega)}}{2\sqrt{2}\bar{c}}\tanh\left(\frac{\theta_\omega}{2}\right)dW_1^*(z) + \frac{\omega\sqrt{\gamma(0)}}{2\bar{c}}dW_0(z) \\
&\quad - \frac{\omega^2\gamma^{(s)}(\omega)}{8\bar{c}^2}dz, \\
d\psi_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}\tanh(\theta_\omega)}dW_1^*(z) - \frac{\omega\sqrt{\gamma(0)}}{\bar{c}}dW_0(z) + \frac{\omega^2\gamma^{(s)}(\omega)}{4\bar{c}^2}dz, \\
d\theta_\omega &= \frac{\omega\sqrt{\gamma(\omega)}}{\sqrt{2}\bar{c}}dW_2^*(z) + \frac{\omega^2\gamma(\omega)}{4\bar{c}^2\tanh(\theta_\omega)}dz.
\end{aligned}
\tag{4.16}
$$

We therefore conclude that in the monochromatic case, the reflection coefficient is given by

$$
R_\omega^\epsilon(0,\hat{L}) = -\frac{\overline{\beta_\omega^\epsilon(0,\hat{L})}}{\overline{\alpha_\omega^\epsilon(0,\hat{L})}} = -\tanh\left(\frac{\theta_\omega(\hat{L})}{2}\right)e^{i(\psi_\omega(\hat{L})+2\phi_\omega(\hat{L}))},
$$

where $(\phi_\omega, \psi_\omega, \theta_\omega)$ satisfy the system (4.2).

In the next section we will continue following [8] to generalize the result to the reflection of incoherent waves.

4.3. Reflection of Incoherent Waves

We start with the linear 1D wave equation

$$
\begin{aligned}
\rho(z)\frac{\partial u^\epsilon}{\partial t} + \frac{\partial p^\epsilon}{\partial z} &= 0, \\
\frac{1}{K(z)}\frac{\partial p^\epsilon}{\partial t} + \frac{\partial u^\epsilon}{\partial z} &= 0.
\end{aligned}
\tag{4.17}
$$

The medium parameters are given by

$$
\begin{aligned}
\frac{1}{K(z)} &= \begin{cases} \frac{1}{\bar{K}}(1+\nu(z/\epsilon^2)) & \text{for } z \in [-\hat{L}, 0], \\ \frac{1}{\bar{K}} & \text{for } z \in (-\infty, -\hat{L}) \cup (0,\infty), \end{cases} \\
\rho(z) &= \bar{\rho} \quad \text{for all } z.
\end{aligned}
$$

The random slab is now on the interval $[-\hat{L}, 0]$ instead of $[0, \hat{L}]$, and the pulse is now incoming from the right, so that the reflected wave travels to the right as well.

The analysis is performed in the strongly heterogeneous white-noise regime, in which he takes the pulse width to be of order ϵ, and the pulse amplitude to have order 1. The pulse is of the form

$$\frac{1}{\sqrt{\epsilon}} f\left(\frac{t}{\epsilon}\right),$$

where f is square-integrable so that

$$\int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{\epsilon}} f\left(\frac{t}{\epsilon}\right)\right]^2 dt = \int_{-\infty}^{\infty} f(u)^2 du < \infty.$$

Now we transform the wave equation (4.3) by introducing the right- and left-going modes

$$\begin{aligned} A^\epsilon(t,z) &= \overline{\zeta}^{1/2} u^\epsilon(t,z) + \overline{\zeta}^{-1/2} p^\epsilon(t,z), \\ B^\epsilon(t,z) &= \overline{\zeta}^{1/2} u^\epsilon(t,z) - \overline{\zeta}^{-1/2} p^\epsilon(t,z), \end{aligned} \tag{4.18}$$

where the effective impedance is $\overline{\zeta} = \sqrt{\overline{K}\overline{\rho}}$. By calculating the derivatives of equation (4.3), using equation (4.3), one arrives at the system

$$\frac{\partial}{\partial z}\begin{bmatrix} A \\ B \end{bmatrix} = -\frac{1}{2\bar{c}}\begin{bmatrix} 2 + \nu(z/\epsilon^2) & \nu(z/\epsilon^2) \\ -\nu(z/\epsilon^2) & -2 - \nu(z/\epsilon^2) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

where the effective speed $\bar{c} = \sqrt{\overline{K}/\bar{\rho}}$. Now, we transform coordinates again:

$$\begin{aligned} a^\epsilon(s,z) &= A^\epsilon(\epsilon s + z/\bar{c}, z), \\ b^\epsilon(s,z) &= B^\epsilon(\epsilon s - z/\bar{c}, z). \end{aligned}$$

Taking the Fourier transform with respect to the time variable s,

$$\hat{a}(\omega, z) = \int e^{i\omega s} a(s, z) ds, \qquad \hat{b}(\omega, z) = \int e^{i\omega s} b(s, z) ds,$$

one arrives at the system

$$\frac{d}{dz} \begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix} = \frac{1}{\epsilon} \mathbf{H}_\omega \left(\frac{z}{\epsilon}, \nu \left(\frac{z}{\epsilon^2} \right) \right) \begin{bmatrix} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{bmatrix}, \tag{4.19}$$

$$\mathbf{H}_\omega(z, \nu) = \frac{i\omega}{2\bar{c}} \nu \begin{bmatrix} 1 & -e^{-2i\omega z/\bar{c}} \\ e^{2i\omega z/\bar{c}} & -1 \end{bmatrix}.$$

The boundary conditions are

$$\hat{b}^\epsilon(\omega, 0) = \frac{1}{\sqrt{\epsilon}} \hat{f}(\omega), \qquad \hat{a}^\epsilon(\omega, -\hat{L}) = 0.$$

Once again, we transform the BVP into an IVP by defining the propagator matrix $\mathbf{P}^\epsilon_\omega$ satisfying

$$\begin{bmatrix} \hat{a}^\epsilon(\omega, z) \\ \hat{b}^\epsilon(\omega, z) \end{bmatrix} = \mathbf{P}^\epsilon_\omega(-\hat{L}, z) \begin{bmatrix} \hat{a}^\epsilon(\omega, -\hat{L}) \\ \hat{b}^\epsilon(\omega, -\hat{L}) \end{bmatrix}.$$

The propagator matrix is of the form

$$\mathbf{P}^\epsilon_\omega(-\hat{L}, z) = \begin{bmatrix} \alpha^\epsilon_\omega(-\hat{L}, z) & \overline{\beta^\epsilon_\omega(-\hat{L}, z)} \\ \beta^\epsilon_\omega(-\hat{L}, z) & \overline{\alpha^\epsilon_\omega(-\hat{L}, z)} \end{bmatrix}, \tag{4.20}$$

where $(\alpha^\epsilon_\omega, \beta^\epsilon_\omega)$ is a solution of equation (4.3) with the initial conditions

$$\alpha^\epsilon_\omega(-\hat{L}, z = -\hat{L}) = 1, \qquad \beta^\epsilon_\omega(-\hat{L}, z = -\hat{L}) = 0.$$

Now define the transmission and reflection coefficients on a slab $[-\hat{L}, z]$ by

$$\mathbf{P}^{\epsilon}_{\omega} \begin{bmatrix} 0 \\ \\ T^{\epsilon}_{\omega}(-\hat{L}, z) \end{bmatrix} = \begin{bmatrix} R^{\epsilon}(-\hat{L}, z) \\ \\ 1 \end{bmatrix}$$

In terms of equation (4.3), the reflection coefficient is given by

$$R^{\epsilon}_{\omega}(-\hat{L}, z) = \overline{\frac{\beta^{\epsilon}_{\omega}(-\hat{L}, z)}{\alpha^{\epsilon}_{\omega}(-\hat{L}, z)}}. \tag{4.21}$$

The reflected wave is given in terms of the reflection coefficient as

$$\hat{a}^{\epsilon}(\omega, 0) = \frac{1}{\sqrt{\epsilon}} \hat{f}(\omega) R^{\epsilon}_{\omega}(-\hat{L}, 0).$$

Now, by differentiating equation (4.3) and using equation (4.3), we find that the Reflection coefficient is given by

$$\begin{aligned} \frac{dR^{\epsilon}_{\omega}}{dz} &= \overline{\frac{1}{\alpha^{\epsilon}_{\omega}} \frac{d\beta^{\epsilon}_{\omega}}{dz} - \frac{\beta^{\epsilon}_{\omega}}{(\alpha^{\epsilon}_{\omega})^2} \frac{d\alpha^{\epsilon}_{\omega}}{dz}} \\ &= -\frac{i\omega}{2\bar{c}\epsilon} \nu \left(\frac{z}{\epsilon^2} \right) (e^{-2i\omega z/(\bar{c}\epsilon)} - 2R^{\epsilon}_{\omega} + (R^{\epsilon}_{\omega})^2 e^{2i\omega z/(\bar{c}\epsilon)}). \end{aligned} \tag{4.22}$$

with initial condition

$$R^{\epsilon}_{\omega}(-\hat{L}, z = -\hat{L}) = 0.$$

The reflected wave at $z = 0$ then has the representation

$$A^{\epsilon}(t, 0) = a^{\epsilon} \left(\frac{t}{\epsilon}, 0 \right) = \frac{1}{2\pi} \int \hat{a}^{\epsilon}(\omega, 0) e^{-i\frac{\omega t}{\epsilon}} d\omega = \frac{1}{2\pi\sqrt{\epsilon}} \int R^{\epsilon}_{\omega}(-\hat{L}, 0) \hat{f}(\omega) e^{-i\frac{\omega t}{\epsilon}} d\omega.$$

We will find the statistical distribution of the reflected wave by finding its moments. The mean amplitude of the reflected wave is

$$\mathbb{E}[A^{\epsilon}(t, 0)] = \frac{1}{2\pi\sqrt{\epsilon}} \int \mathbb{E}[R^{\epsilon}_{\omega}(-\hat{L}, 0)] \hat{f}(\omega) e^{-i\frac{\omega t}{\epsilon}} d\omega.$$

The second moment is

$$A^\epsilon(t,0)^2 = \frac{1}{4\pi^2\epsilon}\left(\int R^\epsilon_{\omega_1}(-\hat{L},0)\hat{f}(\omega_1)e^{-\frac{i\omega_1 t}{\epsilon}}d\omega_1\right)\left(\int \overline{R^\epsilon_{\omega_2}(-\hat{L},0)\hat{f}(\omega_2)}e^{\frac{i\omega_2 t}{\epsilon}}d\omega_2\right)$$
$$= \frac{1}{4\pi^2\epsilon}\int\int R^\epsilon_{\omega_1}(-\hat{L},0)\overline{R^\epsilon_{\omega_2}(-\hat{L},0)}\hat{f}(\omega_1)\overline{\hat{f}(\omega_2)}e^{i\frac{(\omega_2-\omega_1)t}{\epsilon}}d\omega_1 d\omega_2.$$

so that the mean intensity is

$$\mathbb{E}[A^\epsilon(t,0)^2] = \frac{1}{4\pi^2\epsilon}\int\int \mathbb{E}[R^\epsilon_{\omega_1}(-\hat{L},0)\overline{R^\epsilon_{\omega_2}(-\hat{L},0)}]\hat{f}(\omega_1)\overline{\hat{f}(\omega_2)}e^{i\frac{(\omega_2-\omega_1)t}{\epsilon}}d\omega_1 d\omega_2.$$

Introducing the change of variables

$$\omega_1 = \omega + \epsilon h/2, \quad \omega_2 = \omega - \epsilon h/2,$$

we have

$$\mathbb{E}[A^\epsilon(t,0)^2] = \frac{1}{4\pi^2}\int\int \mathbb{E}[R^\epsilon_{\omega+\epsilon h/2}(-\hat{L},0)\overline{R^\epsilon_{\omega-\epsilon h/2}(-\hat{L},0)}]\hat{f}(\omega+\epsilon h/2)\overline{\hat{f}(\omega-\epsilon h/2)}e^{-iht}d\omega dh$$

We wish to solve the Ricatti equation (4.3) for the Reflection coefficient. To do so, we introduce

$$U^\epsilon_{p,q}(\omega,h,z) = (R^\epsilon_{\omega+\epsilon h/2}(-\hat{L},z))^p(\overline{R^\epsilon_{\omega-\epsilon h/2}(-\hat{L},z)})^q, \quad p,q\in\mathbb{N} \tag{4.23}$$

By using the Ricatti equation (4.3) we see that the family $(U^\epsilon_{p,q})_{p,q\in\mathbb{N}}$ satisfies

$$\frac{\partial U^\epsilon_{p,q}}{\partial z} = \frac{i\omega}{\bar{c}\epsilon}\nu(p-q)U^\epsilon_{p,q} + \frac{i\omega}{2\bar{c}\epsilon}\nu e^{\frac{2i\omega z}{\bar{c}\epsilon}}(qe^{-\frac{ihz}{\bar{c}}}U^\epsilon_{p,q-1} - pe^{\frac{ihz}{\bar{c}}}U^\epsilon_{p+1,q})$$
$$+\frac{i\omega}{2\bar{c}\epsilon}\nu e^{-\frac{2i\omega z}{\bar{c}\epsilon}}(qe^{\frac{ihz}{\bar{c}}}U^\epsilon_{p,q+1} - pe^{\frac{-ihz}{\bar{c}}}U^\epsilon_{p-1,q}), \qquad -\hat{L}\le z\le 0.$$

To remove the slow components $\exp(\pm ihz/\bar{c})$, we take the shifted and scaled Fourier transform with respect to h:

$$V^\epsilon_{p,q}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih(\tau-(p+q)z/\bar{c})}U^\epsilon_{p,q}(\omega,h,z)dh.$$

The system of equations satisfied by $(V^\epsilon_{p,q})_{p,q\in\mathbb{N}}$ is

$$\begin{aligned}\frac{\partial V^\epsilon_{p,q}}{\partial z} &= -\frac{p+q}{\bar{c}}\frac{\partial V^\epsilon_{p,q}}{\partial \tau} + \frac{i\omega}{\bar{c}\epsilon}\nu(p-q)V^\epsilon_{p,q} + \frac{i\omega}{2\bar{c}\epsilon}\nu e^{\frac{2i\omega z}{\bar{c}\epsilon}}(qV^\epsilon_{p,q-1} - pV^\epsilon_{p+1,q}) \\ &\quad + \frac{i\omega}{2\bar{c}\epsilon}\nu e^{-\frac{2i\omega z}{\bar{c}\epsilon}}(qV^\epsilon_{p,q+1} - pV^\epsilon_{p-1,q})\end{aligned} \tag{4.24}$$

with initial condition

$$V^\epsilon_{p,q}(\omega,\tau,z=-\hat{L}) = \begin{cases}\delta(\tau), & \text{for } p,q=0\\ 0 & \text{otherwise.}\end{cases}$$

Now we apply another diffusion approximation theorem, described in [17], an infinite-dimensional analogue of equation (4.1) to arrive at the SDE

$$\begin{aligned}dV_{p,q} = &-\frac{q+p}{\bar{c}}\frac{\partial V_{p,q}}{\partial\tau}dz + \frac{i\sqrt{\gamma}\omega}{\bar{c}}(p-q)V_{p,q}dW_0(z) \\ &+ \frac{i\sqrt{\gamma}\omega}{2\sqrt{2}\bar{c}}(qV_{p,q-1} - pV_{p+1,q} + qV_{p,q+1} - pV_{p-1,q})dW_1(z) \\ &+ \frac{\sqrt{\gamma}\omega}{2\sqrt{2}\bar{c}}(qV_{p,q-1} - pV_{p+1,q} - qV_{p,q+1} + pV_{p-1,q})dW_2(z) \\ &+ \frac{\gamma\omega^2}{4\bar{c}^2}[pq(V_{p+1,q+1} + V_{p-1,q-1} - 2V_{p,q}) - 3(p-q)^2V_{p,q}]dz,\end{aligned} \tag{4.25}$$

where $W_j,\ j=0,1,2$ are three independent Brownian motions and γ is the integrated covariance of the process ν:

$$\gamma = \int_0^\infty \mathbb{E}[\nu(0)\nu(z)]dz.$$

Taking the expectation of equation (4.3), we find that the moments satisfy

$$\begin{aligned}\frac{\partial\mathbb{E}[V_{p,q}]}{\partial z} = &-\frac{q+p}{\bar{c}}\frac{\partial\mathbb{E}[V_{p,q}]}{\partial\tau} - \frac{3\gamma\omega^2}{4\bar{c}^2}(p-q)^2\mathbb{E}[V_{p,q}] \\ &+ \frac{\gamma\omega^2}{4\bar{c}^2}pq(\mathbb{E}[V_{p+1,q+1}] + \mathbb{E}[V_{p-1,q-1}] - 2\mathbb{E}[V_{p,q}]).\end{aligned}$$

The family of moments $f_p(\omega, \tau, z) = \mathbb{E}[V_{p+1,p}(\omega, \tau, z)], p \in \mathbb{N}$ satisfies

$$\frac{\partial f_p}{\partial z} = -\frac{2p+1}{\bar{c}}\frac{\partial f_p}{\partial \tau} + \frac{\gamma\omega^2}{4\bar{c}^2}[p(p+1)(f_{p+1} + f_{p-1} - 2f_p) - 3f_p],$$

starting from $f_p(\omega, \tau, z = -\hat{L}) = 0$. This is a linear system of transport equations starting from a zero initial condition, so $f_p = 0$ for all p. The same is true of the family of moments $f_p(\omega, \tau, z) = \mathbb{E}[V_{p+n_0,p}(\omega, \tau, z)], p \in \mathbb{N}$. Therefore,

$$\mathbb{E}[U^{\epsilon}_{p,q}(\omega, h, 0)] \to 0$$

as $\epsilon \to 0$ for $p \neq q$.

The diagonal family of moments $g_p(\omega, \tau, z) = \mathbb{E}[V_{p,p}(\omega, \tau, z)]$, $p \in \mathbb{N}$ satisfy

$$\frac{\partial g_p}{\partial z} = -\frac{2p}{\bar{c}}\frac{\partial g_p}{\partial \tau} + \frac{\gamma\omega^2}{4\bar{c}^2}p^2(g_{p+1} + g_{p-1} - 2g_p)$$

starting from $g_p(\omega, \tau, z = -\hat{L}) = \delta(\tau)$ for $p = 0$, 0 otherwise. If $\mathcal{W}_p(\omega, \tau, -\hat{L}, z)$ denotes the solution of this system of transport equations, then

$$\begin{aligned}\mathbb{E}[U^{\epsilon}_{1,1}(\omega, h, z)] &= \mathbb{E}[R^{\epsilon}_{\omega+\epsilon h/2}(-\hat{L}, z)\overline{R^{\epsilon}_{\omega-\epsilon h/2}(-\hat{L}, z)}] \\ &= \int \mathbb{E}[V^{\epsilon}_{11}(\omega, \tau, 0)]e^{ih\tau}d\tau \to \int \mathcal{W}_1(\omega, \tau, -\hat{L}, 0)e^{ih\tau}d\tau.\end{aligned}$$

To summarize, we have on page 257 of [8],

Proposition 1 *The expectation of the product of two reflection coefficients at two nearby frequencies,*

$$\mathbb{E}[(R^{\epsilon}_{\omega+\epsilon h/2}(-\hat{L}, 0))^p(\overline{R^{\epsilon}_{\omega-\epsilon h/2(-\hat{L},0)}})^q],$$

has the following limit as $\epsilon \to 0$:

(1) If $p \neq q$, then it converges to 0.

(2) If $p = q$, then it converges to

$$\int \mathcal{W}_p(\omega, \tau, -\hat{L}, 0) e^{ih\tau} d\tau.$$

The analysis of higher moments of the reflection coefficient is very similar. The result is on page 265 of [8]

Proposition 2 *The expectation of the product of $2n$ reflection coefficients*

$$\mathbb{E}\left[\prod_{j=1}^{n} R^{\epsilon}_{\omega_j+\epsilon h_j/2}(-\hat{L}, 0)\overline{R^{\epsilon}_{\omega_j-\epsilon h_j/2}(-\hat{L}, 0)}\right],$$

where n is a positive integer, $(\omega_j)_{1\leq j\leq n} \in \mathbb{R}^n$ are all distinct, and $(h_j)_{1\leq j\leq n} \in \mathbb{R}^n$, converges as $\epsilon \to 0$ to the limit

$$\prod_{j=1}^{n} \int e^{ih_j\tau_j} \mathcal{W}_1(\omega_j, \tau_j, -\hat{L}, 0) d\tau_j.$$

If there is one or several unmatched frequencies in the product of reflection coefficients, then the limit of the moment is zero.

The mean amplitude of the reflected wave is

$$\mathbb{E}[A^{\epsilon}(t, 0)] = \frac{1}{2\pi\sqrt{\epsilon}} \int \mathbb{E}[R^{\epsilon}_{\omega}(-\hat{L}, 0)] \hat{f}(\omega) e^{-i\frac{\omega t}{\epsilon}} d\omega$$

We have

$$\mathbb{E}[A^{\epsilon}(t, 0)] \to 0$$

as $\epsilon \to 0$.

The mean intensity of the reflected wave is

$$I(t) = \lim_{\epsilon\to 0}\mathbb{E}[A^\epsilon(t,0)^2] = \lim_{\epsilon\to 0}\frac{1}{4\pi^2}\int\int \mathbb{E}[U_{11}^\epsilon(\omega,h,0)]\hat{f}(\omega+\epsilon h/2)\overline{\hat{f}(\omega-\epsilon h/2)}e^{-iht}d\omega dh \tag{4.26}$$

$$= \frac{1}{4\pi^2}\int\int\int \mathcal{W}_1(\omega,\tau,-\hat{L},0)|\hat{f}(\omega)|^2 e^{ih(\tau-t)}dhd\tau d\omega \tag{4.27}$$

$$= \frac{1}{2\pi}\int \mathcal{W}_1(\omega,t,-\hat{L},0)|\hat{f}(\omega)|^2 d\omega. \tag{4.28}$$

4.4. Monte Carlo Solution of the Transport Equation

In the previous section, the reflected wave was shown to have intensity given by

$$I(t) = \frac{1}{2\pi}\int \mathcal{W}_1(\omega,t,-\hat{L},0)|\hat{f}(\omega)|^2 d\omega.$$

$\mathcal{W}_1$ is the solution of the transport equation

$$\frac{\partial \mathcal{W}_p}{\partial z} + \frac{2p}{\bar{c}}\frac{\partial \mathcal{W}_p}{\partial \tau} = (\mathcal{L}_\omega \mathcal{W})_p, \qquad z \geq -\hat{L},\ \tau\in\mathbb{R},\ p\in\mathbb{N}, \tag{4.29}$$
$$(\mathcal{L}_\omega\phi)_p = \frac{1}{L_{\text{loc}}(\omega)}p^2(\phi_{p+1}+\phi_{p-1}-2\phi_p),$$

starting from

$$\mathcal{W}_p(\omega,\tau,-\hat{L},z=-\hat{L}) = \delta(\tau)\mathbf{1}_0(p),$$

where $L_{\text{loc}}(\omega)$ is the localization length defined by

$$L_{\text{loc}}(\omega) = \frac{4\bar{c}^2}{\gamma\omega^2},$$

and γ is the autocovariance of the process describing the random medium fluctuations, given by

$$\gamma = \int_{-\infty}^{\infty}\mathbb{E}[\nu(0)\nu(z)]dz. \tag{4.30}$$

To solve the equations (4.4), [8] uses the following probabilistic representation.

First, introduce the jump Markov process $(N_z)_{z\geq -\hat{L}}$ with state space $\mathbb{N}$ and infinitesimal generator $\mathcal{L}_\omega$ given by equation (4.4). A Markov jump process is a piecewise-constant stochastic process, with a state transition matrix and a time process governing the times the process jumps to a new state. The transition time process $\{T_k\}_{k\in\mathbb{N}}$ is given by a sum of exponential random variables

$$T_{k+1} = T_k + \tau_k$$

where τ_k is an exponential random variable with parameter λ, so that its density, defined on the positive reals, is given by

$$f(s) = \lambda e^{-\lambda s}$$

The value of λ may depend on the current state. The transition matrix K describes the probability of going to another state, given the current state, so that the Jump Markov process is given by

$$X(t) = K_{N(t)}.$$

The jump Markov process has infinitesimal generator A given by

$$A(x,y) = \begin{cases} -\lambda(x) & x = y \\ \lambda(x)K(x,y) & x \neq y \end{cases}$$

where $\lambda(x)$ is the parameter λ for state x. Thus, the jump Markov process $(N_z)_{z\geq -\hat{L}}$ can be constructed to have infinitesimal generator $\mathcal{L}_\omega$ if $\lambda(n) = 2n^2/L_{\text{loc}}(\omega)$ and for $x, y \geq 1$,

$$K(x,y) = \begin{cases} 2, & x = y \\ -1, & |x-y| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

The state $n = 0$ is an absorbing state, so that when the process reaches it, it remains at that state for all further time.

Now, define the process

$$\frac{\partial \mathcal{T}_z}{\partial z} = -\frac{2}{\bar{c}} N_z,$$

with $\mathcal{T}_{-\hat{L}} = \tau$. The process $(N_z, \mathcal{T}_z)_{z \geq -\hat{L}}$ is a Markov process with generator

$$\mathcal{L}_\omega - \frac{2n}{\bar{c}} \frac{\partial}{\partial \tau}.$$

The backward Kolmogorov equation, after the transformation $z \to -z$ is

$$\frac{\partial u}{\partial z} = \left(\mathcal{L}_\omega - \frac{2n}{\bar{c}} \frac{\partial}{\partial \tau} \right) u, \quad z > -\hat{L}, \quad u(n, \tau, z = -\hat{L}) = u_0(n, \tau). \tag{4.31}$$

The solution of the Kolmogorov equation is

$$\begin{aligned} u(n, \tau, z) &= \mathbb{E}[u_0(N_z, \mathcal{T}_z) \mid N_{-\hat{L}} = n, \mathcal{T}_{-\hat{L}} = \tau] \\ &= \mathbb{E}\left[u_0 \left(N_z, \tau - \frac{2}{\bar{c}} \int_{-\hat{L}}^{z} N_{z'} dz' \right) \mid N_{-\hat{L}} = n \right]. \end{aligned}$$

The Kolmogorov equation (4.4) is the same form as equations (4.4), so after integrating in τ, we have

$$\int_{\tau_0}^{\tau_1} \mathcal{W}_p(\omega, \tau, -\hat{L}, 0) d\tau = \mathbb{P}\left(N_0 = 0, \frac{2}{\bar{c}} \int_{-\hat{L}}^{0} N_{z'} dz' \in [\tau_0, \tau_1] \mid N_{-\hat{L}} = p \right)$$

Therefore, $\mathcal{W}_p$ can be calculated by running Monte Carlo to the Markov jump process (N_z) described above.

$\mathcal{W}_1$ is plotted below in Figure (4.2).

4.5. Comparison of Analytical Results with Random Boundary Condition

Now we will compare the analytical reflected wave intensity, given by

$$I(t) = \frac{1}{2\pi} \int \mathcal{W}_1(\omega, t, -\hat{L}, 0) |\hat{f}(\omega)|^2 d\omega. \tag{4.32}$$

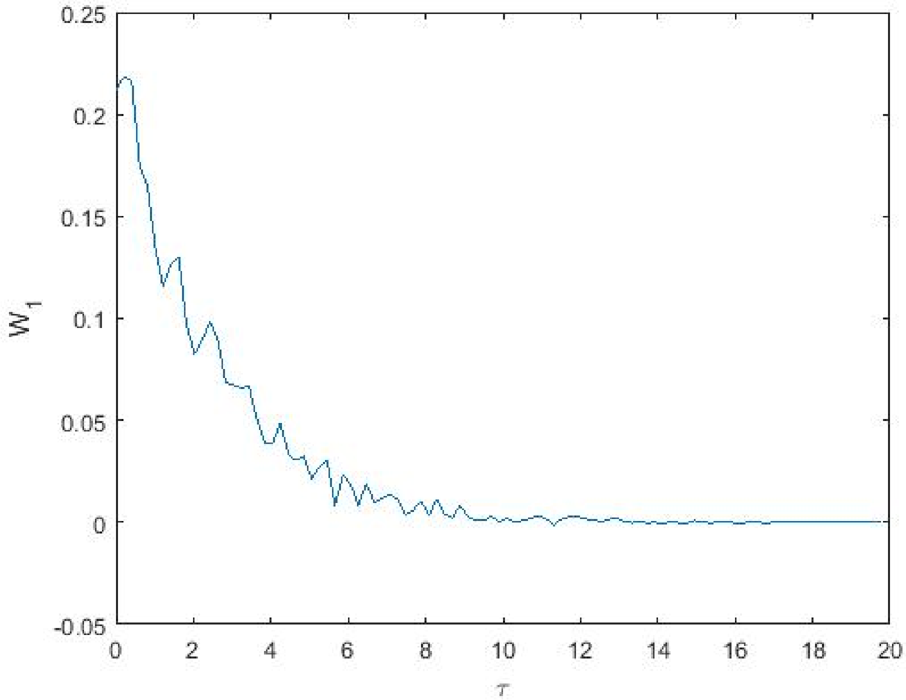

Figure 4.2. Decay of Reflection Coefficient

to the reflected wave intensity found by solving the wave equation with random boundary condition derived in Chapter 3. The quantity $\mathcal{W}_1$ is the solution of the stochastic transport equations (4.4). We select the stochastic process ν to be the Whittle-Matérn process with $p = 3/2$, given by equation (4.1). For this process, the autocovariance (4.4) is calculated to be $\gamma = 1.26$. Using the following form of the incoming pulse $f(t)$:

$$f(t) = (2c_\infty^4 t^2 - c_\infty^2)\exp(-(c_\infty t)^2) \tag{4.33}$$
$$|\hat{f}(\omega)|^2 = \frac{\pi\omega^4}{4c_\infty^2} e^{-\omega^2/(2c_\infty^2)},$$

the asymptotic result $I(t)$ is given in Figure (4.3), calculated by the method presented in the previous section with 100000 Monte Carlo samples and 1000 terms for τ evenly spaced on the interval $[0, 2]$.

In order to compare the computation using the random boundary condition given by

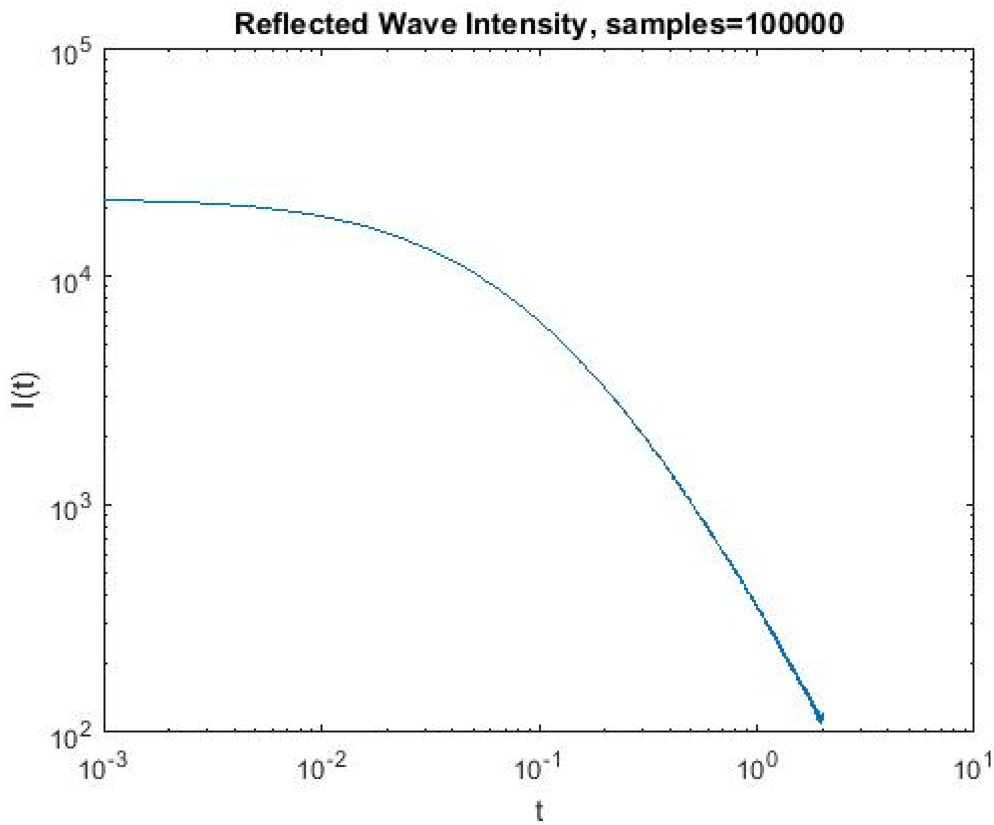

Figure 4.3. Asymptotic Result for Reflected Wave Intensity

(3.1) we must make sure that the scaling of the incoming pulse and random fluctuations of the wave speed are scaled properly. The scaling regime in the asymptotic analysis above is such that the random medium $\nu(z)$ is scaled as $\nu(z/\epsilon^2)$, the incoming pulse $f(t)$ is scaled as $(1/\epsilon)f(t/\epsilon)$, the width of the incoming pulse is ϵ, and the amplitude of the random perturbations is $\nu \sim 1$.

The system to be solved with these scalings in mind is

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) + f(x,t), \qquad x \in [-L, L] \\ u(x,0) &= \frac{1}{\epsilon}u_0(x/\epsilon), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \\ &\frac{\partial u}{\partial x} + \frac{1}{c_\infty}\frac{\partial u}{\partial t} = g(x/\epsilon^2, t, \omega), \quad x = L \\ &\frac{\partial u}{\partial x} - \frac{1}{c_\infty}\frac{\partial u}{\partial t} = 0, \quad x = -L \\ &\frac{d^2\phi_j}{dt^2} + B_j^2\phi_j = B_j^2 u. \end{aligned} \tag{4.34}$$

where $g(x,t,\omega)$ is given by (3.1). Writing the system (4.5) in weak form, we obtain

$$\int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dx = -\int_{-L}^{L} \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}(x,t)\right)v(x,t)dx + \int_{-L}^{L} f(x,t)v(x,t)dx$$

for test functions $v \in C^\infty(-L, L)$. Using integration by parts,

$$\begin{aligned} \int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c^2(x)\frac{\partial u}{\partial x}(x,t)v(x,t)\mid_{-L}^{L} \\ &\quad + \int_{-L}^{L} f(x,t)v(x,t)dx \\ \int_{-L}^{L} \frac{\partial^2 u}{\partial t^2}(x,t)v(x,t)dt &= -\int_{-L}^{L} c^2(x)\frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\ &\quad - c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t) + \int_{-L}^{L} f(x,t)v(x,t)dx \end{aligned}$$

We use the Galerkin approximation

$$u(x,t,\omega) \approx \sum_{k=0}^{n_x} u_k(t,\omega)\psi_k(x) \tag{4.35}$$

where ψ_k are the Galerkin difference basis functions as in [3]. These are piecewise polynomials defined by values on a uniform grid whose restriction to any interval bounded by grid points is the Lagrange interpolant of the nodal data. Near boundaries we simply take the values of the solution at external ghost points to be free, called the ghost basis method in [3]. Here we take the local polynomial degrees to be 3 and the grid spacing to be $\Delta x = 1/100$. This seems sufficient to resolve the waves to the accuracy provided by the linearized approximate boundary condition, but some discretization errors are noticeable for the more accurate quadratic approximation when the amplitude of the perturbation is very small.

$$\begin{aligned}\sum_{k=1}^{n_x} M_{jk}\frac{d^2u_k}{dt^2} = \sum_{k=1}^{n_x}\left(-S_{jk}\frac{du_k}{dt} + M_{jk}f_k\right) + c_\infty^2 v(L,t)\left(g - \frac{1}{c_\infty}\frac{\partial u}{\partial t}(L,t)\right) \\ - c_\infty v(-L,t)\frac{\partial u}{\partial t}(-L,t)\end{aligned} \tag{4.36}$$

where

$$M_{jk} = \int_{-L}^{L} \psi_j(x)\psi_k(x)dx, \qquad S_{jk} = \int_{-L}^{L} c^2(x)\frac{d\psi_j}{dx}(x)\frac{d\psi_k}{dx}(x)dx$$

The standard 4th-order Runge-Kutta method is used to discretize the time-variable with $\Delta t = 1/10000$. The source term $f(x,t) = 0$, and the initial condition $\frac{1}{\epsilon}u_0(x/\epsilon) = \frac{1}{\epsilon}f(x/\epsilon)$ to match the asymptotic calculation, where f is as in equation (4.5). Results have not yet been obtained which match the asymptotic result given in Figure (4.3).

Chapter 5

APPROXIMATION TO THE DTN MAP IN TWO DIMENSIONS

5.1. DtN Map 2D

In this chapter we will derive an approximation to the nonreflecting boundary condition for the 2D wave equation with a computational boundary at $x = a$ and Neumann conditions in y. See Figure (5.1).

The 2D Wave Equation is

$$\frac{\partial}{\partial x}\left(c^2(x,y,\omega)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c^2(x,y,\omega)\frac{\partial u}{\partial y}\right) - \frac{\partial^2 u}{\partial t^2} = f(x,y,t)$$
$$u(x,y,0,\omega) = u_0(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0,\omega) = v_0(x,y)$$
$$\frac{\partial u}{dy}(x,0,t,\omega) = \frac{\partial u}{dy}(x,H,t,\omega) = 0.$$

for $(x,y) \in (-L,M) \times (0,H)$ and $t \in (0,T)$. In the exterior region $\Sigma = (L,M) \times (0,H)$ the wave speed is given by

$$c(x,y,\omega) = c_\infty + \tilde{c}(x,y,\omega)$$

where c_∞ is a constant deterministic quantity and $\tilde{c}$ is a mean-zero stochastic process satisfying

$$c_0 \leq c(x,y,\omega) \leq c_1$$

almost everywhere and almost surely. Taking the Laplace transform and using the fact that $u_0(x,y) = v_0(x,y) = f(x,y,t) = 0$ in the exterior region, we have

$$\frac{\partial}{\partial x}\left((c_\infty + \tilde{c}(x,y,\omega))^2\frac{\partial \hat{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left((c_\infty + \tilde{c}(x,y,\omega))^2\frac{\partial \hat{u}}{\partial y}\right) - s^2\hat{u} = 0$$

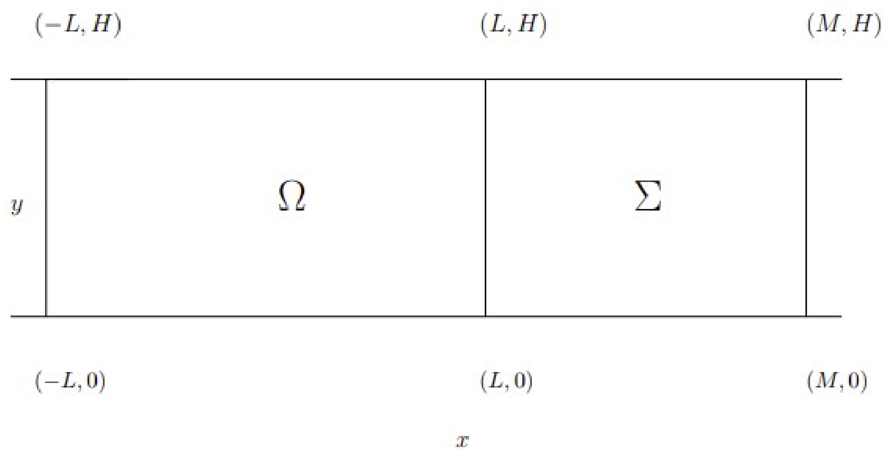

Figure 5.1. Setting of 2D Problem. $\tilde{c} = 0$ for $(x, y) \in \Omega$, random perturbations in wave speed in region Σ. Neumann boundary conditions for $y = 0, H$.

for $(x, y) \in (L, M) \times (0, H)$.

Assuming $\tilde{c} = 0$ for $x > M$ we have that the DtN map at $x = M$ can be directly written using a Fourier cosine series in y (or more properly diagonalized in the Fourier basis):

$$\begin{aligned} \hat{u}(M, y, \omega) &= \sum_{k=1}^{\infty} u_k \cos\left(\frac{k\pi y}{H}\right), \\ \frac{\partial \hat{u}}{\partial x}(M, y, \omega) &= -\sum_{k=1}^{\infty} \gamma_k u_k \cos\left(\frac{k\pi y}{H}\right), \\ \gamma_k &= \left(\frac{s^2}{c_\infty^2} + \frac{k^2\pi^2}{H^2}\right)^{1/2} \end{aligned}$$

and the branch is chosen so that $\mathscr{R}\gamma_k > 0$ if $\mathscr{R}s > 0$.

To represent the DtN map at $x = L$ we can rewrite the system in (infinite) matrix form by introducing

$$C_{k\ell}(x) = \frac{4}{c_\infty H}\int_0^H \tilde{c}(x,y)\cos\left(\frac{k\pi y}{H}\right)\cos\left(\frac{\ell\pi y}{H}\right)dy,$$
$$P_{k\ell}(x) = \frac{2}{c_\infty^2 H}\int_0^H \tilde{c}^2(x,y)\cos\left(\frac{k\pi y}{H}\right)\cos\left(\frac{\ell\pi y}{H}\right)dy,$$
$$B_{k\ell}(x) = -\frac{4\pi\ell}{c_\infty H^2}\int_0^H \frac{\partial\tilde{c}}{\partial y}(x,y)\cos\left(\frac{k\pi y}{H}\right)\sin\left(\frac{\ell\pi y}{H}\right)dy,$$
$$Q_{k\ell}(x) = -\frac{4\pi\ell}{c_\infty^2 H^2}\int_0^H \frac{\partial\tilde{c}}{\partial y}(x,y)\tilde{c}(x,y)\cos\left(\frac{k\pi y}{H}\right)\sin\left(\frac{\ell\pi y}{H}\right)dy.$$

If we now define

$$u_k(x) = \frac{2}{H}\int_0^H \hat{u}(x,y)\cos\left(\frac{k\pi y}{H}\right)dy$$

then we have the equation:

$$\frac{d^2u_k}{dx^2} - \gamma_k^2 u_k + \sum_\ell (C_{k\ell} + P_{k\ell})\left(\frac{d^2u_\ell}{dx^2} - \frac{\ell^2\pi^2}{H^2}u_\ell\right) + \sum_\ell\left(\frac{dC_{k\ell}}{dx} + \frac{dP_{k\ell}}{dx}\right)\frac{du_\ell}{dx}$$
$$+ \sum_\ell (B_{k\ell} + Q_{k\ell})u_\ell = 0.$$

Similarly we write the DtN map in matrix form:

$$\frac{du_k}{dx} = -\gamma_k u_k - \sum_\ell G_{k\ell}(x)u_\ell.$$

Then we differentiate and deduce:

$$\frac{d^2u_k}{dx^2} = \gamma_k^2 u_k + \sum_\ell\left((\gamma_k + \gamma_\ell)G_{k\ell} - \frac{dG_{k\ell}}{dx}\right)u_\ell + \sum_{\ell m} G_{km}G_{m\ell}u_\ell.$$

Substituting these expressions into the equation we derive the Ricatti equation for $G_{k\ell}$:

$$\begin{aligned}\frac{dG_{k\ell}}{dx} + \sum_m (C_{km} + P_{km})\frac{dG_{m\ell}}{dx} = {} & (\gamma_k + \gamma_\ell)G_{k\ell} + \sum_m G_{km}G_{m\ell} + \gamma_\ell^2 (C_{k\ell} + P_{k\ell}) \\ & + \sum_m (C_{km} + P_{km})(\gamma_m + \gamma_\ell)G_{m\ell} \\ & + \sum_{m,j} (C_{kj} + P_{kj})G_{jm}G_{m\ell} - \gamma_\ell \left(\frac{dC_{k\ell}}{dx} + \frac{dP_{k\ell}}{dx}\right) \\ & - \sum_m \left(\frac{dC_{km}}{dx} + \frac{dP_{km}}{dx}\right) G_{m\ell} \\ & + B_{k\ell} + Q_{k\ell} - \ell^2 (C_{k\ell} + P_{k\ell}).\end{aligned}$$

If we linearize the problem we can compute a first approximation to $G_{k\ell}$. Note that in the linearization both P and Q are removed:

$$\frac{dG_{k\ell}^{(1)}}{dx} = (\gamma_k + \gamma_\ell)G_{k\ell}^{(1)} + \gamma_\ell^2 C_{k\ell} - \gamma_\ell \frac{dC_{k\ell}}{dx} + B_{k\ell} - \ell^2 C_{k\ell}.$$

Since $G_{k\ell}(M) = 0$ we can write down $G_{k\ell}^{(1)}(L)$ as an integral:

$$G_{k\ell}^{(1)} = -\int_L^M e^{(\gamma_k+\gamma_\ell)(L-x)} \left(\frac{s^2}{c_\infty^2} C_{k\ell} + B_{k\ell} - \gamma_\ell \frac{dC_{k\ell}}{dx}\right) dx.$$

To acquire a closed-form expression for the DtN map, we will choose the following two-dimensional analogue of the process (2.1) which vanishes at $x = L, M$ and $y = 0, H$:

$$\tilde{c}(x, y) = \sum_{i,j=1}^{\infty} \frac{\xi_{ij}}{\pi^4 i^r j^r} \sin\left(\frac{i\pi(x - L)}{M - L}\right) \sin\left(\frac{j\pi y}{H}\right) \tag{5.1}$$

Then we have

$$
\begin{aligned}
C_{kl} &= \sum_{i,j=1}^{\infty} \frac{\xi_{ij}}{c_\infty \pi^5 i^r j^r} \sin\left(\frac{i\pi(x-L)}{M-L}\right) A_{jkl} \\
B_{kl} &= 0 \\
\frac{dC_{k\ell}}{dx} &= \sum_{i,j=1}^{\infty} \frac{\xi_{ij}/(M-L)}{c_\infty \pi^4 i^{r-1} j^r} \cos\left(\frac{i\pi(x-L)}{M-L}\right) A_{jkl} \\
A_{jkl} &= \left(\frac{1-(-1)^{j+k+\ell}}{j+k+\ell} + \frac{1-(-1)^{j+k-\ell}}{j+k-\ell} + \frac{1-(-1)^{j-k+\ell}}{j+k-\ell} + \frac{1-(-1)^{j-k-\ell}}{j-k-\ell}\right)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int_L^M e^{-(\nu_k+\nu_\ell)(x-L)} \sin\left(\frac{i\pi(x-L)}{M-L}\right) dx &= \frac{i\pi c_\infty^2/(2(M-L))}{s^2 + \frac{\pi^2 c_\infty^2 (k^2+\ell^2)}{2H^2} + c_\infty^2 \gamma_k \gamma_\ell + \frac{i^2\pi^2}{(M-L)^2}} \\
\int_L^M e^{-(\nu_k+\nu_\ell)(x-L)} \cos\left(\frac{i\pi(x-L)}{M-L}\right) dx &= \frac{c_\infty^2(\gamma_k+\gamma_\ell)/2}{s^2 + \frac{\pi^2 c_\infty^2 (k^2+\ell^2)}{2H^2} + c_\infty^2 \gamma_k \gamma_\ell + \frac{i^2\pi^2}{(M-L)^2}}
\end{aligned}
$$

Thus,

$$
G_{k\ell}^{(1)} = \sum_{i,j=1}^{\infty} D_{ijk\ell} \frac{c_\infty^2 \ell^2 \pi^2/H^2 + c_\infty^2 \gamma_\ell \gamma_k}{s^2 + \frac{\pi^2 c_\infty^2 (k^2+\ell^2)}{2H^2} + c_\infty^2 \gamma_k \gamma_\ell + \frac{i^2\pi^2}{(M-L)^2}}
$$

where

$$
D_{ijk\ell} = \frac{\xi_{ij} A_{jk\ell}/(2(M-L))}{c_\infty \pi^4 i^{r-1} j^r}
$$

The DtN map is then

$$
\frac{du_k}{dx} = -\gamma_k u_k - \sum_{i,j,\ell} D_{ijk\ell} u_\ell \frac{c_\infty^2 \ell^2 \pi^2/H^2 + c_\infty^2 \gamma_\ell \gamma_k}{s^2 + \frac{\pi^2 c_\infty^2 (k^2+\ell^2)}{2H^2} + c_\infty^2 \gamma_k \gamma_\ell + \frac{i^2\pi^2}{(M-L)^2}}
$$

The square root operator

$$
\gamma_k = \left(\frac{s^2}{c_\infty^2} + \frac{k^2\pi^2}{H^2}\right)^{1/2}
$$

is approximated via the least-squares algorithm given in [1].

$$\gamma_k = \frac{s}{c_\infty} - \sum_{j=1}^{Q} \frac{sc\lambda_k^2 a_j - c^2\lambda_k^3 d_j}{s^2 - sc\lambda_k b_j + c^2\lambda_k^2 g_j} := \frac{s}{c_\infty} - \sum_{j=1}^{Q} \frac{sA_{jk} - B_{jk}}{s^2 - sC_{jk} + D_{jk}} \tag{5.2}$$

where $A_{jk} = c_\infty\lambda_k^2 a_j$, $B_{jk} = c_\infty^2\lambda_k^3 d_j$, $C_{jk} = c_\infty\lambda_k b_j$, and $D_{jk} = c_\infty^2\lambda_k^2 g_j$. The values of a_j, b_j, d_j, g_j are given in table. As shown in [1], the approximation (5.1) gives for $Q = 31$ an error less than 10^{-6} for $T \leq 10^4$. Thus,

j	a_j	b_j	d_j	g_j
1	-1.44973E-7	-4.59136E-5	-1.45333E-7	1.0000000005
2	-7.52363E-7	-2.04653E-4	-7.53785E-7	0.99999989
3	-2.52811E-6	-5.48932E-4	-2.53264E-6	0.9999997
4	-7.47593E-6	-1.23706E-3	-7.511476E-6	0.99999919
5	-2.10610E-5	-2.58602E-3	-2.12561E-6	0.9999963
6	-5.80557E-5	-5.21498E-3	-5.882134E-6	0.9999889
7	-1.58151E-4	-1.03277E-2	-1.6120948E-5	0.9999737
8	-4.27342E-4	-2.02676E-2	-4.402573E-4	0.9999418
9	-1.14369E-3	-3.95893E-2	-1.20115E-3	0.9998850
10	-3.00129E-3	-7.71083E-2	-3.286218E-3	0.9997697
11	-7.55569E-3	-0.149759	-9.049647E-3	0.9994074
12	-1.72246E-2	-0.28939	-2.513037E-2	0.9981144
13	-2.87246E-2	-0.55146	-6.98155E-2	0.9938616
14	1.02309E-2	-1.00527	-0.184833E-2	0.9817475
15	0.27071	-1.60827	-0.398150	0.95722285
16	0.27739	-0.969474	0	0

Table 5.1. Coefficients of 31-pole Approximation of γ

$$\frac{du_k}{dx} + \frac{s}{c_\infty}u_k = \sum_{j=1}^{Q} \frac{sc_\infty\lambda_k^2 a_j - c_\infty^2\lambda_k^3 d_j}{s^2 - sc_\infty\lambda_k b_j + c_\infty^2\lambda_k^2 g_j} - \frac{1}{2}\sum_{i,j,\ell} D_{ijk\ell}u_\ell \left(1 + \frac{2I_\ell - J_{k\ell} + X_{k\ell}(s) - \frac{i^2\pi^2}{(M-L)^2}}{2s^2 + J_{k\ell} + X_{k\ell}(s) + \frac{i^2\pi^2}{(M-L)^2}}\right)$$

where $I_\ell = c_\infty^2 \ell^2 \pi^2 / H^2,\ J_{k\ell} = \pi^2 c_\infty^2 (k^2 + \ell^2)/(2H^2)$ and

$$X_{k\ell}(s) = -s \sum_m \frac{sc_\infty \lambda_k^2 a_m - c_\infty^2 \lambda_k^3 d_m}{s^2 - sc_\infty \lambda_k b_m + c_\infty^2 \lambda_k^2 g_m} - s \sum_m \frac{sc_\infty \lambda_\ell^2 a_m - c_\infty^2 \lambda_\ell^3 d_m}{s^2 - sc_\infty \lambda_\ell b_m + c_\infty^2 \lambda_\ell^2 g_m}$$
$$+ \sum_{m,n} \frac{sc_\infty \lambda_\ell^2 a_n - c_\infty^2 \lambda_\ell^3 d_n}{s^2 - sc_\infty \lambda_\ell b_n + c_\infty^2 \lambda_\ell^2 g_n} \frac{sc_\infty \lambda_k^2 a_m - c_\infty^2 \lambda_k^3 d_m}{s^2 - sc_\infty \lambda_k b_m + c_\infty^2 \lambda_k^2 g_m}$$

Taking the inverse Laplace Transform we have

$$c_\infty \frac{du_k}{dx} + \frac{du_k}{dt} = c_\infty \sum_{j=1}^{Q} \phi_{jk}^{(6)} - \frac{1}{2} \sum_{i,j,\ell} c_\infty D_{ijk\ell} \left(u_\ell + \phi_{k\ell}^{(1)} \right) \tag{5.3}$$

where $\phi_{jk}^{(6)}$ satisfies the auxiliary ODE

$$\frac{d^2 \phi_{jk}^{(6)}}{dt^2} = b_j c_\infty \lambda k \frac{d\phi_{jk}^{(6)}}{dt} + c^2 \lambda_k^2 \left(-g_j \phi_{jk}^{(6)} + a_j \frac{du_k}{dt} - c_\infty \lambda_k d_j u_k \right)$$

and $\phi_{ik\ell}^{(1)}$ satisfies the auxiliary ODE

$$2\frac{d^2 \phi_{ik\ell}^{(1)}}{dt^2} + \left(J_{k\ell} + \frac{i^2 \pi^2}{(M-L)^2} \right) \phi_{ik\ell}^{(1)} - c_\infty \sum_m A_{mk} ((\phi_{ik\ell}^{(1)} - u_\ell) + \phi_{k\ell m}^{(2)}) - c_\infty \sum_m A_{m\ell} ((\phi_{ik\ell}^{(1)} - u_\ell) + \phi_{k\ell m}^{(3)})$$
$$+ c_\infty^2 \sum_{m,n} A_{mk} A_{n\ell} (\phi_{mk\ell}^{(4)} + \phi_{k\ell mn}^{(5)}) = \left(2I_\ell - J_{k\ell} - \frac{i^2 \pi^2}{(M-L)^2} \right) u_\ell$$

where we have

$$\frac{d^2 \phi_{klm}^{(2)}}{dt^2} - C_{mk} \frac{d\phi_{klm}^{(2)}}{dt} + D_{mk} \phi_{klm}^{(2)} = (C_{mk} - B_{mk}/A_{mk}) \left(\frac{d\phi_{ik\ell}^{(1)}}{dt} - \frac{du_\ell}{dt} \right) - D_{mk} (\phi_{k\ell} - u_\ell)$$
$$\frac{d^2 \phi_{klm}^{(3)}}{dt^2} - C_{m\ell} \frac{d\phi_{klm}^{(3)}}{dt} + D_{m\ell} \phi_{klm}^{(3)} = (C_{m\ell} - B_{m\ell}/A_{m\ell}) \left(\frac{d\phi_{ik\ell}^{(1)}}{dt} - \frac{du_\ell}{dt} \right) - D_{m\ell} (\phi_{k\ell} - u_\ell)$$

$$\frac{d^2 \phi_{k\ell m}^{(4)}}{dt^2} - C_{mk} \frac{d\phi_{k\ell m}^{(4)}}{dt} + D_{mk} \phi_{k\ell m}^{(4)} = \phi_{ik\ell}^{(1)} - u_\ell$$

$$\frac{d^4\phi^{(5)}_{k\ell mn}}{dt^4} - (C_{mk}+C_{n\ell})\frac{d^3\phi^{(5)}_{k\ell mn}}{dt^3} + (C_{mk}C_{n\ell}+D_{n\ell}+D_{mk})\frac{d^2\phi^{(5)}_{k\ell mn}}{dt^2} - (C_{n\ell}D_{mk}+C_{mk}D_{n\ell})\frac{d\phi^{(5)}_{k\ell mn}}{dt}$$
$$+ D_{mk}D_{n\ell}\phi^{(5)}_{k\ell mn} = (C_{mk} - \frac{A_{n\ell}B_{mk}+A_{mk}B_{n\ell}}{A_{mk}A_{n\ell}})\frac{du_\ell}{dt} + \left(\frac{B_{mk}B_{n\ell}}{A_{mk}A_{n\ell}} - D_{mk}\right)u_\ell$$

5.2. Galerkin Discretization in Two Space Dimensions

The system that we are solving is

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(c^2(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t)\\
&(x,y)\in(-L,L)\times(0,H), \quad 0<t<T,\\
&u(x,y,0,\omega) = u_0(x,y), \quad \frac{\partial u}{\partial t}(x,y,0,\omega) = v_0(x,y)\\
&\frac{\partial u}{\partial y}(x,0,t,\omega) = \frac{\partial u}{\partial y}(x,H,t,\omega) = 0, \qquad \frac{\partial u}{\partial t}(L,y,t,\omega) + c\frac{\partial u}{\partial x}(L,y,t,\omega) = g,\\
&\frac{\partial u}{\partial t}(-L,y,t,\omega) - c\frac{\partial u}{\partial x}(-L,y,t,\omega) = h,
\end{aligned}$$

with auxiliary ODEs $\phi^{(1)} - \phi^{(6)}$ given above with zero initial conditions. g is the inverse Fourier transform of the right-hand side of equation (5.1) and h is the inverse Fourier transform of $c_\infty \sum_{j=1}^{Q}\phi^{(6)}_{jk}$.

To discretize we use the Galerkin difference approximation

$$u(x,y,t,\omega) \approx \sum_{k=0}^{n_x}\sum_{r=0}^{n_y} u_{kr}(t,\omega)\zeta_k(x)\psi_r(y)$$

where ζ_k, ψ_r are Galerkin difference basis functions. This leads to the system of ordinary differential equations

$$\sum_{k=0}^{n_x}\sum_{r=0}^{n_y} M^{(x)}_{jk}M^{(y)}_{qr}\frac{d^2u_{kr}}{dt^2} = \sum_{k=0}^{n_x}\sum_{r=0}^{n_y}(S_{jkqr}u_{kr} + M^{(x)}_{jk}M^{(y)}_{qr}f_{kr})$$
$$+ \delta_{j0}\sum_{r=0}^{n_y} B_{0,qr}\left(g_{0,r} - \frac{du_{0r}}{dt}\right) + \delta_{jn_x}\sum_{r=0}^{n_y} B_{1,qr}\left(g_{1,r} - \frac{du_{n_xr}}{dt}\right),$$

for $j = 0, \ldots, n_x$, $q = 0, \ldots, n_y$. Here we have

$$M_{jk}^{(x)} = \int_{-L}^{L} \zeta_j(x)\zeta_k(x)dx, \qquad M_{qr}^{(y)} = \int_0^H \psi_q(y)\psi_r(y)dy,$$

$$S_{jkqr} = \int_{-L}^{L}\int_0^H c^2(x,y)\left(\frac{d\zeta_j}{dx}(x)\frac{d\zeta_k}{dx}(x)\psi_q(y)\psi_r(y) + \zeta_j(x)\zeta_k(x)\frac{d\psi_q}{dy}(y)\frac{d\psi_r}{dy}(y)\right)dydx,$$

$$B_{0,qr} = \int_0^H c(x_0,y)\psi_q(y)\psi_r(y)dy, \qquad B_{1,qr} = \int_0^H c(x_1,y)\psi_q(y)\psi_r(y)dy.$$

In the future we plan to carry out experiments analogous to those in Section (2.3).

Chapter 6

CONCLUSION

We have derived random boundary conditions for the 1D and 2D wave equations with nonreflecting boundary by solving the DtN map with variable wave speed given by an orthogonal expansion. The method converged with expected linear and quadratic rates for the 1D experiments when the resulting Ricatti equation is given a linear and quadratic approximation, respectively.

We have also completed the same procedure using a stationary stochastic process with properties consistent with analysis done by Papanicolaou and his co-authors in [8]. We propose a numerical experiment to compare the asymptotic result in [8] to the random boundary condition derived in Chapter 3.

Possible improvements in the future would include a more extensive uncertainty quantification experiment using the method, and an implementation of an algorithm to improve the efficiency of the method. One possibility would be the inclusion of a pole-reduction algorithm to compress the boundary condition.

Least-squares pole-reduction algorithms are given in [2], [1]. Balanced truncation algorithms for pole-reduction are found in [14], [21], and pole-reduction algorithms using Prony's method are found in [13] and [4].

In future work we will also conduct a more extensive analysis of the effect of truncation on the eigenfunction expansion, complete the numerical experiments in 2D proposed in Chapter 5, and complete the comparison with the asymptotic results proposed in Chapter 4.

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