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ESTIMATION OF PARAMETERS OF GAMMA AND
GENERALIZED GAMMA DISTRIBUTIONS BASED ON
CENSORED EXPERIMENTAL DATA

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ESTIMATION OF PARAMETERS OF GAMMA AND
GENERALIZED GAMMA DISTRIBUTIONS BASED ON
CENSORED EXPERIMENTAL DATA

A Dissertation Presented to the Graduate Faculty of the

Dedman College

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in

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for the degree of

Doctor of Philosophy

with a

Major in Statistics

by

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August 4, 2021

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Estimation of Parameters of Gamma and
Generalized Gamma Distributions Based on
Censored Experimental Data

Advisor: Dr. Hon Keung Tony Ng

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In time-to-event data analysis, censoring is one of the unique features that restricts our ability to observe the time-to-events and poses difficulties for statistical analysis. Censoring occurs when the exact time-to-event cannot be observed for some or all observations. In this thesis, we study the parameter estimation methods for a two-parameter gamma distribution and a three-parameter generalized gamma distribution based on different kinds of censored data arising from life-testing experiments.

We first study the parameter estimation of a three-parameter generalized gamma distribution based on left-truncated and right-censored data. It is well known that the maximum likelihood estimates of the parameters for the generalized gamma distribution may not be stable, especially when the data is incomplete. A stochastic version of the expectation-maximization (EM) algorithm is proposed as an alternative method to compute approximate maximum likelihood estimates. The proposed estimation procedure is compared with some existing estimation procedures based on the maximum likelihood method, such as the direct optimization method, the profile likelihood method, and the EM algorithm, in terms of accuracy and stability. Two different methods to obtain reliable initial estimates of the parameters required for the iterative algorithms are also proposed. Interval estimation based on a parametric bootstrap method is discussed. The proposed methodologies are illustrated with a numerical example. Then, a Monte Carlo simulation study is used to evaluate the performance of the proposed estimation procedures. Based on the simulation results, we

make some recommendations about which estimation procedures are more appropriate in practice.

Then, we consider the parameter estimation and reliability analysis based on one-shot device testing data in a realistic situation where defectives are produced in the manufacturing process. For one-shot device testing, all the lifetimes of the devices are either left-censored or right-censored due to the destructive nature of the test. Unlike non-destructive testing of products with continuous monitoring, defective one-shot devices will not be detected until the time of usage or testing. Moreover, in the presence of defectives, if a one-shot device does not work at the time of testing, we may not be able to distinguish whether the particular device is a defective or a device that has a lifetime smaller than the testing time. A maximum likelihood approach and a Bayesian approach are proposed for the point and interval estimation of the parameters and reliability indices under different scenarios. The proposed methodologies are illustrated with a numerical example when the lifetimes of the devices follow a two-parameter gamma distribution. A Monte Carlo simulation study is used to evaluate the performance of the proposed estimation procedures under different settings. Based on the simulation results, some practical guidelines are provided.

Finally, some possible future research directions based on this thesis are discussed.

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Dedicated to my family, friends and cats.

Chapter 1

Introduction

1.1. Time-to-event Data

Time-to-event data are the measurement of time that elapses until an event occurs. An event could be the healing of a wound, the death of a patient, or the failure of a machine. With a well-defined time origin, the time-to-event is defined as the time from the origin to the time of the occurrence of the event of interest. For example, in reliability engineering, an energy company is interested in the lifetimes of its fleet of high-voltage transmission and distribution transformers. The time-to-event is defined as the time from the installation of the transformer to the failure of the transformer ([Hong et al., 2009](#)). For statistical analysis of time-to-event data, also known as survival analysis or reliability analysis, special techniques and methods are required due to the distinct features of time-to-event data. Censoring and truncation are two specific features of time-to-event data that lead to the incompleteness of the data. More details on censoring and truncation will be discussed in [Sections 1.3.1 and 1.3.2](#).

1.2. Generalized Gamma Distribution

The generalized gamma distribution can be traced back to [Amoroso \(1925\)](#) and [D'Addario \(1932\)](#), which can be regarded as a special case of a function introduced for analyzing economic income data. [Stacy \(1962\)](#) and [Stacy and Mihram \(1965\)](#) introduced the widely-used three-parameter generalized gamma distribution and examined the elementary properties of the distribution. As the name itself implies, the generalized gamma distribution is a

generalization of the two-parameter gamma distribution, which is also related to other commonly used parametric models in survival analysis, such as the exponential distribution, the Weibull distribution, and the chi-squared distribution. The generalized gamma distribution has gained more attention recently because of its flexibility.

1.2.1. Characteristics

There are several ways to parameterize the generalized gamma distribution. The probability density function (pdf) of the generalized gamma distribution can be expressed as

$$f(x) = \frac{\eta}{\alpha\Gamma(\kappa)} \left(\frac{x}{\alpha}\right)^{\kappa\eta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^\eta\right\}, \quad x > 0, \quad (1.1)$$

where $\Gamma(p) = \int_0^\infty u^{p-1}e^{-u} du$ is the gamma function, and $\alpha > 0$, $\eta > 0$, $\kappa > 0$ are the parameters. The cumulative distribution function (cdf), the survival function and the hazard function of the generalized gamma distribution are, respectively,

$$F(x) = \frac{1}{\Gamma(\kappa)} \gamma\left(\kappa, \left(\frac{x}{\alpha}\right)^\eta\right), \quad x > 0, \quad (1.2)$$

$$S(x) = \frac{1}{\Gamma(\kappa)} \Gamma\left(\kappa, \left(\frac{x}{\alpha}\right)^\eta\right), \quad x > 0,$$

$$h(x) = \frac{\eta}{\alpha\Gamma\left(\kappa, \left(\frac{x}{\alpha}\right)^\eta\right)} \left(\frac{x}{\alpha}\right)^{\kappa\eta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^\eta\right\}, \quad x > 0,$$

where $\gamma(p, t) = \int_0^t u^{p-1}e^{-u} du$ is the lower incomplete gamma function and $\Gamma(p, t) = \int_t^\infty u^{p-1}e^{-u} du$ is the upper incomplete gamma function.

1.2.2. Relationship with other distributions

A unique feature of the generalized gamma distribution is that many well-known distributions are special cases of it (Stacy and Mihram, 1965). This feature allows us to use the generalized gamma distribution for model selection. Table 1.1 summarizes the relationships between the generalized gamma distribution and some commonly used lifetime distributions.

Table 1.1: Relationships between the generalized gamma distribution and some commonly used lifetime distributions

Distribution	pdf	Parameters
Exponential	$f(x) = \frac{1}{\alpha} \exp\{-\frac{x}{\beta}\}$	$\eta = 1, \kappa = 1$
Gamma	$f(x) = \frac{1}{\alpha^\kappa \Gamma(\kappa)} x^{\kappa-1} \exp\{-\frac{x}{\alpha}\}$	$\eta = 1$
Weibull	$f(x) = \frac{\eta}{\alpha} (\frac{x}{\alpha})^{\eta-1} \exp\{-(\frac{x}{\alpha})^\eta\}$	$\kappa = 1$
Chi-squared	$f(x) = \frac{1}{2^{n/2} \Gamma(n/2)} x^{n/2-1} \exp\{-\frac{x}{2}\}$	$\alpha = 2, \eta = 1, \kappa = \frac{n}{2}$
Half-normal	$f(x) = \frac{2}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$	$\alpha = \sqrt{2}, \eta = 2, \kappa = \frac{1}{2}$
Rayleigh	$f(x) = \frac{x}{\sigma^2} \exp\{-\frac{x^2}{2\sigma^2}\}$	$\alpha = \sigma\sqrt{2}, \eta = 2, \kappa = 1$

1.2.3. Parameter estimation

Stacy and Mihram (1965) discussed the maximum likelihood estimation method and derived the method of moments estimators for the parameters of the generalized gamma distribution. Parr and Webster (1965) derived the maximum likelihood estimators (MLEs) of the parameters and showed that the MLEs are asymptotically multivariate normally distributed. Harter (1967) considered a four-parameter generalized gamma distribution with an additional location parameter compared to the three-parameter generalized gamma distribution and formulated an iterative procedure for the maximum likelihood estimation. Hager and Bain (1970) explored the distributional properties of some statistics related to hypothesis testings for the generalized gamma distribution. Since different sets of parameters may result in pdfs of generalized gamma distributions that look alike, estimating the parameters

becomes difficult. To address this issue, [Prentice \(1974\)](#) and [Farewell and Prentice \(1977\)](#) considered reparameterizing the generalized gamma distribution. [Lawless \(1980\)](#) considered the construction of confidence intervals and tests of significance related to the parameters in the generalized gamma distribution. [Hirose \(1995\)](#) discussed another reparameterization of the generalized gamma distribution and developed a predictor-corrector method to estimate the parameters. [Huang and Hwang \(2006\)](#) proposed a procedure to obtain the estimators for the reparameterized form of the generalized gamma distribution using a moment estimation approach. [Gomès et al. \(2008\)](#) proposed a method based on goodness-of-fit test by transforming the generalized gamma distribution to the gamma distribution with a finite number of values of the parameter η . [Balakrishnan and Mitra \(2014\)](#) explored the expectation-maximization algorithm to obtain the maximum likelihood estimates under a left-truncated and right-censored scenario.

1.3. Censoring and Truncation

Censoring and truncation are two special features of time-to-event data ([Meeker and Escobar, 1998](#)). Censoring and truncation restrict our ability to observe the time-to-events and hence they pose difficulties for statistical analysis. In this section, we discuss different kinds of censoring and truncation and introduce left-truncated and right-censored data and one-shot device testing data.

1.3.1. Censoring

Censoring occurs when the exact time-to-event cannot be observed for some or all observations. For example, it could happen when some units accidentally broke, when some subjects in medical studies dropped out, or when a machine had not failed when the life testing experiment terminated. Different types of censoring can occur in a study of time-to-events.

- **Right Censoring:** A time-to-event is said to be right-censored if the event of interest occurs after the termination of the study. It occurs when an observation has not failed

by the last inspection. We only know that the failure time is greater than the time of the last inspection. For example, we consider a life testing experiment on integrated circuits and the experiment is set to be terminated at 1370 hours. Those integrated circuits which have not failed by 1370 hours are right-censored.

- **Left Censoring:** A time-to-event is said to be left-censored if the event of interest occurs before the starting of the study. It occurs when an observation has failed at the time of its first inspection. We only know that the observation fails before the inspection time. For example, we consider a life testing experiment of car airbags. If the car airbag fails at the time of the inspection, then the observation is left-censored. Left censoring also arises when the exact value of a response has not been observed and we only have an upper bound on that response. For example, we consider an ultrasonic measuring system that lacks the sensitivity needed to measure observations below a noise floor. When the measurement is taken, if the signal is below the noise floor, all we know is that the measurement is less than the threshold.
- **Interval Censoring:** A time-to-event is said to be interval-censored if the event of interest is only known to have occurred in a given time interval in the study. It occurs when the failure time of an observation is known to have happened between two inspections. In fact, right censoring and left censoring can be considered as special cases of interval censoring. If an observation has failed at its first inspection, then it is the same as a left-censored observation. If an observation has not failed by the time of the last inspection, then it is right-censored.

1.3.2. Truncation

The selection process of subjects that is inherent in the study design gives rise to truncated observations. When truncation is present, the observations come from a conditional distribution. In other words, truncation occurs when time-to-events can only be observed when they are in a particular range. In general, two kinds of truncation can be considered.

- **Right Truncation:** The observations that are greater than a certain value are excluded in the selection process of the units for the study. For example, in a study of the degree of porosity in castings for automotive engine mounts, the units are inspected by X-ray to make sure there are no large internal pores. Since pores larger than 10 microns can be easily detected, the inspection process eliminates the units containing pores greater than 10 microns. Thus, the distribution of pore size of the units passing the inspection is a right-truncated distribution with support $(0, 10)$ microns.
- **Left Truncation:** The observations that are less than a certain value are excluded in the selection process of the units for the study. For example, in a field-tracking study of circuit packs, the units from a certain vendor had already undergone a 1000-hour burn-in process at the manufacturing plant. Thus, the circuit packs after the burn-in process are left-truncated and they follow a left-truncated distribution with support $(1000, \infty)$ hours.

1.3.3. Left-truncated and right-censored data

Truncation and censoring can occur at the same time in a time-to-event study. One situation is that left truncation and right censoring occur simultaneously. For example, [Hong et al. \(2009\)](#) discussed the data of power transformers of an energy company. The failure time of a unit is observed only if it failed after 1980, as detailed record keeping on the lifetime of transformers was started only in 1980. Information on the lifetime of units that were installed after 1980 is available, while for units that were installed before 1980, lifetimes are available only if they failed after 1980. No information is available on units that were installed and failed before 1980. Thus the units that were installed before 1980 and failed after 1980 are left-truncated. Also, the study ended in 2008, which incorporates right censoring into the data. The units that failed before 2008 are completely observed, while the units that were still in service in 2008 are right-censored.

1.3.4. One-shot device testing (Left- and right-censored) data

In a time-to-event study, it is possible that all the time-to-events are unobservable, and they are either left-censored or right-censored. “One-shot device” testing is an example that all the lifetimes of the devices are either left-censored or right-censored due to the destructive nature of the test. [Bain and Engelhardt \(1991\)](#) considered a device that is destroyed after its use and performs its intended function only once, regardless of whether the device performs its intended function properly. Such a device is a so-called “one-shot device” or “single-shot device”.

Automatic weapons ([Valis et al., 2008](#)) and electro-explosive devices ([Fan et al., 2009](#)) are examples of one-shot devices. For instance, an automatic weapon could fire all the rounds placed in a magazine or in an ammunition feed belt without any external intervention. Such devices will usually get destroyed and can therefore perform their intended function only once. On the other hand, an electro-explosive device inducts a current to excite inner powder and makes it explode. Naturally, we can only observe it by detonating it directly. Since the device cannot be used anymore after a successful detonation, we will not know the failure time of the device if the detonation becomes a failure. Thus, data of one-shot devices are binary (a success or a failure). In the same manner, one-shot device testing data also arise in destructive inspection procedures, wherein each device is allowed for only a single inspection because the test itself results in its destruction. [Morris \(1987\)](#) presented a study of 52 Li/SO₂ storage batteries under destructive discharge. Each battery was tested at one of three inspection times and then classified as acceptable or unacceptable according to a critical capacity value.

Instead of observing exact failure times obtained by continuous monitoring, we could only observe either a success or a failure at the inspection times for one-shot devices. The corresponding binary data consequently result in less precise inference. In this manner, one-shot device testing data differ from typical data obtained by measuring lifetimes in standard life-tests. If a successful test occurs, it implies that the lifetime is beyond the inspection time and therefore right-censored. Furthermore, the lifetime is smaller than the inspection time

and therefore left-censored when a test results in a failure. Consequently, all lifetimes are either left-censored or right-censored. The nature of one-shot devices results in very limited lifetime data and therefore poses a unique challenge in analysis from a statistical point of view. For a comprehensive review of data collection and statistical analysis of one-shot devices, one can refer to the book by [Balakrishnan et al. \(2021\)](#).

1.4. Maximum Likelihood Estimation Method

1.4.1. Maximum likelihood estimator

Suppose $\mathbf{X} = (X_1, \dots, X_n)$ is a random sample from a population with pdf $f(x|\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of parameters. Based on the observed value of the sample $\mathbf{x} = (x_1, \dots, x_n)$, the likelihood function is defined as

$$L(\boldsymbol{\theta}|\mathbf{x}) = \prod_{i=1}^n f(x_i|\boldsymbol{\theta}). \quad (1.3)$$

The MLE of the parameter $\boldsymbol{\theta}$ based on a sample \mathbf{x} , denoted as $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\mathbf{x})$, is the value of $\boldsymbol{\theta}$ at which $L(\boldsymbol{\theta}|\mathbf{x})$ attains its maximum, i.e., $\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} L(\boldsymbol{\theta}|\mathbf{x})$.

The MLE has many desirable properties, however, obtaining the MLE can be a challenging task in practice. There are many situations that the analytical form the MLE cannot be obtained and numerical methods are required to find the value of $\boldsymbol{\theta}$ that obtains the global maximum. The numerical methods can be unstable and sensitive to the initial value of the numerical algorithm as well as some extreme observations. Moreover, these numerical methods may be trapped in a local maximum or a saddle point instead of the global maximum. Therefore, computational algorithms that provide reliable estimates based on the maximum likelihood method are desired ([Casella and Berger, 2002](#)).

1.4.2. Asymptotic properties of MLE

Let $l(\boldsymbol{\theta}) = \ln L(\boldsymbol{\theta})$ denote the logarithm of the likelihood in Eq. (1.3). Denote $l_i(\boldsymbol{\theta})$ as the contribution of X_i to the total log-likelihood, i.e., $l(\boldsymbol{\theta}) = \sum_{i=1}^n l_i(\boldsymbol{\theta})$. Let $\mathcal{I}(\boldsymbol{\theta})$ denote the large-sample (or limiting) average amount of information per observation. Then, under some regularity conditions which will be discussed below,

$$\mathcal{I}(\boldsymbol{\theta}) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \mathbb{E} \left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \right] \right\} = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[-\frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \right] \right\}, \quad (1.4)$$

where the expectation is with respect to the as of yet unobserved data \mathbf{X} . For large samples, the matrix $I(\boldsymbol{\theta}) = n\mathcal{I}(\boldsymbol{\theta})$ approximately quantifies the amount of information that we expect to obtain from our future data. For most of the statistical models, $I(\boldsymbol{\theta})$ can be simplified to the Fisher information matrix for $\boldsymbol{\theta}$, i.e.,

$$I(\boldsymbol{\theta}) = \mathbb{E} \left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \right] = \sum_{i=1}^n \mathbb{E} \left[-\frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \right].$$

When data are available, the observed information matrix for $\boldsymbol{\theta}$ is defined as

$$\hat{I}(\hat{\boldsymbol{\theta}}) = -\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} = -\sum_{i=1}^n \frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}. \quad (1.5)$$

To discuss the asymptotic results of the MLE, we first need to define a set of conditions on the model, which are usually referred to as the regularity conditions. Regularity conditions mainly relate to the differentiability of the density and the ability to interchange differentiation and integration. The usual regularity conditions are:

- The point x at which $f(x|\boldsymbol{\theta}) > 0$ does not depend on $\boldsymbol{\theta}$.
- The parameters are identifiable; that is, if $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, then $f(x|\boldsymbol{\theta}) \neq f(x|\boldsymbol{\theta}')$.
- The true parameter value $\boldsymbol{\theta}$ is in the interior of the parameter space Θ .

- The density $f(x|\boldsymbol{\theta})$ has third mixed partial derivatives with respect to the elements of $\boldsymbol{\theta}$ in a neighborhood of the true $\boldsymbol{\theta}$. Each one of these derivatives is bounded by a function that has finite expectations with respect to $f(x|\boldsymbol{\theta})$.
- For all $\boldsymbol{\theta}$ in a neighborhood of the true $\boldsymbol{\theta}$,

$$\mathbb{E} \left[\frac{\partial^2 \ln f(x|\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \right] = \frac{\partial^2 \mathbb{E}[\ln f(x|\boldsymbol{\theta})]}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top},$$

where the expectations are with respect to the data from $f(x|\boldsymbol{\theta})$.

- The elements of $\mathcal{I}(\boldsymbol{\theta})$ in (1.4) are finite and $\mathcal{I}(\boldsymbol{\theta})$ is positive definite.

Let $\hat{\boldsymbol{\theta}}_n$ denote the MLE of $\boldsymbol{\theta}$ obtained from a sample of size n , and let $g(\boldsymbol{\theta})$ be a continuous function of $\boldsymbol{\theta}$. Then the limit of the $\hat{\boldsymbol{\theta}}_n$ has the following properties (Meeker and Escobar, 1998):

- *Consistency*: Under the regularity conditions on $f(x|\boldsymbol{\theta})$, for every $\epsilon > 0$ and every $\boldsymbol{\theta} \in \Theta$,

$$\lim_{n \rightarrow \infty} P_{\boldsymbol{\theta}}(|g(\hat{\boldsymbol{\theta}}_n) - g(\boldsymbol{\theta})| \geq \epsilon) = 0.$$

That is, $g(\hat{\boldsymbol{\theta}}_n)$ is a consistent estimator of $g(\boldsymbol{\theta})$.

- *Asymptotic Normality*: Under the regularity conditions on $f(x|\boldsymbol{\theta})$,

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \rightarrow N(\mathbf{0}, \mathcal{I}(\boldsymbol{\theta})^{-1}),$$

where $N(\mathbf{0}, \mathcal{I}(\boldsymbol{\theta})^{-1})$ is the normal distribution, with mean $\mathbf{0}$ and variance-covariance matrix $\mathcal{I}(\boldsymbol{\theta})^{-1}$, and $\mathcal{I}(\boldsymbol{\theta})$ is defined as in (1.4).

- *Efficiency*: The variance of $\hat{\boldsymbol{\theta}}$ achieves the $\mathcal{I}(\boldsymbol{\theta})^{-1}$ when the sample size tends to infinity. No consistent estimator has a lower asymptotic mean squared error than the MLE.

1.5. Expectation-Maximization and Stochastic Expectation-Maximization Algorithms

1.5.1. Expectation-maximization algorithm

The expectation-maximization (EM) algorithm proposed by [Dempster et al. \(1977\)](#) is an iterative procedure designed to find maximum likelihood estimates (MLEs) of parametric models where incomplete data are observed. The EM algorithm is based on the idea of replacing one difficult likelihood maximization with a sequence of easier maximizations whose limit is the answer to the original problem. The EM algorithm intends to converge to the MLE ([Casella and Berger, 2002](#); [McLachlan and Krishnan, 2007](#)).

Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ denote the incomplete or observed data, and $\mathbf{X} = (X_1, \dots, X_m)$ be the augmented or missing data. Then, (\mathbf{Y}, \mathbf{X}) is the complete data. The densities $g(\cdot|\boldsymbol{\theta})$ of \mathbf{Y} and $f(\cdot|\boldsymbol{\theta})$ of (\mathbf{Y}, \mathbf{X}) have the relationship

$$g(\mathbf{y}|\boldsymbol{\theta}) = \int f(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta}) d\mathbf{x},$$

where $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{x} = (x_1, \dots, x_m)$ are the realizations of \mathbf{Y} and \mathbf{X} respectively, and $\boldsymbol{\theta}$ is the vector of the parameters of interest. For the discrete case, the integrals are replaced by summations.

Let $L(\boldsymbol{\theta}|\mathbf{y}) = g(\mathbf{y}|\boldsymbol{\theta})$ be the incomplete-data likelihood and $L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}) = f(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta})$ be the complete-data likelihood. The goal is to estimate $\boldsymbol{\theta}$ by finding $\hat{\boldsymbol{\theta}}$ which maximizes $L(\boldsymbol{\theta}|\mathbf{y})$. The EM algorithm allows us to maximize $L(\boldsymbol{\theta}|\mathbf{y})$ by working with $L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}) = f(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta})$ and the conditional pdf or probability mass function (pmf) of \mathbf{X} given \mathbf{y} and $\boldsymbol{\theta}$. Define the conditional density of \mathbf{x} given \mathbf{y} as

$$h(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y}) = \frac{f(\mathbf{y}, \mathbf{x}|\boldsymbol{\theta})}{g(\mathbf{y}|\boldsymbol{\theta})},$$

then we can obtain

$$\ln L(\boldsymbol{\theta}|\mathbf{y}) = \ln L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}) - \ln h(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y}). \quad (1.6)$$

As we do not observe \mathbf{x} , we replace the terms on the right side of Eq. (1.6) with their expectation with respect to \mathbf{x} given the current estimate of $\boldsymbol{\theta}$, denoted as $\boldsymbol{\theta}^{(k)}$, i.e.,

$$\ln L(\boldsymbol{\theta}|\mathbf{y}) = \mathbb{E} \left[\ln L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}) | \boldsymbol{\theta}^{(k)}, \mathbf{y} \right] - \mathbb{E} \left[\ln h(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y}) | \boldsymbol{\theta}^{(k)}, \mathbf{y} \right].$$

The EM algorithm consists of two steps – the E-step and the M-step. The E-step calculates the expected log-likelihood, and the M-step maximizes the expected log-likelihood with respect to the parameters. In the E-step, given the current estimate of $\boldsymbol{\theta}$, denoted as $\boldsymbol{\theta}^{(k)}$, we obtain the objective function

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}) = \mathbb{E} \left[\ln L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{X}) | \boldsymbol{\theta}^{(k)}, \mathbf{y} \right]. \quad (1.7)$$

Then, in the M-step, we maximize the objective function in Eq. (1.7) with respect to $\boldsymbol{\theta}$ to obtain the updated estimate as

$$\boldsymbol{\theta}^{(k+1)} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)}).$$

The EM algorithm is particularly suited to missing data problems. Filling in the missing data and maximizing the likelihood based on complete data often simplify the computation. The EM algorithm also has many other desirable and advantageous properties. However, the sequence $\{\boldsymbol{\theta}^{(k)}\}$ might not converge to the global maximum of $L(\boldsymbol{\theta})$ when there are several stationary points in the likelihood function. The convergence of the EM algorithm can be extremely slow in some applications. In some situations, particularly with high-dimensional data, the conditional expectation is a high-dimensional integral or an integral over an irregular region, which might not be easy to calculate explicitly. Moreover, the

limiting position of $\{\boldsymbol{\theta}^{(k)}\}$ also greatly depends on the initial value in many cases. For more details of the EM algorithm and its properties, one may refer to [Celeux et al. \(1995\)](#); [Nielsen \(2000\)](#); [Casella and Berger \(2002\)](#).

1.5.2. Stochastic EM algorithm

The stochastic EM (SEM) algorithm is a stochastic version of the EM algorithm to address the limitations of the EM algorithm. The basic idea underlying the SEM algorithm is to avoid the computation of the conditional expectation involved in $Q(\boldsymbol{\theta}, \boldsymbol{\theta}^{(k)})$ in Eq. (1.7) by simulating the unobserved sample. Based on the simulated pseudo-complete sample, we obtain the updated estimate $\boldsymbol{\theta}^{(k+1)}$ ([Celeux et al., 1995](#)).

The SEM algorithm replaces the E-step of the EM algorithm with a stochastic step. S-step: Instead of taking the expected value with respect to some σ -finite measure $d\mathbf{x}$ of the missing data in Eq. (1.7), we randomly draw values of $\mathbf{x}^{(k)} = (x_1^{(k)}, \dots, x_m^{(k)})$ from the conditional distribution $h(\mathbf{x}|\boldsymbol{\theta}, \mathbf{y})$ in Eq. (1.6) given the current estimate of $\boldsymbol{\theta}$, denoted as $\boldsymbol{\theta}^{(k)}$, and the observed data \mathbf{y} . Thus, we obtain the objective function

$$Q(\boldsymbol{\theta}, \mathbf{x}^{(k)}) = \ln L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{x}^{(k)}). \quad (1.8)$$

M-step: Maximize the objective function in Eq. (1.8) with respect to $\boldsymbol{\theta}$ to obtain the updated estimate as

$$\boldsymbol{\theta}^{(k+1)} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}} Q(\boldsymbol{\theta}, \mathbf{x}^{(k)}).$$

The random drawings prevent the sequence $\{\boldsymbol{\theta}^{(k)}\}$ generated by the SEM algorithm from converging to the first stationary point of the likelihood function that the algorithm encounters. At each iteration, there is a non-zero probability of accepting an updated estimate $\boldsymbol{\theta}^{(k+1)}$. This is the reason why the SEM algorithm can avoid the convergence to a saddle point or a local maximum of the likelihood function. The sequence $\{\boldsymbol{\theta}^{(k)}\}$ does not converge pointwise, however, the sequence is a time-homogeneous Markov chain when the observed

data are fixed. If the Markov chain is ergodic, the algorithm will converge in the sense that as the number of iterations tends to infinity, $\boldsymbol{\theta}^{(k)}$ converges in distribution to a random variable, $\tilde{\boldsymbol{\theta}}$, where $\tilde{\boldsymbol{\theta}}$ is distributed according to the stationary distribution of the Markov chain (Celeux et al., 1995; Nielsen, 2000).

To obtain an estimate of $\boldsymbol{\theta}$ based on SEM algorithm, we obtain $\{\boldsymbol{\theta}^{(k)}\}$, $k = 1, 2, \dots, K$, discard the first B iterations as burn-in, and average over the estimates from the remaining iterations, i.e.,

$$\hat{\boldsymbol{\theta}} = \frac{1}{K - B} \sum_{k=B+1}^K \boldsymbol{\theta}^{(k)}.$$

Since the SEM algorithm can be viewed as a stochastic extension of the EM algorithm, it is still directed by the EM algorithm dynamics. Thus, the SEM algorithm can be expected to detect the most stable fixed point of the EM algorithm in a comparatively small number of iterations. In some situations, the M-step of the EM algorithm is not analytically tractable while the SEM algorithm does not involve such difficulties.

1.6. Goodness-of-fit Test

The goodness-of-fit of a statistical model describes how well the model fits a set of observations. Measures of goodness-of-fit typically summarize the discrepancy between the observed data and the expected values under the model in consideration. Such measures can be used in a statistical hypothesis testing procedure to test whether the observed data follow a specified statistical distribution. A goodness-of-fit test can be used to test the following hypotheses:

$$\begin{aligned} H_0 : F(x) &= F_0(x) \\ \text{against } H_a : F(x) &\neq F_0(x). \end{aligned} \tag{1.9}$$

There are many classical goodness-of-fit tests for testing the hypotheses in Eq. (1.9), for example, the Anderson-Darling test, the Pearson's chi-squared test and the Kolmogorov-Smirnov test (D'Agostino and Stephens, 1986; Huber-Carol et al., 2002). In this thesis, we consider some estimation procedures based on the Kolmogorov-Smirnov (K-S) test. The K-S test is a nonparametric hypothesis test that measures the probability that a chosen univariate dataset is drawn from a continuous model. The test is based on the K-S statistic that measures the supremum distance between the empirical distribution function of a univariate dataset and the cumulative distribution function of the target distribution. Specifically, given a random sample $\{x_1, \dots, x_n\}$, the empirical cdf $F_n(x)$ is defined as

$$F_n(x) = \frac{1}{n} \sum_1^n I(x_i \leq x),$$

and the of K-S test statistic to test the hypotheses in Eq. (1.9) is defined as

$$D_n(x) = \sup_x (|F_n(x) - F_0(x)|).$$

The limiting form of the distribution function of D_n under H_0 is

$$\lim_{n \rightarrow \infty} \Pr(\sqrt{n}D_n \leq \delta) = 1 - 2 \sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2\delta^2}.$$

Thus, the p -value of the K-S test can be approximated as,

$$\Pr(D_n > D_{n,obs}) = 2 \sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2nD_{n,obs}^2},$$

where $D_{n,obs}$ is the observed value of the K-S statistic. Marsaglia et al. (2003) provided a numerical method to compute the p -value of the K-S test.

1.7. Scope of the Thesis

In this thesis, we study the parameter estimation methods for gamma and generalized gamma distributions based on different kinds of censored data arising from life-testing experiments. The rest of this thesis is organized as follows. In Chapter 2, we focus on the parameter estimation of the generalized gamma distribution using the maximum likelihood estimation method based on left-truncated and right-censored data with direct optimization methods, the EM algorithm, and the stochastic EM algorithm. In Chapter 3, we study the parameter estimation based on one-shot device testing data. Both frequentist approach and Bayesian approach are considered. These methodologies are illustrated by considering the lifetimes of the one-shot devices that follow gamma distributions. In Chapter 4, some possible extensions based on the results presented in this thesis are described as future research directions. All the detailed proofs and derivations are relegated to the appendices for conciseness in the presentation of the main text of the thesis.

Chapter 2

Estimation of Parameters for Generalized Gamma Distributions Based on Left-Truncated and Right-Censored Data

2.1. Introduction

In this chapter, we study the parameter estimation of the generalized gamma distribution based on left-truncated and right-censored (LTRC) data. It is well known that the maximum likelihood estimates of the parameters for the generalized gamma distribution may not be stable, especially when the data is incomplete. Different estimation procedures are proposed and their performances are evaluated by means of a Monte Carlo simulation study. These estimation procedures are also compared with some existing estimation procedures based on the maximum likelihood method, such as the direct optimization method, the profile likelihood method and the expectation maximization algorithm, in terms of accuracy and stability. Based on the simulation results, we make some recommendations about which estimation procedures are more appropriate in practice.

This chapter is organized as follows. In Section 2.2, we describe the data structure and the notations used in this chapter. In Section 2.3 we review some existing parameter estimation procedures and propose several novel parameter estimation procedures. In Section 2.4, a method based on the goodness-of-fit test is proposed to obtain an initial estimate for the iterative parameter estimation procedures. In Section 2.6, a numerical example is presented to illustrate the parameter estimation procedures described in Section 2.3. In Section 2.7, a Monte Carlo simulation study is used to evaluate the performances of the parameter estimation procedures considered here. Some concluding remarks are provided in Section 2.8.

2.2. Data Description and Notation

Suppose we are interested in studying the lifetime distribution of a manufacturing product with random installation time. Field failure time data were collected for this purpose. We assume that the failure time of a unit is observed only if it failed after the time τ^* , which is the time we started collecting the field failure data. When the failure time of a unit is observed, the installation time of that product can also be observed. Then, the lifetime of a product is defined as the difference between the failure time and the installation time. For those units that failed before τ^* , no information related to the failure time and the installation time can be obtained. Hence, they are not included in the study. This results in left-truncated data. In addition, the lifetimes of the units were followed until the end of the study, c^* , which incorporates right-censoring into the data. In other words, the units that failed before c^* are completely observed, while the units that were still working at c^* are right-censored. For those right-censored observations, we assume that the installation times are known.

Suppose we obtain the information of the installation time and the failure/censoring time of n products. Let δ_i ($i = 1, \dots, n$) denote the censoring indicator, i.e.,

$$\delta_i = \begin{cases} 1, & \text{if the product failed before } c^*, \\ 0, & \text{otherwise,} \end{cases}$$

and v_i ($i = 1, \dots, n$) denote the truncation indicator, i.e.,

$$v_i = \begin{cases} 1, & \text{if the product is installed after } \tau^*, \\ 0, & \text{otherwise.} \end{cases}$$

Let the random installation time of the i -th product be W_i and the time of failure of the i -th product be Y_i . We can summarize the four types of field failure data being observed as follows:

1. Installed after τ^* and failed before c^* (i.e., $W_i > \tau^*$ and $Y_i < c^*$): The lifetime of the product, $X_i = Y_i - W_i$, is observed. The observation is neither truncated nor censored, i.e., $v_i = 1$, $\delta_i = 1$.
2. Installed after τ^* and failed after c^* (i.e., $W_i > \tau^*$ and $Y_i > c^*$): The lifetime of the product is right-censored with censoring time $c^* - W_i$. The observation is not truncated but it is right-censored, i.e., $v_i = 1$, $\delta_i = 0$.
3. Installed before τ^* and failed before c^* (i.e., $W_i < \tau^*$ and $Y_i < c^*$): The lifetime of the product, $X_i = Y_i - W_i$, is observed. The observation is not censored but it is left-truncated, i.e., $v_i = 0$, $\delta_i = 1$.
4. Installed before τ^* and failed after c^* (i.e., $W_i < \tau^*$ and $Y_i > c^*$): The lifetime of the product is right-censored with censoring time $c^* - W_i$. The observation is left-truncated and right-censored, i.e., $v_i = 0$, $\delta_i = 0$.

In Figure 2.1, we illustrate the four types of field failure data being observed and those products that are not observed. We further define $T_i = \min\{Y_i, c^*\} - W_i$ and $\mathcal{T}_i = \max\{0, \tau^* - W_i\}$, and their realizations as t_i and τ_i , respectively, for $i = 1, \dots, n$. Therefore, the field failure data that we observed are $(\delta_i, v_i, t_i, \tau_i)$, $i = 1, \dots, n$.

Suppose the lifetime of the manufacturing product of interest follows the $GG(\alpha, \eta, \kappa)$ distribution in Eqs. (1.1) and (1.2), based on the observed left-truncated and right-censored data $(\delta_i, v_i, t_i, \tau_i)$, $i = 1, \dots, n$, the log-likelihood function

$$\begin{aligned}
\ln L(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i v_i \ln f_X(t_i; \boldsymbol{\theta}) + \sum_{i=1}^n (1 - \delta_i) v_i \ln(1 - F_X(t_i; \boldsymbol{\theta})) \\
&\quad + \sum_{i=1}^n \delta_i (1 - v_i) \ln \left(\frac{f_X(t_i; \boldsymbol{\theta})}{1 - F_X(\tau_i; \boldsymbol{\theta})} \right) \\
&\quad + \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \ln \left(\frac{1 - F_X(t_i; \boldsymbol{\theta})}{1 - F_X(\tau_i; \boldsymbol{\theta})} \right),
\end{aligned}$$

where $\boldsymbol{\theta} = (\alpha, \eta, \kappa)$ is the vector of parameters. The log-likelihood function can be expressed as

$$\begin{aligned}
\ln L(\boldsymbol{\theta}) = & \sum_{i=1}^n \delta_i v_i \left[\ln \eta - \ln t_i + \kappa \eta (\ln t_i - \ln \eta) - \left(\frac{t_i}{\alpha} \right)^\eta - \ln \Gamma(\kappa) \right] \\
& + \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha} \right)^\eta \right) - \ln \Gamma(\kappa) \right] \\
& + \sum_{i=1}^n \delta_i v_i \left[\ln \eta - \ln t_i + \kappa \eta (\ln t_i - \ln \eta) - \left(\frac{t_i}{\alpha} \right)^\eta - \ln \Gamma \left(\kappa, \left(\frac{\tau_i}{\alpha} \right)^\eta \right) \right] \\
& + \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha} \right)^\eta \right) - \ln \Gamma \left(\kappa, \left(\frac{\tau_i}{\alpha} \right)^\eta \right) \right]. \tag{2.1}
\end{aligned}$$

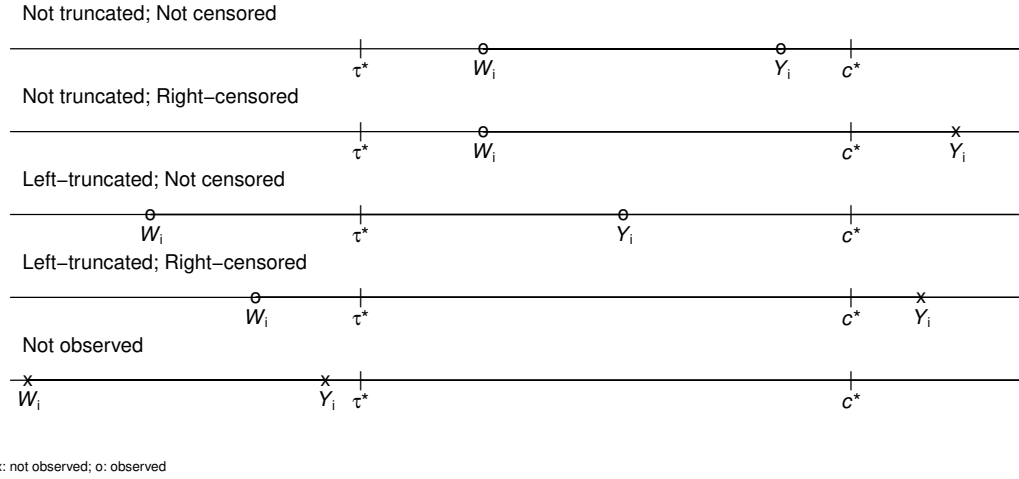


Figure 2.1: Illustration of the four types of field failure data.

2.3. Parameter Estimation Procedures

2.3.1. Direct optimization methods

Direct optimization methods that maximize the log-likelihood function using numerical methods are considered here. The following methods require a vector of initial estimates of $\boldsymbol{\theta}$ denoted as $\boldsymbol{\theta}^{(0)} = (\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)})$.

2.3.1.1. Direct optimization using Nelder-Mead method (DONM)

To obtain the MLE of $\boldsymbol{\theta}$, denoted as $\hat{\boldsymbol{\theta}}^* = (\hat{\alpha}^*, \hat{\eta}^*, \hat{\kappa}^*)$, numerical methods can be used to maximize the log-likelihood function in Eq. (2.1). In our study, we use the Nelder-Mead method to maximize the likelihood function with respect to $\boldsymbol{\theta}$ under the constraints $\alpha > 0$, $\eta > 0$ and $\kappa > 0$. Note that the constraint optimization algorithm based on the Nelder-Mead method is implemented in the R function `constrOptim` (R Core Team, 2020).

2.3.1.2. Direct optimization based on three-stage profile likelihood approach (DOPL1)

Instead of maximizing the likelihood function in Eq. (2.1) with respect to α , η and κ simultaneously which requires dealing with a three-dimensional optimization problem, we consider a three-stage profile likelihood approach which only requires to deal with a one-dimensional optimization problem repeatedly. Specifically, suppose $\boldsymbol{\theta}^{(k)} = (\alpha^{(k)}, \eta^{(k)}, \kappa^{(k)})$ is the estimate in the k -th iteration, the following algorithm is used to obtain the MLE of $\boldsymbol{\theta}$:

Step 1. Consider $\eta^{(k)}$ and $\kappa^{(k)}$ to be fixed, use a numerical optimization method to maximize the following function with respect to α in order to obtain $\alpha^{(k+1)}$:

$$\begin{aligned}
& \ln L(\alpha|\eta^{(k)}, \kappa^{(k)}) \\
&= \sum_{i=1}^n \delta_i v_i \left[\ln \frac{\eta^{(k)}}{t_i} - \ln \Gamma(\kappa^{(k)}) + \kappa^{(k)} \eta^{(k)} \ln \frac{t_i}{\alpha} - \left(\frac{t_i}{\alpha} \right)^{\eta^{(k)}} \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^{\eta^{(k)}} \right) - \ln \Gamma(\kappa^{(k)}) \right] \\
&+ \sum_{i=1}^n \delta_i (1 - v_i) \left[\ln \frac{\eta^{(k)}}{t_i} + \kappa^{(k)} \eta^{(k)} \ln \frac{t_i}{\alpha} - \left(\frac{t_i}{\alpha} \right)^{\eta^{(k)}} - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^{\eta^{(k)}} \right) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^{\eta^{(k)}} \right) - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{\tau_i}{\alpha} \right)^{\eta^{(k)}} \right) \right]. \quad (2.2)
\end{aligned}$$

Step 2. Consider $\alpha^{(k+1)}$ and $\kappa^{(k)}$ to be fixed, use a numerical optimization method to maximize the following function with respect to η in order to obtain $\eta^{(k+1)}$:

$$\begin{aligned}
& \ln L(\eta|\alpha^{(k+1)}, \kappa^{(k)}) \\
&= \sum_{i=1}^n \delta_i v_i \left[\ln \frac{\eta}{t_i} - \ln \Gamma(\kappa^{(k)}) + \kappa^{(k)} \eta \ln \frac{t_i}{\alpha^{(k+1)}} - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta} \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta} \right) - \ln \Gamma(\kappa^{(k)}) \right] \\
&+ \sum_{i=1}^n \delta_i (1 - v_i) \left[\ln \frac{\eta}{t_i} + \kappa^{(k)} \eta \ln \frac{t_i}{\alpha^{(k+1)}} - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta} - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta} \right) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta} \right) - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{\tau_i}{\alpha^{(k+1)}} \right)^{\eta} \right) \right]. \quad (2.3)
\end{aligned}$$

Step 3. Consider $\alpha^{(k+1)}$ and $\eta^{(k+1)}$ to be fixed, use a numerical optimization method to maximize the following function with respect to κ in order to obtain $\kappa^{(k+1)}$:

$$\begin{aligned}
& \ln L(\kappa | \alpha^{(k+1)}, \eta^{(k+1)}) \\
= & \sum_{i=1}^n \delta_i v_i \left[\ln \frac{\eta^{(k+1)}}{t_i} - \ln \Gamma(\kappa) + \kappa \eta^{(k+1)} \ln \frac{t_i}{\alpha^{(k+1)}} - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right] \\
& + \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) - \ln \Gamma(\kappa) \right] \\
& + \sum_{i=1}^n \delta_i (1 - v_i) \left[\ln \frac{\eta^{(k+1)}}{t_i} + \kappa \eta^{(k+1)} \ln \frac{t_i}{\alpha^{(k+1)}} \right. \\
& \quad \left. - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} - \ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) \right] \\
& + \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) - \ln \Gamma \left(\kappa, \left(\frac{\tau_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) \right]. \quad (2.4)
\end{aligned}$$

Step 4. Repeat Steps 1–3 until the convergence is achieved to the desired level of accuracy.

Here, we define that the convergence occurs as

$$\max\{|\alpha^{(k+1)} - \alpha^{(k)}|, |\eta^{(k+1)} - \eta^{(k)}|, |\kappa^{(k+1)} - \kappa^{(k)}|\} < \epsilon,$$

where ϵ is the preset tolerance level.

In our study, we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method to maximize the functions in Eqs. (2.2) – (2.4). Note that the BFGS quasi-Newton method is implemented in the R function `optim` (R Core Team, 2020).

2.3.1.3. Direct optimization based on two-stage profile likelihood approach (DOPL2)

We consider a two-stage profile likelihood approach. We update (α, η) simultaneously while we update κ individually as κ is involved with the gamma function. Specifically, suppose $\boldsymbol{\theta}^{(k)} = (\alpha^{(k)}, \eta^{(k)}, \kappa^{(k)})$ is the estimate in the k -th iteration, the following algorithm is used to obtain the MLE of $\boldsymbol{\theta}$:

Step 1. Consider $\kappa^{(k)}$ to be fixed, use a numerical optimization method to maximize the following function with respect to (α, η) in order to obtain $(\alpha^{(k+1)}, \eta^{(k+1)})$:

$$\begin{aligned}
& \ln L(\alpha, \eta | \kappa^{(k)}) \\
&= \sum_{i=1}^n \delta_i v_i \left[\ln \frac{\eta}{t_i} - \ln \Gamma(\kappa^{(k)}) + \kappa^{(k)} \eta \ln \frac{t_i}{\alpha} - \left(\frac{t_i}{\alpha} \right)^\eta \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^\eta \right) - \ln \Gamma(\kappa^{(k)}) \right] \\
&+ \sum_{i=1}^n \delta_i (1 - v_i) \left[\ln \frac{\eta}{t_i} + \kappa^{(k)} \eta \ln \frac{t_i}{\alpha} - \left(\frac{t_i}{\alpha} \right)^\eta - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^\eta \right) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa^{(k)}, \left(\frac{t_i}{\alpha} \right)^\eta \right) - \ln \Gamma \left(\kappa^{(k)}, \left(\frac{\tau_i}{\alpha} \right)^\eta \right) \right]. \quad (2.5)
\end{aligned}$$

Step 2. Consider $\alpha^{(k+1)}$ and $\eta^{(k+1)}$ to be fixed, use a numerical optimization method to maximize the following function with respect to κ in order to obtain $\kappa^{(k+1)}$:

$$\begin{aligned}
& \ln L(\kappa | \alpha^{(k+1)}, \eta^{(k+1)}) \\
&= \sum_{i=1}^n \delta_i v_i \left[\ln \frac{\eta^{(k+1)}}{t_i} - \ln \Gamma(\kappa) + \kappa \eta^{(k+1)} \ln \frac{t_i}{\alpha^{(k+1)}} - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) v_i \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) - \ln \Gamma(\kappa) \right] \\
&+ \sum_{i=1}^n \delta_i (1 - v_i) \left[\ln \frac{\eta^{(k+1)}}{t_i} + \kappa \eta^{(k+1)} \ln \frac{t_i}{\alpha^{(k+1)}} - \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right. \\
&\quad \left. - \ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) (1 - v_i) \left[\ln \Gamma \left(\kappa, \left(\frac{t_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) - \ln \Gamma \left(\kappa, \left(\frac{\tau_i}{\alpha^{(k+1)}} \right)^{\eta^{(k+1)}} \right) \right]. \quad (2.6)
\end{aligned}$$

Step 3. Repeat Steps 1–2 until the convergence is achieved to the desired level of accuracy.

Here, we define that the convergence occurs as

$$\max\{|\alpha^{(k+1)} - \alpha^{(k)}|, |\eta^{(k+1)} - \eta^{(k)}|, |\kappa^{(k+1)} - \kappa^{(k)}|\} < \epsilon,$$

where ϵ is the preset tolerance level.

In our study, we use the Newton-Raphson method to maximize the functions in Eqs. (2.5) – (2.6). Note that the Newton-Raphson method is implemented in the R function `maxNR` of package `maxLik` (Henningsen and Toomet, 2011).

2.3.2. Expectation-maximization algorithm (EMPL2)

Balakrishnan and Mitra (2014) studied the EM algorithm for $GG(\alpha, \eta, \kappa)$ distribution based on LTRC data. In the E-step, the objective function $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(k)})$ in Eq. (1.7) can be expressed as

$$\begin{aligned} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(k)}) &= n(\ln \eta - \ln \alpha) + \sum_{i=1}^n \delta_i (\ln t_i - \ln \alpha) + \sum_{i=1}^n (1 - \delta_i) (E_i^{(k)} - \ln \alpha) \\ &\quad - \sum_{i=1}^n \delta_i \left(\frac{t_i}{\alpha}\right)^\eta - \sum_{i=1}^n (1 - \delta_i) \left(\frac{\alpha^{(k)}}{\alpha}\right)^\eta \frac{\Gamma\left(\kappa^{(k)} + \frac{\eta}{\eta^{(k)}}, \left(\frac{t_i}{\alpha^{(k)}}\right)^\eta\right)}{\Gamma\left(\kappa^{(k)}, \left(\frac{t_i}{\alpha^{(k)}}\right)^\eta\right)} \\ &\quad - \sum_{i=1}^n \left[v_i \ln \Gamma(\kappa) + (1 - v_i) \ln \Gamma\left(\kappa, \left(\frac{t_i}{\alpha}\right)^\eta\right) \right], \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} E_i^{(k)} &= \mathbb{E} [\ln T_i | T_i > t_i] \\ &= \ln \alpha^{(k)} + \frac{1}{\eta^{(k)} \Gamma\left(\kappa^{(k)}, \left(\frac{t_i}{\alpha^{(k)}}\right)^\eta\right)} \int_{\left(\frac{t_i}{\alpha^{(k)}}\right)^\eta}^{\infty} u^{\kappa^{(k)}-1} \ln u e^{-u} du. \end{aligned}$$

In the M-step, a process similar to DOPL2 is used. Due to the difficulties in maximizing Eq. (2.7), Balakrishnan and Mitra (2014) proposed using EM-gradient algorithm (see also,

Lange, 1995) to obtain the updated estimates. Specifically, the algorithm is described as follows:

- Step 1. Given the value of $\kappa^{(k)}$, perform a one-step Newton-Raphson on the Q -function in Eq. (2.7) to obtain $\alpha^{(k+1)}$ and $\eta^{(k+1)}$.
- Step 2. Given the values of $\alpha^{(k+1)}$ and $\eta^{(k+1)}$, perform a one-step Newton-Raphson on the Q -function in Eq. (2.7) to obtain $\kappa^{(k+1)}$. Then, go to the next cycle of the E-step with $(\alpha^{(k+1)}, \eta^{(k+1)}, \kappa^{(k+1)})$.

The algorithm is continued until convergence is achieved to the desired level of accuracy. This method requires a vector of initial estimates of $\boldsymbol{\theta}$ denoted as $\boldsymbol{\theta}^{(0)} = (\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)})$. See Balakrishnan and Mitra (2014) for more details.

2.3.3. Stochastic EM algorithm (SEM)

The SEM algorithm is essentially using imputation of the unobserved data to approximate the MLE. The SEM algorithm for $GG(\alpha, \eta, \kappa)$ distribution is based on imputing the censored observations with the current parameter estimates $\boldsymbol{\theta}^{(k)}$ of $\boldsymbol{\theta}$ from the k -th SEM cycle, then estimate the parameter $\boldsymbol{\theta}$ based on the pseudo-complete data (see, for example, Ng and Ye, 2014). The SEM algorithm is described as follows:

In the S-step, a random sample for the censored observations is drawn from the conditional distribution in Eq. (1.6), to generate the pseudo-complete data. If the lifetime for the i -th unit is censored, the lifetime of the i -th unit is imputed from the distribution with pdf

$$f_{X_i|X_i>t_i}(x) = \frac{\eta}{\alpha\Gamma(\kappa, (t_i/\alpha)^\eta)} \left(\frac{x}{\alpha}\right)^{\kappa\eta-1} \exp\left(-\left(\frac{x}{\alpha}\right)^\eta\right), \quad x > t_i,$$

and cdf

$$\begin{aligned} F_{X_i|X_i>t_i}(x) &= \int_{t_i}^x \frac{\eta}{\alpha\Gamma(\kappa, (t_i/\alpha)^\eta)} \left(\frac{u}{\alpha}\right)^{\kappa\eta-1} \exp\left(-\left(\frac{u}{\alpha}\right)^\eta\right) du \\ &= \frac{\Gamma(\kappa)}{\Gamma(\kappa, (t_i/\alpha)^\eta)} [F_X(x) - F_X(t_i)], \quad x > t_i, \end{aligned}$$

where $F_X(\cdot)$ is the cdf of $GG(\alpha, \eta, \kappa)$ in Eq. (1.2). The imputation is implemented by generating the random variate using the inverse transformed method. For a given value of (α, η, κ) , the imputed lifetime x_i of the item with censoring time t_i can be obtained by:

Step 1. Generate u from the uniform distribution in $(0, 1)$.

Step 2. Let $z = \frac{\Gamma(\kappa, (t_i/\alpha)^\eta)}{\Gamma(\kappa)}u + F_X(t_i)$, obtain $x = F_X^{-1}(z)$, where $F_X^{-1}(\cdot)$ is the inverse cdf of the generalized gamma distribution defined in Eq. (1.2).

Let $x_i^{(k)}$ denote the imputed lifetime of the i -th unit if the unit is censored, given the current estimate of the parameters $\boldsymbol{\theta}^{(k)} = (\alpha^{(k)}, \eta^{(k)}, \kappa^{(k)})$. After the k -th S-step, the lifetime of the pseudo-complete data $\mathbf{t}^{(k)} = (t_1^{(k)}, \dots, t_n^{(k)})$ can be expressed as

$$t_i^{(k)} = \begin{cases} t_i, & \text{if } \delta_i = 1, \\ x_i^{(k)}, & \text{if } \delta_i = 0. \end{cases}$$

In the M-step, we maximize the log-likelihood function based on the pseudo-complete data $\mathbf{t}^{(k)} = (t_1^{(k)}, \dots, t_n^{(k)})$. The Q -function in Eq. (1.8) can be expressed as

$$\begin{aligned} Q(\boldsymbol{\theta}|\mathbf{x}^{(k)}) &= \ln L(\boldsymbol{\theta}|\mathbf{t}^{(k)}) \\ &= \sum_{i=1}^n v_i \ln f_X(t_i^{(k)}) + \sum_{i=1}^n (1 - v_i) \ln \frac{f_X(t_i^{(k)})}{1 - F_X(\tau_i)} \\ &= \sum_{i=1}^n v_i \left[\ln \frac{\eta}{t_i^{(k)}} - \ln \Gamma(\kappa) + \kappa \eta \ln \frac{t_i^{(k)}}{\alpha} - \left(\frac{t_i^{(k)}}{\alpha} \right)^\eta \right] \\ &\quad + \sum_{i=1}^n (1 - v_i) \left[\ln \frac{\eta}{t_i^{(k)}} + \kappa \eta \ln \frac{t_i^{(k)}}{\alpha} - \left(\frac{t_i^{(k)}}{\alpha} \right)^\eta \right. \\ &\quad \left. - \ln \Gamma \left(\kappa, \left(\frac{t_i^{(k)}}{\alpha} \right)^\eta \right) + \ln \Gamma(\kappa) \right]. \end{aligned} \tag{2.8}$$

Then, use an appropriate optimization method to obtain $\boldsymbol{\theta}^{(k+1)}$ which maximizes the Q -function in Eq. (2.8). We consider several different optimization methods as described in Sections 2.3.3.1 – 2.3.3.3.

Let $\{\boldsymbol{\theta}^{(k)}\}$, $k = 1, 2, \dots, K$ denote the SEM chain. The first B iterations are discarded as the burn-in period. The average of the rest $(K - B)$ iterations will be used as the final estimate, i.e.,

$$\hat{\boldsymbol{\theta}} = \frac{1}{K - B} \sum_{k=B+1}^K \boldsymbol{\theta}^{(k)}.$$

Similar to the procedures described in Sections 2.3.1 and 2.3.2, the SEM algorithms presented below require a vector of initial estimate $\boldsymbol{\theta}^{(0)} = (\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)})$.

2.3.3.1. SEM using Nelder-Mead method (SEMNM)

For a given $\boldsymbol{\theta}^{(k)} = (\alpha^{(k)}, \eta^{(k)}, \kappa^{(k)})$, we use the Nelder-Mead method to maximize the Q -function in Eq. (2.8) with respect to $\boldsymbol{\theta}$ under the constraints $\alpha > 0$, $\eta > 0$ and $\kappa > 0$ in the M-step of the SEM algorithm. This maximization method is the same as the one presented in Section 2.3.1.1.

2.3.3.2. SEM based on three-stage profile likelihood approach (SEMPL1)

Similar to the optimization method presented in Section 2.3.1.2, we consider a profile likelihood approach which deals with a one-dimensional optimization problem repeatedly in the M-step of the SEM algorithm. Due to the difficulties in maximizing Eq. (2.8), we propose using a variation of the EM-gradient algorithm (Lange, 1995) in our SEM algorithm. In the k -th cycle of the SEM algorithm,

- Step 1. Consider $\eta^{(k)}$ and $\kappa^{(k)}$ to be fixed, perform a one-step BFGS quasi Newton method on Eq. (2.8) to obtain the updated value $\alpha^{(k+1)}$.
- Step 2. Consider $\alpha^{(k+1)}$ and $\kappa^{(k)}$ to be fixed, perform a one-step BFGS quasi Newton method on Eq. (2.8) to obtain the updated value $\eta^{(k+1)}$.
- Step 3. Consider $\alpha^{(k+1)}$ and $\eta^{(k+1)}$ to be fixed, perform a one-step BFGS quasi Newton method on Eq. (2.8) to obtain the updated value $\kappa^{(k+1)}$.

Thus, we can obtain the estimate of the parameter vector $\boldsymbol{\theta}^{(k+1)} = (\alpha^{(k+1)}, \eta^{(k+1)}, \kappa^{(k+1)})$ and go to the $(k + 1)$ -th cycle of the SEM algorithm.

2.3.3.3. SEM based on two-stage profile likelihood approach (SEMPL2)

Similar to the optimization method presented in Section 2.3.1.3, we consider a two-stage profile likelihood approach in the M-step of the SEM algorithm. Due to the difficulties in maximizing Eq. (2.8), we propose using a variation of the EM-gradient algorithm (Lange, 1995) in our SEM algorithm. In the k -th cycle of the SEM algorithm,

1. Consider $\kappa^{(k)}$ to be fixed, perform a one-step Newton-Raphson method to maximize Eq. (2.8) to obtain the updated value $(\alpha^{(k+1)}, \eta^{(k+1)})$.
2. Consider $\alpha^{(k+1)}$ and $\eta^{(k+1)}$ to be fixed, perform a one-step Newton-Raphson method to maximize Eq. (2.8) to obtain the updated value $\kappa^{(k+1)}$.

Thus, we can obtain the estimate of the parameter vector $\boldsymbol{\theta}^{(k+1)} = (\alpha^{(k+1)}, \eta^{(k+1)}, \kappa^{(k+1)})$ and go to the $(k + 1)$ -th cycle of the SEM algorithm.

2.4. Initial Estimate for Iterative Procedures

One key component in all the aforementioned methods is the initial estimate of the iterative algorithm. Finding a feasible and reliable initial value is crucial in obtaining the parameter estimates. As mentioned in Casella and Berger (2002), when the analytic solution to the MLE is not possible, using a numerical method to find MLE might not be stable due to the initial value. Balakrishnan and Mitra (2014) used the true value of the parameters in the simulation study as the initial value, however, we do not have the information about the true value of the parameters in practice. Therefore, we propose to use the estimation method proposed by (Gomès et al., 2008) which utilizes a goodness-of-fit test to obtain an initial estimate of the parameter vector based on LTRC data.

2.4.1. Goodness-of-fit-based estimation method (GOFT)

Suppose $X \sim GG(\alpha, \eta, \kappa)$, when η is known, $Y = X^\eta$ follows the two-parameter gamma distribution with the shape parameter K and the scale parameter A , where

$$K = \kappa \text{ and } A = \alpha^\eta. \quad (2.9)$$

Therefore, given an estimate of η , we can transform the generalized gamma distribution into a two-parameter gamma distribution. Suppose $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ is a random sample from a two-parameter gamma distribution with the shape parameter K and the scale parameter A , the parameters K and A can be estimated using the method of moments as

$$\hat{K} = \frac{\hat{\mu}_{\mathbf{Y}}^2}{\hat{\sigma}_{\mathbf{Y}}^2} \text{ and } \hat{A} = \frac{\hat{\mu}_{\mathbf{Y}}}{\hat{\sigma}_{\mathbf{Y}}^2}, \quad (2.10)$$

where $\hat{\mu}_{\mathbf{Y}} = \frac{1}{n} \sum_{i=1}^n Y_i$ and $\hat{\sigma}_{\mathbf{Y}}^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \hat{\mu}_{\mathbf{Y}})^2$.

To obtain a reasonable estimate of η , goodness-of-fit-based procedures can be considered. The p -value of the K-S test discussed in Section 1.6 can be used in a generalized gamma distribution to show how compatible the data is with a particular choice of η . In practice, a reasonable value of η is no more than 20 and the accuracy of the tenth decimal is usually sufficient (Gomès et al., 2008). We apply the K-S test to all $\hat{\eta}$ from 0.1(0.1)20.0. For each $\hat{\eta}$, a K-S test is performed on the transformed two-parameter gamma distribution with the shape parameter \hat{K} and the scale parameter \hat{A} calculated from Eq. (2.10). We pick the $\hat{\eta}$ with the largest p -value since this estimate is most compatible with the data. Then, calculate α and κ based on Eq. (2.9). Note that \hat{K} and \hat{A} are explicit and do not require a numerical method to get the estimate.

Applying the K-S test to truncated and censored data can be complicated, therefore, we consider the truncated data as non-truncated and apply the following two ways to perform the K-S test based on LTRC data.

2.4.2. Initial values based on uncensored data

The first way is to apply the K-S test to the uncensored observations by discarding the censored data. The detailed algorithm to obtain the initial estimate can be described as follows:

Step 1. Let $\mathbf{x} = (x_1, x_2, \dots, x_m)$ denote the lifetime of the units which are not censored. For $\forall \eta_i \in H = \{0.1, 0.2, \dots, 20\}$, let $\mathbf{y}^{[i]} = (y_1^{[i]}, y_2^{[i]}, \dots, y_m^{[i]}) = (x_1^{\eta_i}, x_2^{\eta_i}, \dots, x_m^{\eta_i})$.

Step 2. Estimate $\hat{K}^{[i]}$ and $\hat{A}^{[i]}$ as shown in Eq. (2.10).

Step 3. A K-S test is performed to test

H_0 : the transformed data $\mathbf{y}^{[i]}$ follows the gamma distribution with the shape parameter $\hat{K}^{[i]}$ and the scale parameter $\hat{A}^{[i]}$

against H_a : the transformed data $\mathbf{y}^{[i]}$ does not follow the gamma distribution with the shape parameter $\hat{K}^{[i]}$ and the scale parameter $\hat{A}^{[i]}$.

Let $p^{[i]}$ denote p -value of the K-S test.

Step 4. Pick η_{i^*} for $p^{[i^*]} = \max_i p^{[i]}$ and the corresponding $\hat{K}^{[i^*]}$ and $\hat{A}^{[i^*]}$. From Eq. (2.9), we can obtain an estimate of $\boldsymbol{\theta}$ as $\hat{\boldsymbol{\theta}} = \left(\left(\hat{A}^{[i^*]} \right)^{\frac{1}{\eta_{i^*}}}, \eta_{i^*}, \hat{K}^{[i^*]} \right)$ which can serve as an initial estimate of the iterative procedures.

2.4.3. Initial values based on pseudo-uncensored data

The second way is to apply the K-S test to all the observations by treating the censoring time as the failure time. We obtain a pseudo-uncensored sample by treating the censoring times as failure times and the conventional K-S test can be applied. The detailed algorithm to obtain the initial estimate based on pseudo-uncensored data can be described as follows:

Step 1. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denote the pseudo-uncensored lifetime data. For $\forall \eta_i \in H = \{0.1, 0.2, \dots, 20\}$, let $\mathbf{y}^{[i]} = (y_1^{[i]}, y_2^{[i]}, \dots, y_n^{[i]}) = (x_1^{\eta_i}, x_2^{\eta_i}, \dots, x_n^{\eta_i})$.

Step 2. Estimate $\hat{K}^{[i]}$ and $\hat{A}^{[i]}$ as shown in Eq. (2.10).

Step 3. A K-S test is performed to test

H_0 : the transformed data \mathbf{y} follows the gamma distribution with the shape parameter $\hat{K}^{[i]}$ and the scale parameter $\hat{A}^{[i]}$

against H_a : the transformed data \mathbf{y} does not follow the gamma distribution with the shape parameter $\hat{K}^{[i]}$ and the scale parameter $\hat{A}^{[i]}$.

Let $p^{[i]}$ denote p -value of the K-S test.

Step 4. Pick η_{i^*} for $p^{[i^*]} = \max_i p^{[i]}$ and the corresponding $\hat{K}^{[i^*]}$ and $\hat{A}^{[i^*]}$. From Eq. (2.9), we can obtain an estimate of $\boldsymbol{\theta}$ as $\hat{\boldsymbol{\theta}} = \left(\left(\hat{A}^{[i^*]} \right)^{\frac{1}{\eta_{i^*}}}, \eta_{i^*}, \hat{K}^{[i^*]} \right)$ which can serve as an initial estimate of the iterative procedures.

2.5. Interval Estimation

For interval estimation, we attempt to construct confidence intervals for the parameters α , η and κ based on the asymptotic normality of the MLEs. However, in a preliminary study, we find that the asymptotic variances obtained by inverting the observed Fisher information matrix (see, for example, [Louis, 1982](#)) may be negative. As pointed out by [Molenberghs and Verbeke \(2007\)](#), the observed Fisher information matrix need not be positive definite which results in negative asymptotic variances. to obtain $100(1 - 2\lambda)\%$ ($0 < \lambda < 1/2$) percentile bootstrap confidence intervals ([Efron and Tibshirani, 1993](#)) for parameters α , η and κ based on different point estimation methods presented in Sections 2.3 – 2.4.

Suppose $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\eta}, \hat{\kappa})$ is the estimate of the parameter vector $\boldsymbol{\theta}$ based on LTRC sample, the following procedures are used to generate the bootstrap sample with truncated and non-truncated observations:

- For non-truncated observations:
 1. Fix the installation time w_i .
 2. Generate a lifetime y_i following $GG(\hat{\boldsymbol{\theta}})$.
 3. If $w_i + y_i < c^*$, the observed lifetime is $t_i = y_i$; otherwise, $t_i = c^* - w_i$.

- For left-truncated observations:
 1. Fix the installation time w_i .
 2. Generate a lifetime y_i following $GG(\hat{\theta})$.
 3. If $w_i + y_i < \tau^*$, discard y_i and go back to Step 2.
 4. If $w_i + y_i < c^*$, the observed lifetime $t_i = y_i$; otherwise, $t_i = c^* - w_i$.

Based on the m -th bootstrap sample $\mathcal{B}^{(m)}$ ($m = 1, 2, \dots, \mathcal{M}$), we obtain the vector of parameter estimates $\hat{\theta}^{(m)} = (\hat{\alpha}^{(m)}, \hat{\eta}^{(m)}, \hat{\kappa}^{(m)})$. Take the parameter α as an example, we denote the \mathcal{M} ordered parameter estimates as $\alpha^{[1]} < \alpha^{[2]} < \dots < \alpha^{[\mathcal{M}]}$, then a $100(1 - 2\lambda)\%$ percentile bootstrap confidence interval for α is $(\alpha^{[int(\mathcal{M}\lambda)]}, \alpha^{[int(\mathcal{M}(1-\lambda))]}),$ where $int(a)$ is the integer part of a .

2.6. Numerical Example

In this section, a numerical example based on LTRC data presented in [Balakrishnan and Mitra \(2014\)](#) is used to illustrate the estimation methods discussed in Sections 2.3 – 2.4 (see also, [Hong et al., 2009](#)). For this dataset, an energy company began careful archival record-keeping on their power transformers in 1980. The dataset contains complete information on all units that were installed after 1980 (i.e., $\tau^* = 1980$). Information on units that were installed before 1980 and failed after 1980 are also available. However, there is no information on units installed and failed before 1980. Thus, transformers that were installed before 1980 must be viewed as transformers sampled from left-truncated distribution(s). The lifetimes of the units are followed until 2008 (i.e., $c^* = 2008$). Units that are still in service have lifetimes that are right-censored. Hence, the data are left-truncated and right-censored. For those units that are left-truncated or right-censored (or both), the truncation times and censoring times differ from unit to unit because of the staggered entry of the units into service. The dataset has $n = 200$ observations with 61 of them are neither truncated nor censored ($\delta_i = 1, v_i = 1$), 99 of them are right-censored but not truncated ($\delta_i = 0, v_i = 1$),

40 of them are left-truncated but not censored ($\delta_i = 1, v_i = 0$), and 0 observation is LTRC ($\delta_i = 0, v_i = 0$). The percentage of censoring is 49.5%.

For the iterative estimation procedures, the tolerance limit required for convergence is taken as $\epsilon = 10^{-5}$. For the SEM algorithms, $K = 2000$ iterations is used and $B = 400$ burn-ins is used for SEMNM and SEMPL2, and $B = 1600$ burn-ins is used for SEMPL1 due to the slow convergence rate. For the initial values, we consider $(\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)}) = (15.0, 3.0, 5.0)$ and the initial estimates obtained from the procedures in Sections 2.4.2 and 2.4.3, $(\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)}) = (11.7, 2.7, 7.1)$ and $(\alpha^{(0)}, \eta^{(0)}, \kappa^{(0)}) = (25.4, 7.3, 0.6)$, respectively. The parameter estimates from the seven estimation methods in Section 3 with different initial values and the corresponding log-likelihood values are presented in Table 2.1. Since different methods have different numbers of iterations, for the sake of comparing the SEM algorithm implementations among each other and with other estimation methods, we also present the computation time for different estimation methods in Table 2.1. When calculating the percentile bootstrap confidence intervals based on the SEM algorithm, different values of K and B are used to ensure the convergence of the SEM algorithm sequences. Specifically, $K = 2000$ and $B = 400$ are used for the SEMNM method, $K = 60000$ and $B = 48000$ are used for the SEMPL1 method, and $K = 10000$ and $B = 2000$ are used for the SEMPL2 method. The bootstrap confidence intervals are computed with $\mathcal{M} = 1000$ bootstrap samples.

From Table 2.1, we observe that the direct optimization methods, the EM algorithm, and the estimation procedures based on the SEM algorithm provide similar estimates. We can see that different implementations of the SEM algorithm provide estimates that are robust to the initial values and the computation time is not depending on the initial value. In Figure 2.2, we present the SEM algorithm sequences $\{\boldsymbol{\theta}^{(k)}, k = 1, 2, \dots, 2000\}$ based on SEMNM, SEMPL1 and SEMPL2 methods. From Figure 2.2, we can see that the sequence of SEMPL1 method becomes stable much slower than the SEMNM and SEMPL2 methods.

In terms of computation time, the direct maximization method takes less computation time compared to the EM algorithm and the SEM algorithm in this example. SEMPL1 is the fastest among the three SEM algorithm implementations considered here in this example.

However, SEMPL1 method requires a much longer chain to achieve the stationary status. We also observe that the computation time of different implementations of the SEM algorithm does not depend on the initial value, while the computation time of the EM algorithm and the direct maximization methods based on profile likelihood does. In this example, the initial value based on the pseudo-uncensored sample proposed in Section 2.4.3 reduces the computation time of the EM algorithm and the direct maximization methods based on profile likelihood.

We also notice that the confidence interval for κ based on EMPL2 method varies with the initial value while the confidence intervals based on other methods are relatively stable for different initial values.

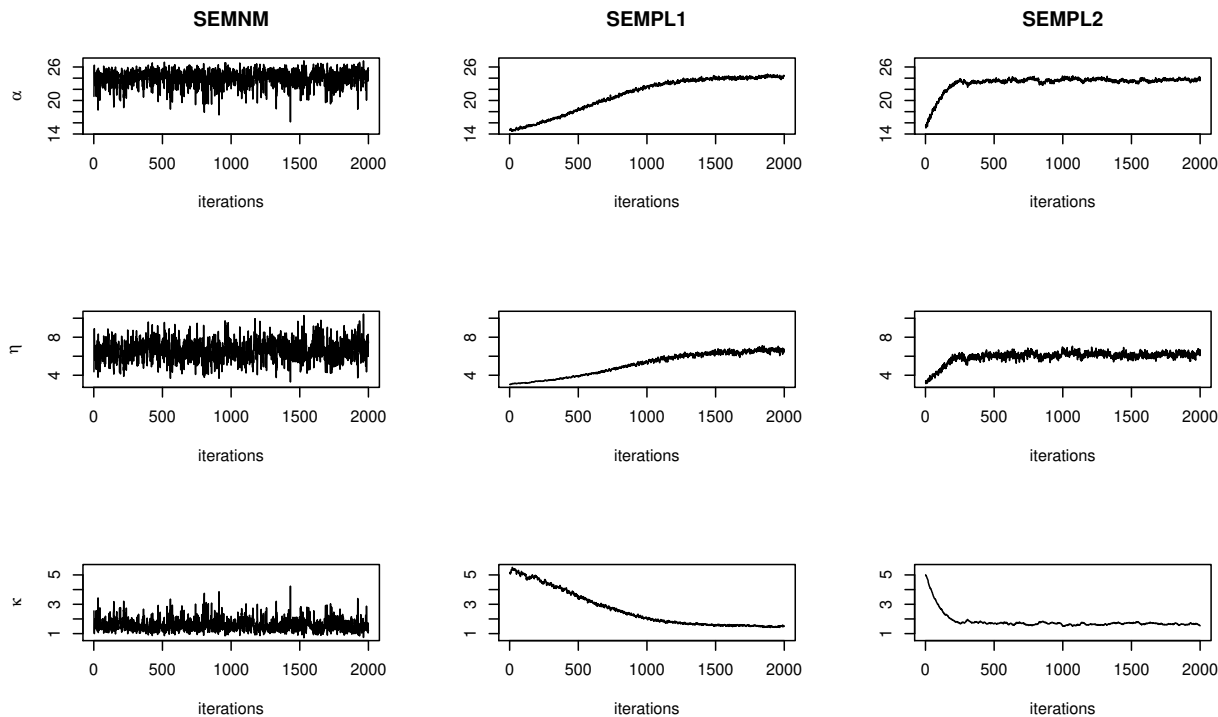


Figure 2.2: Stochastic expectation-maximization (SEM) iterative chains of SEM using Nelder-Mead method and SEM based on profile likelihood approach of two parameters methods for the numerical example

Table 2.1: Estimates of the parameters of $GG(\alpha, \eta, \kappa)$ and the corresponding 95% confidence intervals, log-likelihood values and the computation time (in seconds) for the numerical example

Initial Value	Method	$\hat{\alpha}$	$\hat{\eta}$	$\hat{\kappa}$	Log-likelihood	Time
(15.0, 3.0, 5.0)	DONM	23.87 (4.76, 28.09)	6.27 (1.71, 16.21)	1.60 (0.44, 16.97)	-296.31	0.08
	DOPL1	23.78 (9.50, 28.02)	6.21 (2.30, 14.98)	1.63 (0.48, 9.93)	-296.31	10.12
	DOPL2	23.79 (9.02, 27.99)	6.22 (2.31, 17.24)	1.62 (0.43, 10.86)	-296.31	8.86
	EMPL2	23.87 (14.86, 28.02)	6.27 (3.13, 15.92)	1.60 (0.45, 5.50)	-296.31	29.81
	SEMNM	24.03 (6.61, 27.88)	6.62 (1.82, 19.66)	1.57 (0.45, 75.26)	-296.49	28.08
	SEMPL1	24.13 (7.85, 28.33)	6.49 (2.15, 17.93)	1.52 (0.41, 11.43)	-296.31	6.44
	SEMPL2	23.63 (15.85, 28.00)	6.14 (3.37, 16.04)	1.67 (0.47, 4.64)	-296.31	15.49
	(11.7, 2.7, 7.1)	DONM	23.88 (4.61, 28.16)	6.27 (1.66, 16.55)	1.60 (0.43, 16.28)	-296.31
DOPL1		23.78 (9.49, 28.12)	6.21 (2.27, 17.68)	1.63 (0.40, 9.38)	-296.31	10.12
DOPL2		23.79 (8.55, 28.09)	6.22 (2.18, 15.17)	1.62 (0.46, 10.99)	-296.31	10.34
EMPL2		23.87 (8.49, 28.10)	6.27 (2.15, 16.59)	1.60 (0.46, 10.38)	-296.31	47.60
SEMNM		23.88 (3.70, 27.93)	6.51 (1.23, 19.31)	1.61 (0.45, 113.8)	-296.48	28.76
SEMPL1		23.85 (5.94, 28.09)	6.27 (1.91, 16.29)	1.61 (0.44, 15.59)	-296.31	6.26
SEMPL2		23.67 (16.50, 27.81)	6.16 (3.43, 14.94)	1.66 (0.51, 4.36)	-296.31	15.52
(25.4, 7.3, 0.6)		DONM	23.88 (5.25, 28.05)	6.27 (1.76, 14.66)	1.60 (0.5, 14.89)	-296.31
	DOPL1	23.96 (9.65, 28.10)	6.33 (2.33, 16.21)	1.57 (0.45, 9.32)	-296.31	3.00
	DOPL2	23.96 (8.91, 28.1)	6.33 (2.29, 16.35)	1.58 (0.43, 10.88)	-296.31	4.23
	EMPL2	23.89 (15.73, 27.89)	6.28 (3.24, 15.04)	1.59 (0.49, 4.83)	-296.31	8.92
	SEMNM	23.98 (2.99, 27.91)	6.55 (1.26, 18.97)	1.59 (0.47, 109.61)	-296.46	28.12
	SEMPL1	24.11 (6.18, 28.15)	6.47 (1.91, 18.16)	1.53 (0.39, 13.53)	-296.31	6.45
	SEMPL2	23.73 (16.32, 27.88)	6.19 (3.45, 16.91)	1.64 (0.45, 4.43)	-296.31	15.63

2.7. Monte Carlo Simulations

2.7.1. Simulation settings

In this section, a Monte Carlo simulation study is used to evaluate the performance of the estimation procedures. The datasets are simulated in a manner that mimics the data collection process for the dataset described in Section 2.6 (Balakrishnan and Mitra, 2014; Hong et al., 2009). The entire set of installation years is divided into two parts: with truncation (1960 – 1979) and without truncation (1980–1995). The sample size is set to be $n = 200$ and we consider 40 of them to be truncated. We simulate the year of installation for truncated observations ($W = 1960, 1961, \dots, 1979$) based on the following distribution:

$$\Pr(W = w) = \begin{cases} 0.15, & 1960 \leq w \leq 1964, \\ 0.25/15, & 1965 \leq w \leq 1979. \end{cases}$$

We simulate the year of installation for non-truncated observations ($W = 1980, 1981, \dots, 1995$) based on the following distribution:

$$\Pr(W = w) = \begin{cases} 0.1, & 1980 \leq w \leq 1985, \\ 0.04 & 1986 \leq w \leq 1995. \end{cases}$$

The lifetime of the power transformers (in years) are simulated from the generalized gamma distribution $GG(\alpha, \eta, \kappa)$ with $\alpha = 15$, $\eta = 3$ and $\kappa = 5$, and hence, the year of failure is the installation year plus the lifetime. If the year of failure is before $\tau^* = 1980$, then the observation is discarded and a new pair of installation year and lifetime is simulated. If the simulated year of failure exceeds the censoring point c^* , the observation is censored at c^* . Three different proportions of censoring are considered in the simulation study:

- High censoring proportion: $c^* = 2004$, the censoring proportion ranges from 64.5% to 78%.

- Medium censoring proportion: $c^* = 2008$, the censoring proportion ranges from 38% to 61.5%.
- Low censoring proportion: $c^* = 2014$, the censoring proportion ranges from 15% to 33.5%.

For each setting, we simulate 1000 LTRC data sets. The tolerance level required for convergence of the algorithms is set to be $\epsilon = 10^{-5}$. For the DONM, DOPL1, DOPL2 and EMPL2 methods, the maximum number of iterations is set to be 10000. For different implementations of the SEM algorithm, we find that the rate of convergence varies from sample to sample for the implementations SEMPL1 and SEMPL2. Therefore, we choose a sufficiently large number of iterations to ensure that the SEM chains converge. Specifically, we consider $K = 2000$ cycles with $B = 400$ burn-ins for the SEMNM method (average computation time 46.97 seconds), $K = 60000$ cycles with $B = 48000$ burn-ins for the SEMPL1 method (average computation time 230.35 seconds), and $K = 10000$ cycles and $B = 2000$ burn-ins for the SEMPL2 method (average computation time 107.56 seconds). The following three choices of initial values are considered:

- (i) the initial estimate $\boldsymbol{\theta}^{(0)} = (15, 3, 5)$, which are the true values of the parameters;
- (ii) the initial estimate obtained from the procedure described in Section 2.4.2;
- (iii) the initial estimate obtained from the procedure described in Section 2.4.3.

If the SEM algorithm fails to obtain an update due to the numerical maximization difficulties, then the imputed sample is discarded and the algorithm goes back to the S-step to impute another sample. If the SEM algorithm still fails to obtain an update after 100 re-imputations, we stop the algorithm and count it as a failure of the algorithm. If the EM algorithm fails to obtain an update due to the numerical maximization difficulties, we stop the algorithm and count it as a failure of the algorithm. Note that we have also considered other simulation settings with different proportions of truncation and different values of true parameters. Since the simulation results provide similar conclusions, we only present part of the simulation results here.

2.7.2. Results and Discussions

In Tables 2.5, 2.6, and 2.7, we present the simulated biases and mean squared errors (MSEs) based on 1000 simulations for different settings with high, medium and low censoring proportions, respectively. Due to the properties of the parameterization of the three-parameter generalized gamma distributions (see, for example, Stacy, 1962; Stacy and Mihram, 1965), there are cases where the estimates are very different, but the likelihood and the fitted pdf are quite close (Prentice, 1974). Therefore, in addition to considering biases and MSEs as the criteria to evaluate the performances of the estimation methods, we also compare the value of the log-likelihood at the final estimate for each dataset. Since the estimates obtained from the direct optimization methods usually lead to the largest likelihood value by their nature, we only compare the log-likelihood values based on the estimates obtained from the EM algorithm and different implementations of the SEM algorithm. The column “LL” in Tables 2.5, 2.6, and 2.7 stands for the proportion of the times that the estimation method gives the largest log-likelihood value among the estimates obtained from the EM algorithm and different implementations of the SEM algorithm.

From Tables 2.5, 2.6, and 2.7, we first notice that the performance of the estimation procedures that rely on solving a three-dimensional constraint optimization problem (i.e., DONM and SEMNM methods) can be poor, especially for the estimation of parameter κ . These results are due to some extreme estimates, where the estimate of α is very close to its boundary (i.e., 0) and the corresponding estimate of κ is a very large value. It is noteworthy that although the estimates of the parameters are extreme, the fitted pdfs and the likelihood values might still be similar. To illustrate this issue, we plotted the fitted pdfs and present the values of the log-likelihood for one of the simulated data sets with extreme parameter estimates with the true pdf (i.e., the pdf of $GG(15, 3, 5)$) in Figure 2.3. From Figure 2.3, we can see that the fitted pdfs and the values of the log-likelihood are similar even when the parameter estimates are very different. This example also illustrates the difficulties in estimating the parameters for the generalized gamma distribution.

From Tables 2.5, 2.6, and 2.7, we also observe that the performance of the direct optimization methods and different implementations of the SEM algorithm are consistent when different initial values are used, while the performance of the EM algorithm depends on the initial values. This observation also indicates that the proposed methods for obtaining initial values for the iterative procedures are quite reliable. In all the settings considered here, the SEMPL2 method provides estimates with the smallest simulated biases and MSEs in most cases. From the likelihood values based on the estimates obtained from the EM algorithm and different implementations of the SEM algorithm, we observe that the SEMPL1 method has higher proportions than the EM algorithm and other SEM implementations in obtaining the largest likelihood value. To further investigate the performance of the EM and SEM algorithms, we report the proportions of times that the numerical optimization fails in the EM or SEM iterations in Table 2.4. From Table 2.4, once again, we observe that the SEMPL1 and SEMPL2 are more reliable alternatives to the EM algorithm.

Based on the simulation results, both the direct optimization method using the three-stage profile likelihood approach (DOPL1) and the proposed SEM algorithm based on the two-stage profile likelihood approach (SEMP2) are superior to other methods considered here for estimating the parameters of the generalized gamma distribution based on LTRC data. The results indicate that the SEM algorithm is a useful alternative to the direct optimization method. Since the SEM algorithm provides an approximate MLE instead of an exact MLE, we recommend using both DOPL1 and SEMPL2 methods in practice in order to verify the estimation results.

2.8. Concluding Remarks

In this chapter, we study the parameter estimation of the generalized gamma distribution based on LTRC data. We show that some commonly used iterative procedures, such as the direct optimization method based on Nelder-Mead algorithm and the EM algorithm, can yield extreme estimates. Therefore, we propose the stochastic EM algorithm as an alternative way to estimate the parameters of the generalized gamma distribution. We also

propose two different ways to obtain reliable initial values for the iterative procedures for obtaining the estimates. The proposed estimation procedures are illustrated by a numerical example and they are compared with the direct optimization method and the EM algorithm using a Monte Carlo simulation study.

Overall speaking, we recommend the use of the SEM algorithm with a two-stage profile likelihood method in the M-step due to its robustness and stability. However, since the SEM algorithm provides an approximate MLE instead of an exact MLE, we recommend using both DOPL1 and SEMPL2 methods in practice in order to verify the estimation results.

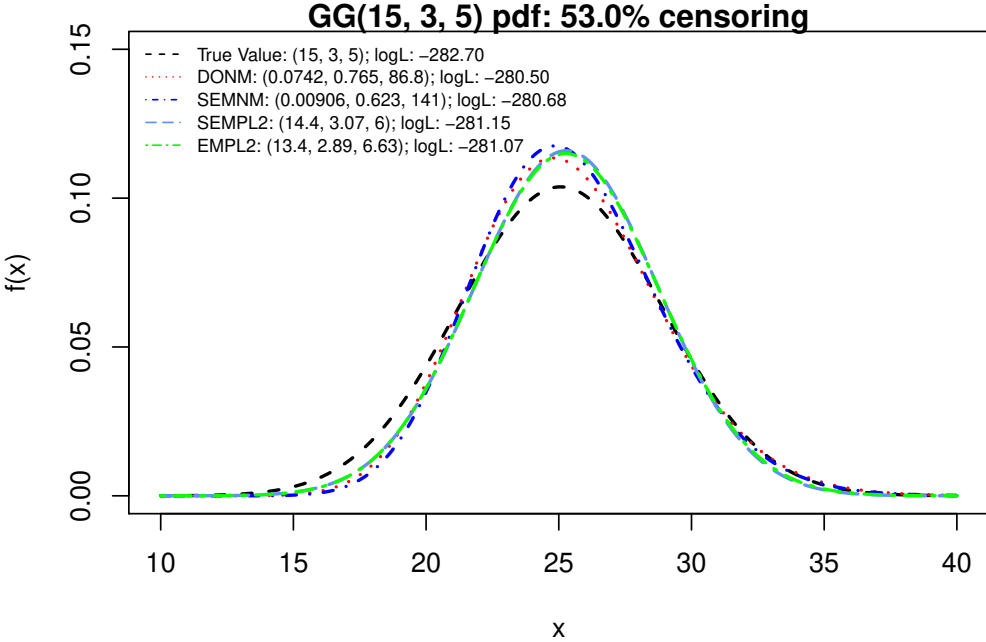


Figure 2.3: Fitted pdfs based on estimates obtained from different methods

Table 2.3: Proportion of times that $\hat{\alpha} < 0.1$ for the direct optimization methods

censoring	method	initial (i)	initial (ii)	initial (iii)
HIGH	DONM	9.3%	9.2%	9.0%
	DOPL1	0%	0%	0%
	DOPL2	0%	0%	0%
MED	DONM	8.9%	8.9%	8.8%
	DOPL1	0%	0%	0%
	DOPL2	0%	0%	0%
LOW	DONM	4.8%	4.9%	4.8%
	DOPL1	0%	0%	0%
	DOPL2	0%	0%	0%

Table 2.4: Proportion of times that the numerical optimization fails

censoring	method	initial (i)	initial (ii)	initial (iii)
HIGH	EMPL2	0.4%	0.6%	0.1%
	SEMNM	3.6%	3.2%	3.4%
	SEMP1	0%	0%	0%
	SEMP2	0.1%	0.1%	0.2%
MED	EMPL2	0.1%	0.3%	0.3%
	SEMNM	3.8%	3.2%	3.5%
	SEMP1	0%	0%	0%
	SEMP2	0.2%	0.2%	0.2%
LOW	EMPL2	1.6%	2.4%	2.8%
	SEMNM	1.6%	1.9%	1.9%
	SEMP1	0%	0%	0%
	SEMP2	2.0%	1.9%	1.8%

Table 2.5: $GG(15, 3, 5)$ HIGH Censoring Percentage: 64.5%-78%, mean=71.0%, median=71%

method	initial (i)						initial (ii)						initial (iii)							
	$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$			
	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	prop	
DONM	-0.71	82.7	0.69	7.46	14.3	1478	-0.71	82.7	0.69	7.46	14.3	1497	-0.71	82.7	0.69	7.46	14.0	1424	-	
DOPL1	0.41	58.2	0.89	6.83	1.79	38.2	-	0.06	61.9	0.84	6.89	2.21	44.4	0.50	57.7	0.90	6.82	1.7	36.9	-
DOPL2	0.29	58.5	0.83	6.52	2.29	58.8	-	0.06	62.0	0.82	6.78	2.52	60.3	0.77	62.0	0.98	7.11	2.09	59.0	-
EMPL2	1.01	21.0	0.64	4.2	-0.048	5.16	27.1%	-0.29	46.2	0.63	6.0	1.39	15.6	5.14	49.0	1.8	7.65	-1.72	12.4	27.7%
SEMINM	-0.20	78.9	1.22	12.4	26.4	3169	0%	-0.29	80.2	1.18	11.1	27.0	3223	-0.29	80.0	1.19	12.2	27.4	3299	0%
SEMPL1	1.14	68.4	1.26	10.7	2.1	62.9	50.6%	1.04	70.0	1.24	10.4	2.31	68.4	1.16	68.1	1.26	10.8	2.06	61.7	63.6%
SEMPL2	4.5	36.6	1.58	6.9	-1.69	5.26	22.3%	4.5	36.7	1.58	6.93	-1.69	5.28	4.49	36.6	1.57	6.72	-1.69	5.26	8.7%

Table 2.6: $GG(15, 3, 5)$ MED Censoring Percentage: 38%-61.5%, mean=52.4%, median=52.5%

method	initial (i)						initial (ii)						initial (iii)							
	$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$			
	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	prop	
DONM	-1.16	77.5	0.50	9.31	13.4	1295	-	-1.16	77.5	0.50	9.29	13.3	1288	-1.16	77.5	0.50	9.3	13.4	1314	-
DOPL1	0.18	49.6	0.71	8.22	1.5	28.5	-	-0.19	53.6	0.66	8.3	1.91	33.6	0.27	49.0	0.72	8.26	1.41	27.4	-
DOPL2	-0.03	51.9	0.64	7.33	2.2	53.1	-	-0.61	57.2	0.57	7.39	2.66	54.9	0.42	55.9	0.78	7.6	2.05	54.2	-
EMPL2	0.69	23.0	0.55	4.97	0.18	6.76	26.1%	-1.84	45.8	0.29	7.53	2.3	21.7	3.91	41.7	1.38	7.25	-1.19	9.47	28.9%
SEMINM	-0.18	69.8	0.94	8.01	21.1	2467	1.6%	-0.38	72.3	0.88	7.76	23.0	2708	-0.30	71.9	0.91	8.01	22.5	2675	1.5%
SEMPL1	0.29	65.7	0.93	14.0	2.79	75.9	49.5%	0.25	66.6	0.93	14.2	2.91	79.8	0.31	65.5	0.94	14.4	2.75	74.9	59.4%
SEMPL2	3.13	29.0	1.12	5.22	-1.09	4.75	22.8%	3.13	29.0	1.12	5.22	-1.09	4.75	3.13	29.0	1.12	5.03	-1.09	4.76	10.2%

Table 2.7: $GG(15, 3, 5)$ LOW Censoring Percentage: 15%-33.5%, mean=23.1%, median=23%

method	initial (i)						initial (ii)						initial (iii)							
	$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$		$\hat{\alpha}$		$\hat{\eta}$		$\hat{\kappa}$			
	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	bias	mse	prop	
DONM	-1.50	62.1	0.08	2.32	9.58	791	-	-1.50	62.0	0.09	2.32	9.69	819	-1.50	62.1	0.09	2.32	9.54	781	-
DOPL1	-0.30	38.3	0.27	1.79	1.43	22.0	-	-0.43	40.0	0.25	1.82	1.6	24.3	-0.54	41.0	0.24	1.84	1.7	25.2	-
DOPL2	-0.49	40.5	0.22	1.74	2.02	41.3	-	-0.49	44.4	0.25	1.92	2.15	43.3	-0.69	45.0	0.22	1.91	2.28	43.6	-
EMPL2	0.47	16.8	0.27	1.08	0.27	12.5	12.0%	1.07	32.1	0.51	1.92	0.22	9.81	0.15	32.4	0.33	1.75	0.80	12.9	15.2%
SEMINM	-0.96	56.5	0.26	2.58	13.2	1357	2.6%	-0.93	56.2	0.26	2.58	13.1	1353	-0.92	56.0	0.26	2.58	13.1	1354	2.6%
SEMPL1	-0.88	54.6	0.23	2.22	3.39	82.4	74.6%	-0.88	54.8	0.23	2.23	3.42	83.7	-0.89	54.8	0.22	2.22	3.43	84.2	71.1%
SEMPL2	1.1	23.1	0.44	1.44	-0.03	6.27	10.8%	1.09	23.2	0.44	1.44	-0.01	6.71	1.07	23.3	0.44	1.44	0.01	6.72	11.1%

Chapter 3

On Reliability Analysis of One-Shot Devices Testing Data with Defectives Based on Gamma Distribution

3.1. Introduction

Reliability analysis of one-shot devices testing data has attracted increasing attention in the past decades. For example, [Fan et al. \(2009\)](#) presented a Bayesian approach for inferring the reliability at a mission time and the mean lifetime of electro-explosive devices under an exponential lifetime distribution based on samples collected from constant-stress accelerated life-tests. In a series of papers, [Balakrishnan and Ling \(2012, 2013, 2014b\)](#) and [Ling et al. \(2015\)](#) presented the maximum likelihood estimation method as well as different confidence intervals under the exponential, Weibull and gamma distributions as well as under the proportional hazards model. Subsequently, [Balakrishnan et al. \(2019a,b,c, 2020\)](#) presented the weighted minimum density power divergence estimators for one-shot device testing data in the presence of outliers under the exponential, Weibull and gamma distributions. [Balakrishnan et al. \(2015, 2016a,b\)](#) considered competing-risk models to analyze one-shot device testing data when the devices contain multiple components and have multiple failure modes. Apart from the works on statistical estimation, some other important aspects in reliability analysis have also been discussed. [Pan and Chu \(2010\)](#) investigated two- and three-stage inspection schemes for assessing one-shot devices in series systems with components following Weibull lifetime distributions. Recently, [Cheng and Elsayed \(2015, 2017, 2018\)](#) examined different approaches to measure the reliability of one-shot devices with a mixture of units under various scenarios and presented reliability metrics of systems with mixtures of nonhomogeneous one-shot units subject to thermal cyclic stresses and further optimal operational use

of such systems. [Balakrishnan and Ling \(2014a\)](#) discussed optimal designs of constant-stress accelerated life-tests for one-shot devices in the presence of budget and time constraints. A recently published book by [Balakrishnan et al. \(2021\)](#) provides a review of data collection, experimental designs, and statistical analysis of one-shot devices.

In a realistic situation of the manufacturing industry, defectives (i.e., devices that fail to function normally or satisfactorily at time $t = 0$, also known as nonconformities) could be produced in the manufacturing process due to different reasons. For instance, [Singh and Nandula \(2020\)](#) pointed out that the root cause for defects in one-shot devices is due to human error, insufficient quality process, inadequate training and failure to address reliability aspects during the design stage, etc. It has been suggested that zero-defect is an impossible goal to achieve or cost-prohibitive ([Raina, 2008](#)). As [Abilia \(2007\)](#) mentioned, zero defects means a defect level of infinity sigma (σ) statistically, which is not possible. Considering defectives in reliability analysis of one-shot devices poses a unique challenge due to the special feature of one-shot devices. Specifically, unlike non-destructive testing of products with continuous monitoring (e.g., batteries and light bulbs), defective one-shot devices will not be detected until the time of usage or testing. Moreover, in the presence of defectives, if a one-shot device does not work at the time of testing, we may not be able to distinguish if the particular device is a defective or a device with a lifetime smaller than the testing time. The model considering defectives in reliability and survival analysis is also known as the infant mortality model, infancy problem model, or birth-death model.

Considering one-shot devices with defectives, [Bain and Engelhardt \(1991\)](#) proposed a mixed Weibull model for the development of reliability test plans for one-shot devices. In this paper, we develop the maximum likelihood approach and the Bayesian approach to estimate the model parameters of gamma distribution and the reliability characteristics based on a sample of one-shot devices in the presence of defectives. [Guikema and Paté-Cornell \(2005\)](#) studied the infancy problems for space launch vehicles and proposed a Bayesian approach to estimate the system reliability. Although the defectives of one-shot devices can be considered as a competing risk, the competing risk model based on continuous distribution cannot be

directly applied here since the failure time distribution of the defectives is a degenerate distribution with probability mass function that equals 1 at time $t = 0$ and 0 elsewhere. Therefore, special treatments for the problem of defectives in one-shot devices are needed.

In this chapter, we aim to provide a comprehensive study of the effect of the presence of defectives in one-shot device testing and reliability analysis. Based on the simulation results, we provide some practical guidelines on product defect tolerances in terms of the effect in statistical analysis. This chapter is organized as follows. In Section 3.2, we describe the data structure and the notations used in this chapter. In Section 3.3, we discuss the maximum likelihood approach and the Bayesian approach for point and interval estimation. In Section 3.4, we illustrate the estimation procedures when the lifetimes of one-shot devices follow a gamma distribution and present a numerical example. In Section 3.5, a Monte Carlo simulation study is used to evaluate the performances of the parameter estimation procedures considered here under different defective proportions and masking rates. Finally, some concluding remarks are provided in Section 3.6.

3.2. Model and Different Scenarios

3.2.1. Model and Notations

Suppose n one-shot devices are available for the life testing experiment, the i -th unit is tested at the inspection time t_i ($i = 1, 2, \dots, n$). Let X_i be the random variable of the lifetime of the i -th unit and x_i be the corresponding realization of the lifetime of the i -th unit ($i = 1, 2, \dots, n$). If the i -th unit fails the test at time t_i , we have $x_i < t_i$, i.e., a left-censored observation. If the i -th unit is functioning at inspection time t_i , we have $x_i \geq t_i$, i.e., a right-censored observation. Let δ_i be the indicator to denote if the i -th unit fails the test or not, i.e.,

$$\delta_i = \begin{cases} 1, & x_i < t_i, \\ 0, & x_i \geq t_i. \end{cases}$$

Thus, the observed data are (t_i, δ_i) , $i = 1, 2, \dots, n$.

Consider that there are defectives in the one-shot devices of interest, we define p as the probability that a randomly selected one-shot device is a defective. Hence, the lifetime X_i has a probability p to be 0 and a probability of $1 - p$ to follow an absolutely continuous lifetime distribution with the cumulative distribution function $F_X(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a k -dimensional vector of parameters. In other words, if the i -th unit is not a defective, then X_i follows $F_X(x; \boldsymbol{\theta})$, and we have

$$\begin{aligned} \Pr(X_i < t_i) &= p + (1 - p)F_X(t_i; \boldsymbol{\theta}) \\ \text{and } \Pr(X_i \geq t_i) &= (1 - p)[1 - F_X(t_i; \boldsymbol{\theta})], \end{aligned}$$

which are the likelihood for left-censored and right-censored units, respectively. We consider three different scenarios regarding our knowledge on p in Sections 3.2.2, 3.2.3, and 3.2.4.

3.2.2. Scenario 1: Not aware of the existence of defectives

In Scenario 1, we consider that the experimenter or data analyst is not aware of the existence of the defectives. In other words, we assume that $p = 0$ in the statistical analysis. Based on the observed data (t_i, δ_i) , $i = 1, 2, \dots, n$, the likelihood and log-likelihood functions are

$$\begin{aligned} L_1(\boldsymbol{\theta}) &= \prod_{i=1}^n F_X(t_i; \boldsymbol{\theta})^{\delta_i} [1 - F_X(t_i; \boldsymbol{\theta})]^{1-\delta_i} \\ \text{and } \ln L_1(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i \ln(F_X(t_i; \boldsymbol{\theta})) + \sum_{i=1}^n (1 - \delta_i) \ln(1 - F_X(t_i; \boldsymbol{\theta})), \end{aligned} \quad (3.1)$$

respectively.

3.2.3. Scenario 2: True proportion of defectives is known

In Scenario 2, we consider that we are aware of the fact that there are defectives. We assume that the true probability that a randomly selected unit is a defective is known. In

other words, we have $p = p_0$, where p_0 is the true value for p . We also assume that we can determine whether a failed device is a defective or not after a careful investigation or autopsy of the unit. In other words, we consider the case with masking, whereas the true causes of failures (i.e., defectives or not) are unknown for some units. For the masking, we denote m_i as the indicator for whether a unit is masked, i.e.,

$$m_i = \begin{cases} 1, & \text{the } i\text{-th unit is masked,} \\ 0, & \text{the } i\text{-th unit is not masked.} \end{cases}$$

For those units which are still functioning at the time of testing (i.e., $\delta_i = 0$), the value of m_i is defined as 0 because those units are not defectives. Denote ν_i as the indicator for whether a unit is defective, i.e.,

$$\nu_i = \begin{cases} 1, & x_i = 0, \\ 0, & x_i > 0. \end{cases}$$

For Scenario 2, the likelihood and log-likelihood functions are

$$\begin{aligned} L_2(\boldsymbol{\theta}) &= \prod_{i=1}^n \left\{ [p_0 + (1 - p_0)F_X(t_i; \boldsymbol{\theta})]^{m_i} [p_0^{\nu_i} [(1 - p_0)F_X(t_i; \boldsymbol{\theta})]^{1-\nu_i}]^{1-m_i} \right\}^{\delta_i} \\ &\quad [(1 - p_0)(1 - F_X(t_i; \boldsymbol{\theta}))]^{(1-\delta_i)} \\ \text{and } \ln L_2(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i m_i \ln(p_0 + (1 - p_0)F_X(t_i; \boldsymbol{\theta})) + \sum_{i=1}^n \delta_i (1 - m_i) \nu_i \ln p_0 \\ &\quad + \sum_{i=1}^n \delta_i (1 - m_i) (1 - \nu_i) [\ln(1 - p_0) + \ln F_X(t_i; \boldsymbol{\theta})] \\ &\quad + \sum_{i=1}^n (1 - \delta_i) [\ln(1 - p_0) + \ln(1 - F_X(t_i; \boldsymbol{\theta}))], \end{aligned} \tag{3.2}$$

respectively.

3.2.4. Scenario 3: Aware of defectives but the true proportion of defectives is unknown

In Scenario 3, similar to Scenario 2, we consider that we are aware of the fact that there are defectives, but the true value of p is unknown. For Scenario 3 with masking, the likelihood and log-likelihood functions are

$$L_{3p}(\boldsymbol{\theta}, p) = \prod_{i=1}^n \left\{ [p + (1-p)F_X(t_i; \boldsymbol{\theta})]^{m_i} [p^{\nu_i} [(1-p)F_X(t_i; \boldsymbol{\theta})]^{1-\nu_i}]^{1-m_i} \right\}^{\delta_i} [(1-p)(1-F_X(t_i; \boldsymbol{\theta}))]^{(1-\delta_i)}$$

and $\ln L_{3p}(\boldsymbol{\theta}, p) = \sum_{i=1}^n \delta_i m_i \ln(p + (1-p)F_X(t_i; \boldsymbol{\theta}))$

$$+ \sum_{i=1}^n \delta_i (1-m_i) \nu_i \ln p$$

$$+ \sum_{i=1}^n \delta_i (1-m_i) (1-\nu_i) [\ln(1-p) + \ln F_X(t_i; \boldsymbol{\theta})]$$

$$+ \sum_{i=1}^n (1-\delta_i) [\ln(1-p) + \ln(1-F_X(t_i; \boldsymbol{\theta}))],$$

respectively. Note that Scenario 1 and Scenario 2 can be considered as special cases of Scenario 3 by setting $p = 0$ and $p = p_0$, respectively.

In our preliminary study, we find that the maximum likelihood estimation method is not numerically stable in some cases, especially when the estimate of p is extremely small. This may be due to the fact that p is a parameter in a bounded range between 0 and 1. Therefore, instead of considering the parameter p in the statistical analysis, we can consider a reparameterization based on the logit transformation, i.e.,

$$\phi = \text{logit}(p) = \ln \left(\frac{p}{1-p} \right)$$

or equivalently

$$p = \text{logit}^{-1}(\phi) = \frac{1}{1 + e^{-\phi}}.$$

Since $\phi \in (-\infty, \infty)$, the reparameterization based on logit transformation provides better numerical stability and some advantages in the maximum likelihood estimation method. The likelihood and log-likelihood function can be expressed in terms of ϕ as

$$\begin{aligned}
L_3(\boldsymbol{\theta}, \phi) &= \prod_{i=1}^n \{ [\text{logit}^{-1}(\phi) + (1 - \text{logit}^{-1}(\phi))F_X(t_i; \boldsymbol{\theta})]^{m_i} \\
&\quad [\text{logit}^{-1}(\phi)^{\nu_i} [(1 - \text{logit}^{-1}(\phi))F_X(t_i; \boldsymbol{\theta})]^{1-\nu_i}]^{1-m_i} \}^{\delta_i} \\
&\quad [(1 - \text{logit}^{-1}(\phi))(1 - F_X(t_i; \boldsymbol{\theta}))]^{(1-\delta_i)} \\
\text{and } \ln L_3(\boldsymbol{\theta}, \phi) &= \sum_{i=1}^n \delta_i m_i \ln (\text{logit}^{-1}(\phi) + (1 - \text{logit}^{-1}(\phi))F_X(t_i; \boldsymbol{\theta})) \\
&\quad + \sum_{i=1}^n \delta_i (1 - m_i) \nu_i \ln (\text{logit}^{-1}(\phi)) \\
&\quad + \sum_{i=1}^n \delta_i (1 - m_i) (1 - \nu_i) [\ln (1 - \text{logit}^{-1}(\phi)) + \ln F_X(t_i; \boldsymbol{\theta})] \\
&\quad + \sum_{i=1}^n (1 - \delta_i) [\ln (1 - \text{logit}^{-1}(\phi)) + \ln (1 - F_X(t_i; \boldsymbol{\theta}))], \tag{3.3}
\end{aligned}$$

respectively.

To incorporate the masking of failed units described in Section 3.2.3 in the statistical analysis, we let w be the proportion of masking of failed units. To facilitate the subsequent discussions, we consider the following three cases under Scenario 2 and Scenario 3 for different values of w :

1. No masking: This is the case that when a one-shot device failed at the inspection time, one can distinguish whether the unit is defective or not for all units, i.e., $w = 0$.
2. Partially masked: This is the case that when a one-shot device failed at the inspection time, one can distinguish whether the unit is defective or not for at least one unit but not for all units, i.e., $0 < w < 1$.
3. Completely masked: This is the case that when a one-shot device fails at the inspection time, one cannot distinguish whether the unit is defective or not, i.e., $w = 1$.

The flowchart presented in Figure 3.1 summarizes the scenarios considered in this chapter.

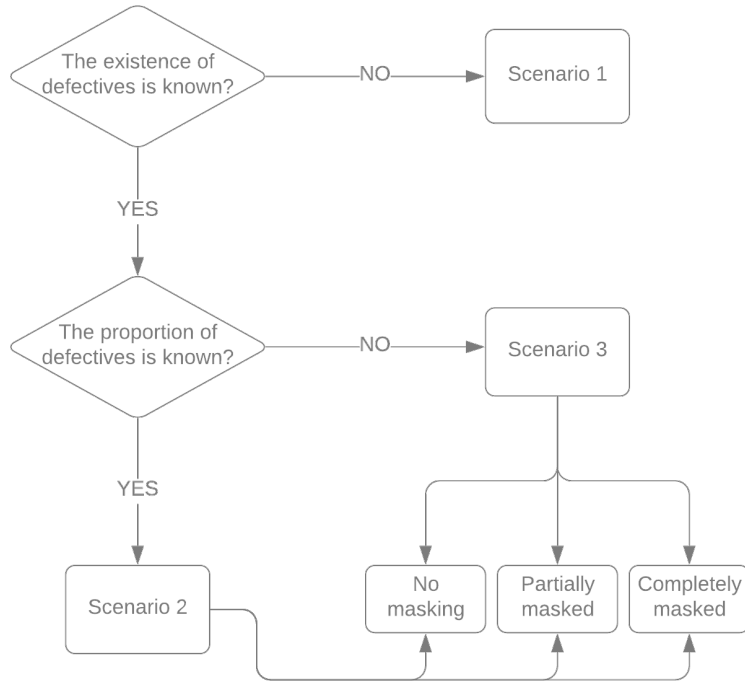


Figure 3.1: Summary of different scenarios considered in this thesis

3.3. Point and Interval Parameter Estimation

In this section, we discuss the point and interval estimation of model parameters under different scenarios presented in Section 3.2. Maximum likelihood estimation method and Bayesian estimation method are considered.

3.3.1. Maximum likelihood estimation

3.3.1.1. Point estimation

To obtain the maximum likelihood estimate (MLE) of the parameter vector $\boldsymbol{\theta}$ in Scenarios 1–3 and the proportion of defectives in the population (p) in Scenario 3, we can maximize the corresponding log-likelihood function directly using the Nelder-Mead method. Specifically, we maximize $\ln L_1(\boldsymbol{\theta})$ in Eq. (3.1) with respect to $\boldsymbol{\theta}$ for Scenario 1, maximize $\ln L_2(\boldsymbol{\theta})$ in Eq.

(3.2) with respect to $\boldsymbol{\theta}$ for Scenario 2, and maximize $\ln L_3(\boldsymbol{\theta}, \phi)$ in Eq. (3.3) with respect to $\boldsymbol{\theta}$ and ϕ for Scenario 3. Let $\hat{\boldsymbol{\theta}}$ and $\hat{\phi}$ be the MLEs of $\boldsymbol{\theta}$ and ϕ , respectively. The MLE of p can be obtained as $\hat{p} = \text{logit}^{-1}(\hat{\phi})$.

3.3.1.2. Interval estimation

For the interval estimation based on MLEs, we consider the normal approximated confidence intervals based on the asymptotic properties of the MLEs introduced in Section 1.4.2. We consider the log-likelihood functions in Eq. (3.1)–(3.3) for Scenarios 1–3 and obtain the observed Fisher information matrix defined in Eq. (1.5) as

$$\hat{I}_1(\hat{\boldsymbol{\theta}}) = -\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_1(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$$

and $\hat{I}_2(\hat{\boldsymbol{\theta}}) = -\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_2(\boldsymbol{\theta}) \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$,

for Scenario 1 and 2, respectively; and

$$\hat{I}_3(\hat{\boldsymbol{\theta}}, \hat{\phi}) = \left(\begin{array}{cc} -\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_3(\boldsymbol{\theta}) & \\ -\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \phi} \ln L_3(\boldsymbol{\theta}, \phi) & -\frac{\partial^2}{\partial \phi^2} \ln L_3(\boldsymbol{\theta}, \phi) \end{array} \right) \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}, \phi=\hat{\phi}},$$

for Scenario 3. Here, $\hat{I}_1(\hat{\boldsymbol{\theta}})$ and $\hat{I}_2(\hat{\boldsymbol{\theta}})$ are $k \times k$ matrices, and $\hat{I}_3(\hat{\boldsymbol{\theta}}, \hat{\phi})$ is a $(k+1) \times (k+1)$ matrix, where k is the number of parameters in the parameter vector $\boldsymbol{\theta}$ (i.e., the dimension of $\boldsymbol{\theta}$). Then, we can obtain the asymptotic variance-covariance matrix of the MLEs of the model parameters (denoted as $\hat{\boldsymbol{\Sigma}}$) by inverting the corresponding observed Fisher information matrix. Specifically, we have

$$\hat{\boldsymbol{\Sigma}}_1(\hat{\boldsymbol{\theta}}) = \hat{I}_1^{-1}(\hat{\boldsymbol{\theta}}), \tag{3.4}$$

$$\hat{\boldsymbol{\Sigma}}_2(\hat{\boldsymbol{\theta}}) = \hat{I}_2^{-1}(\hat{\boldsymbol{\theta}}), \tag{3.5}$$

$$\text{and } \hat{\boldsymbol{\Sigma}}_3(\hat{\boldsymbol{\theta}}, \hat{\phi}) = \hat{I}_3^{-1}(\hat{\boldsymbol{\theta}}, \hat{\phi}) = \begin{pmatrix} \hat{\boldsymbol{\Sigma}}_{3,11}(\hat{\boldsymbol{\theta}}, \hat{\phi}) \\ \hat{\boldsymbol{\Sigma}}_{3,21}(\hat{\boldsymbol{\theta}}, \hat{\phi}) & \hat{\boldsymbol{\Sigma}}_{3,22}(\hat{\boldsymbol{\theta}}, \hat{\phi}) \end{pmatrix}, \tag{3.6}$$

where $\hat{\Sigma}_1(\hat{\boldsymbol{\theta}})$, $\hat{\Sigma}_2(\hat{\boldsymbol{\theta}})$, and $\hat{\Sigma}_{3,11}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ are $k \times k$ matrices containing the asymptotic variance-covariance of $\hat{\boldsymbol{\theta}}$, $\hat{\Sigma}_{3,21}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ is a $1 \times k$ vector containing the asymptotic covariance of $\hat{\boldsymbol{\theta}}$ and $\hat{\phi}$, and $\hat{\Sigma}_{3,22}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ is a scalar containing the asymptotic variance of $\hat{\phi}$,

The relevant second-order derivatives of the log-likelihood functions under different scenarios can be obtained as follows.

- Scenario 1:

$$\begin{aligned} \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_1(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln [F_X(t_i; \boldsymbol{\theta})] \\ &\quad + \sum_{i=1}^n (1 - \delta_i) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln [1 - F_X(t_i; \boldsymbol{\theta})]. \end{aligned}$$

- Scenario 2:

$$\begin{aligned} &\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_2(\boldsymbol{\theta}) \\ &= \sum_{i=1}^n \delta_i m_i \frac{(1 - p_0)(p_0 + (1 - p_0)F_X(t_i; \boldsymbol{\theta})) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} F_X(t_i; \boldsymbol{\theta}) - (1 - p_0)^2 (\frac{\partial}{\partial \boldsymbol{\theta}} F_X(t_i; \boldsymbol{\theta}))^2}{(p_0 + (1 - p_0)F_X(t_i; \boldsymbol{\theta}))^2} \\ &\quad + \sum_{i=1}^n \delta_i (1 - m_i)(1 - \nu_i) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln F_X(t_i; \boldsymbol{\theta}) \\ &\quad + \sum_{i=1}^n (1 - \delta_i) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln (1 - F_X(t_i; \boldsymbol{\theta})). \end{aligned}$$

- Scenario 3 with parameter p :

$$\begin{aligned} &\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_{3p}(\boldsymbol{\theta}, p) \\ &= \sum_{i=1}^n \delta_i m_i \frac{(1 - p)(p + (1 - p)F_X(t_i; \boldsymbol{\theta})) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} F_X(t_i; \boldsymbol{\theta}) - (1 - p)^2 (\frac{\partial}{\partial \boldsymbol{\theta}} F_X(t_i; \boldsymbol{\theta}))^2}{(p + (1 - p)F_X(t_i; \boldsymbol{\theta}))^2} \\ &\quad + \sum_{i=1}^n \delta_i (1 - m_i)(1 - \nu_i) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln F_X(t_i; \boldsymbol{\theta}) \\ &\quad + \sum_{i=1}^n (1 - \delta_i) \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln (1 - F_X(t_i; \boldsymbol{\theta})), \end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial \boldsymbol{\theta} \partial p} \ln L_{3p}(\boldsymbol{\theta}, p) &= \sum_{i=1}^n -\delta_i m_i \frac{\frac{\partial}{\partial \boldsymbol{\theta}} F_X(t_i; \boldsymbol{\theta})}{(p + (1-p)F_X(t_i; \boldsymbol{\theta}))^2}, \\ \frac{\partial^2}{\partial p^2} \ln L_{3p}(\boldsymbol{\theta}, p) &= \sum_{i=1}^n -\delta_i m_i \frac{(1 - F_X(t_i; \boldsymbol{\theta}))^2}{(p + (1-p)F_X(t_i; \boldsymbol{\theta}))^2} \\ &\quad + \sum_{i=1}^n -\delta_i (1 - m_i) \nu_i \left(\frac{1}{p^2}\right) \\ &\quad + \sum_{i=1}^n -\delta_i (1 - m_i) (1 - \nu_i) \frac{1}{(1-p)^2} \\ &\quad + \sum_{i=1}^n -(1 - \delta_i) \frac{1}{(1-p)^2}.\end{aligned}$$

- Scenario 3 with the reparameterization ϕ :

$$\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_3(\boldsymbol{\theta}, \phi) = \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \ln L_{3p}(\boldsymbol{\theta}, p),$$

$$\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \phi} \ln L_3(\boldsymbol{\theta}, \phi) = \frac{\partial^2 \ln L_{3p}(\boldsymbol{\theta}, p)}{\partial \boldsymbol{\theta} \partial p} \frac{dp}{d\phi},$$

$$\frac{\partial^2}{\partial \phi^2} \ln L_3(\boldsymbol{\theta}, \phi) = \frac{\partial^2 \ln L_{3p}(\boldsymbol{\theta}, p)}{\partial p^2} \left(\frac{dp}{d\phi}\right)^2 + \frac{\partial \ln L_{3p}(\boldsymbol{\theta}, p)}{\partial p} \frac{d^2 p}{d\phi^2},$$

where $\frac{dp}{d\phi} = \frac{e^\phi}{(1+e^\phi)^2}$ and $\frac{d^2 p}{d\phi^2} = \frac{e^\phi - e^{2\phi}}{(1+e^\phi)^3}$.

In reliability analysis, the focus is not usually on the estimation of the model parameters but on the estimation of some reliability indices such as the mean time to failure (*MTTF*), the quantiles of the lifetime distribution and reliability function (i.e., probability that a randomly selected unit survives until a specific time t). Here, for illustrative purposes, we focus on estimating *MTTF* of the non-defective units, denoted as

$$MTTF = \mathbb{E}(X) = g(\boldsymbol{\theta}),$$

where X is the random variable of the lifetime of non-defective units, and $g(\cdot)$ is a real function. Hence, the MLE of $MTTF$ can be obtained as $\widehat{MTTF} = g(\hat{\boldsymbol{\theta}})$. Note that the methodologies described below can be easily modified for the other reliability indices since those indices can also be expressed as functions of the model parameters.

Based on the asymptotic properties of the MLE, \widehat{MTTF} is asymptotically distributed as normal with mean $MTTF$ and variance $Var(\widehat{MTTF})$. Since the asymptotic variance-covariance matrix of the model parameters can be obtained, we can apply the delta method (see, for example, [Meeker and Escobar, 1998](#)) to obtain an estimate of the asymptotic variance of \widehat{MTTF} , i.e.,

$$\widehat{Var}(\widehat{MTTF}) = \nabla g(\hat{\boldsymbol{\theta}})^T \hat{\boldsymbol{\Sigma}} \nabla g(\hat{\boldsymbol{\theta}}),$$

where $\hat{\boldsymbol{\Sigma}}$ is $\hat{\boldsymbol{\Sigma}}_1(\hat{\boldsymbol{\theta}})$ and $\hat{\boldsymbol{\Sigma}}_2(\hat{\boldsymbol{\theta}})$ defined in Eqs. (3.4) and (3.5) for Scenario 1 and 2, respectively, and $\hat{\boldsymbol{\Sigma}}_{3,11}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ defined in Eq. (3.6) for Scenario 3. Since $MTTF$ is non-negative, in order to ensure a non-negative lower confidence limit for $MTTF$, [Bishop et al. \(2007\)](#) suggested a log-transformation approach to construct the confidence interval for $MTTF$ by assuming that $\ln \widehat{MTTF}$ is asymptotically distributed as normal with mean $\ln MTTF$ and variance $Var(\ln \widehat{MTTF})$. Once again, we can use the delta method to obtain an estimate of the asymptotic variance of $\ln \widehat{MTTF}$ as

$$\widehat{Var}(\ln \widehat{MTTF}) = \frac{\widehat{Var}(\widehat{MTTF})}{\widehat{MTTF}^2}.$$

Let $\widehat{SE}(\ln \widehat{MTTF}) = \sqrt{\widehat{Var}(\ln \widehat{MTTF})}$ be the standard error of $\ln \widehat{MTTF}$. Then, a $100(1 - \tau)\%$ confidence interval for $MTTF$ of non-defective items can be constructed as

$$(\widehat{MTTF} \cdot e^{-z_{\tau/2} \widehat{SE}(\ln \widehat{MTTF})}, \widehat{MTTF} \cdot e^{z_{\tau/2} \widehat{SE}(\ln \widehat{MTTF})}),$$

where z_{τ} is the 100τ -th upper percentile of the standard normal distribution.

Under Scenario 3, when we are aware of the existence of defectives and the proportion of defectives is unknown, we can also consider the interval estimation of the mean time to failure of all units (i.e., including the defective units) and interval estimation of the defective proportion p . To distinguish the mean time to failure of non-defective units, we denote $MTTF_p$ as the mean time to failure of all units including defectives. We have

$$MTTF_p = (1 - p)MTTF = g_p(\boldsymbol{\theta}, p),$$

where $g_p(\cdot, \cdot)$ is a real function of $\boldsymbol{\theta}$ and p . Thus, the MLE of $MTTF_p$ is $\widehat{MTTF}_p = (1 - \hat{p})\widehat{MTTF} = g_p(\hat{\boldsymbol{\theta}}, \hat{p})$. With the reparameterization ϕ , we have

$$MTTF_p = (1 - \text{logit}^{-1}(\phi))MTTF = g_\phi(\boldsymbol{\theta}, \phi),$$

where $g_\phi(\cdot, \cdot)$ is a real function of $\boldsymbol{\theta}$ and ϕ . Thus, the MLE of $MTTF_p$ is $\widehat{MTTF}_p = (1 - \text{logit}^{-1}(\hat{\phi}))\widehat{MTTF} = g_\phi(\hat{\boldsymbol{\theta}}, \hat{\phi})$. The estimate of the asymptotic variance of $MTTF_p$ can be obtained as

$$\widehat{Var}(\widehat{MTTF}_p) = \nabla g_\phi(\hat{\boldsymbol{\theta}}, \hat{\phi})^\top \hat{\Sigma}_3(\hat{\boldsymbol{\theta}}, \hat{\phi}) \nabla g_\phi(\hat{\boldsymbol{\theta}}, \hat{\phi}).$$

An estimate of the asymptotic variance of $\ln \widehat{MTTF}_p$ can be obtained by the delta method as

$$\widehat{Var}(\ln \widehat{MTTF}_p) = \frac{\widehat{Var}(\widehat{MTTF}_p)}{\widehat{MTTF}_p^2}.$$

Let $\widehat{SE}(\ln \widehat{MTTF}_p) = \sqrt{\widehat{Var}(\ln \widehat{MTTF}_p)}$ be the standard error of $\ln \widehat{MTTF}_p$. Then, a $100(1 - \tau)\%$ confidence interval for $MTTF_p$ can be constructed as

$$(\widehat{MTTF}_p \cdot e^{-z_{\frac{\tau}{2}} \widehat{SE}(\ln \widehat{MTTF}_p)}, \widehat{MTTF}_p \cdot e^{z_{\frac{\tau}{2}} \widehat{SE}(\ln \widehat{MTTF}_p)}).$$

As for the interval estimation for p , let $\widehat{Var}(\hat{\phi}) = \hat{\Sigma}_{3,22}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ defined in Eq. (3.6), and $\widehat{SE}(\hat{\phi}) = \sqrt{\widehat{Var}(\hat{\phi})}$ be the standard error of $\hat{\phi}$. A $100(1 - \tau)\%$ confidence interval for ϕ can be constructed as

$$(\phi_L, \phi_U) = (\hat{\phi} - z_{\frac{\tau}{2}} \widehat{SE}(\hat{\phi}), \hat{\phi} + z_{\frac{\tau}{2}} \widehat{SE}(\hat{\phi})),$$

and hence, the corresponding $100(1 - \tau)\%$ confidence interval for p is

$$\left(\frac{e^{\phi_L}}{1 + e^{\phi_L}}, \frac{e^{\phi_U}}{1 + e^{\phi_U}} \right).$$

3.3.2. Bayesian Estimation

Under the Bayesian framework, we consider that the unknown model parameters are random variables that follow some prior distributions. In this section, we discuss the Bayesian estimation method for the one-shot device testing data with defectives by assuming the defective proportion p is a random variable and imposing a prior distribution on p .

3.3.2.1. Point estimation

When there is informative prior information about the defective proportion, we can use Bayesian estimation method to estimate the defective proportion p . In practical applications, the proportion of defectives is usually small (says $< 5\%$) for a stable manufacturing process. Therefore, one can consider a prior distribution with probability density concentrated near 0. In this section, we restrict our focus on putting a prior on the defective proportion in order to study the effect of the prior information on the performance of estimation procedures for Scenario 3. Note that one can also impose a prior distribution on the parameter vector $\boldsymbol{\theta}$ and perform the corresponding Bayesian analysis.

In the following subsections, we describe the beta prior distribution for p and the way to select the hyper-parameters in the prior distribution.

3.3.2.2. Beta prior for defective proportion

We consider the beta distribution with parameters a and b , denoted as $Beta(a, b)$, as the prior distribution of p . The probability density function of the $Beta(a, b)$ prior distribution for p is

$$\pi(p) = \frac{p^{a-1}(1-p)^{b-1}}{B(a, b)},$$

where $B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function. With the beta prior, the posterior distribution of p can be expressed as

$$\begin{aligned} \pi(p|\boldsymbol{\theta}, \mathbf{x}) = C p^{a-1}(1-p)^{b-1} \prod_{i=1}^n \{ [p + (1-p)F_X(t_i; \boldsymbol{\theta})]^{m_i} [p^{\nu_i} [(1-p)F_X(t_i; \boldsymbol{\theta})]^{1-\nu_i}]^{1-m_i} \}^{\delta_i} \\ [(1-p)(1-F_X(t_i; \boldsymbol{\theta}))]^{(1-\delta_i)}, \end{aligned} \quad (3.7)$$

where C is a constant that does not depend on the parameter p and $\boldsymbol{\theta}$. The logarithm of the posterior distribution of p is

$$\begin{aligned} \ln \pi(p|\boldsymbol{\theta}, \mathbf{x}) = \ln C + (a-1) \ln p + (b-1) \ln(1-p) \\ + \sum_{i=1}^n \delta_i m_i \ln(p + (1-p)F_X(t_i; \boldsymbol{\theta})) + \sum_{i=1}^n \delta_i (1-m_i) \nu_i \ln p \\ + \sum_{i=1}^n \delta_i (1-m_i) (1-\nu_i) [\ln(1-p) + \ln(1-F_X(t_i; \boldsymbol{\theta}))] \\ + \sum_{i=1}^n (1-\delta_i) [\ln(1-p) + \ln(1-F_X(t_i; \boldsymbol{\theta}))]. \end{aligned} \quad (3.8)$$

Maximizing Eq. (3.8) with respect to p and $\boldsymbol{\theta}$ gives the posterior mode of p and an estimate of $\boldsymbol{\theta}$.

3.3.2.3. Selection of the hyper-parameters in the beta prior

For $p \sim \text{Beta}(a, b)$, we have the mode and variance of the random variable p as

$$\text{Mode}(p) = \frac{a - 1}{a + b - 2}$$

and

$$\text{Var}(p) = \frac{ab}{(a + b)^2(a + b + 1)},$$

respectively. Here, we propose to determine the hyper-parameters a and b when the mode and the variance of the prior distribution of p are fixed.

Suppose we have accurate prior information on p , we consider setting $\text{Mode}(p) = p_0$. For a preset value c for the variance of p , i.e., $\text{Var}(p) = c$, we have the following equations in terms of a and b :

$$ca^3 + (c(7p_0 - 3) - p_0^2(1 - p_0))a^2 + (2p_0 - 1)(c(8p_0 - 3) - p_0^2)a + c(2p_0 - 1)^2(3p_0 - 1) = 0, \quad (3.9)$$

$$b = \frac{(1 - p_0)a + 2p_0 - 1}{p_0}. \quad (3.10)$$

For specific values of p_0 and c , by solving Eqs. (3.9) and (3.10), we can obtain the values of the hyper-parameters a and b .

3.3.2.4. Interval Estimation

The interval estimation for $MTTF$ and $MTTF_p$ in the Bayesian framework can be obtained by using the normal approximated confidence intervals described in in Section 3.3.1.2.

A $100(1 - \tau)\%$ credible interval for p , denoted as (p_L, p_U) can be obtained by computing the $100(\tau/2)$ -th and $100(1 - \tau/2)$ -th percentiles of the posterior distribution of p in Eq. (3.7),

i.e., solving the following equations for p_L and p_U :

$$\begin{aligned}\Pr(p < p_L) &= \int_0^{p_L} \pi(p|\boldsymbol{\theta}, \mathbf{x}) dp = \frac{\tau}{2}, \\ \Pr(p < p_U) &= \int_0^{p_U} \pi(p|\boldsymbol{\theta}, \mathbf{x}) dp = 1 - \frac{\tau}{2}.\end{aligned}$$

3.4. Illustration: Gamma Distributed Lifetime

In this section, we illustrate the statistical inferential methods described in the previous sections by assuming that the one-shot devices have gamma distributed lifetimes with parameter vector $\boldsymbol{\theta} = (\alpha \ \beta)$ and $k = 2$. In other words, we assume that the lifetime of the i -th one-shot device, if it is not a defective, follows a gamma distribution with the shape parameter α and the scale parameter β (denoted as $X_i \sim \text{Gamma}(\alpha, \beta)$) with pdf

$$f_X(x; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right), \quad x > 0,$$

and cdf

$$F_X(x; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{x}{\beta}\right), \quad x > 0.$$

Based on the observed data (t_i, δ_i) , $i = 1, 2, \dots, n$, the log-likelihood functions for different scenarios with gamma distributed lifetimes are given as follows.

- Scenario 1:

$$\begin{aligned}\ln L_1(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i \left[\ln \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right] \\ &\quad + \sum_{i=1}^n (1 - \delta_i) \left[\ln \Gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right].\end{aligned}\tag{3.11}$$

- Scenario 2:

$$\begin{aligned}
\ln L_2(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i m_i \ln \left[p_0 + (1 - p_0) \frac{\gamma\left(\alpha, \frac{t_i}{\beta}\right)}{\Gamma(\alpha)} \right] \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) \nu_i \ln p_0 \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) (1 - \nu_i) \left[\ln(1 - p_0) + \ln \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) \left[\ln(1 - p_0) + \ln \Gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right]. \tag{3.12}
\end{aligned}$$

- Scenario 3 with parameter p :

$$\begin{aligned}
\ln L_{3p}(\boldsymbol{\theta}) &= \sum_{i=1}^n \delta_i m_i \ln \left[p + (1 - p) \frac{\gamma\left(\alpha, \frac{t_i}{\beta}\right)}{\Gamma(\alpha)} \right] \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) \nu_i \ln p \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) (1 - \nu_i) \left[\ln(1 - p) + \ln \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) \left[\ln(1 - p) + \ln \Gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right].
\end{aligned}$$

- Scenario 3 with parameter ϕ :

$$\begin{aligned}
\ln L_3(\boldsymbol{\theta}, \phi) &= \sum_{i=1}^n \delta_i m_i \ln \left[\text{logit}^{-1}(\phi) + (1 - \text{logit}^{-1}(\phi)) \frac{\gamma\left(\alpha, \frac{t_i}{\beta}\right)}{\Gamma(\alpha)} \right] \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) \nu_i \ln [\text{logit}^{-1}(\phi)] \\
&+ \sum_{i=1}^n \delta_i (1 - m_i) (1 - \nu_i) \left[\ln(1 - \text{logit}^{-1}(\phi)) + \ln \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right] \\
&+ \sum_{i=1}^n (1 - \delta_i) \left[\ln(1 - \text{logit}^{-1}(\phi)) + \ln \Gamma\left(\alpha, \frac{t_i}{\beta}\right) - \ln \Gamma(\alpha) \right]. \tag{3.13}
\end{aligned}$$

The MLEs of the parameters α , β , and ϕ , denoted as $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\phi}$, respectively, can be obtained by maximizing the corresponding log-likelihood functions in Eqs. (3.11)–(3.13) for Scenario 1–3. The MLE of the parameter p can be obtained as $\hat{p} = \text{logit}^{-1}(\hat{\phi})$. The second derivatives of the log-likelihood functions involved in the observed Fisher information matrix and the asymptotic variance-covariance matrix for the MLEs are presented in Appendix A.

For $X_i \sim \text{Gamma}(\alpha, \beta)$, $MTTF$ and $MTTF_p$ are

$$MTTF = g(\boldsymbol{\theta}) = \alpha\beta$$

and $MTTF_p = g_\phi(\boldsymbol{\theta}, \phi) = \alpha\beta(1 - \text{logit}^{-1}(\phi))$,

respectively. From Section 3.3.1.2, we can obtain

$$\widehat{Var}(\ln \widehat{MTTF}) = \begin{pmatrix} \frac{1}{\hat{\alpha}} & \frac{1}{\hat{\beta}} \end{pmatrix} \hat{\Sigma} \begin{pmatrix} \frac{1}{\hat{\alpha}} \\ \frac{1}{\hat{\beta}} \end{pmatrix},$$

where $\hat{\Sigma}$ is $\hat{\Sigma}_1(\hat{\boldsymbol{\theta}})$ defined in Eq. (3.4) for Scenario 1, $\hat{\Sigma}_2(\hat{\boldsymbol{\theta}})$ defined in Eq. (3.5) for Scenario 2, and $\hat{\Sigma}_{3,11}(\hat{\boldsymbol{\theta}}, \hat{\phi})$ defined in Eq. (3.6) for Scenario 3.

We can also obtain

$$\widehat{Var}(\ln \widehat{MTTF}_p) = \begin{pmatrix} \frac{1}{\hat{\alpha}} & \frac{1}{\hat{\beta}} & -\text{logit}^{-1}(\hat{\phi}) \end{pmatrix} \hat{\Sigma}_3(\hat{\boldsymbol{\theta}}, \hat{\phi}) \begin{pmatrix} \frac{1}{\hat{\alpha}} \\ \frac{1}{\hat{\beta}} \\ -\text{logit}^{-1}(\hat{\phi}) \end{pmatrix}.$$

Then, the confidence intervals for $MTTF$ and $MTTF_p$ can be constructed by using the formulas provided in Section 3.3.1.2.

To further illustrate the statistical inferential procedures proposed in Sections 3.2 and 3.3 for gamma distributed lifetimes, we present here a numerical example based on the simulated one-shot device testing data when the lifetime distribution is $\text{Gamma}(4, 8)$. We consider

the inspection time points at 20, 35 and 50, and 100 units are inspected at each inspection. We simulate the data with the proportion of defectives $p = 0.01, 0.1$, and masking rate $w = 0, 0.5, 0.8, 0.9, 1.0$. The detailed information of the simulated one-shot device testing data is shown in Table 3.1. In this numerical example, the true value for $MTTF$ is 32 and the true values of $MTTF_p$ are 31.7 and 28.8 for $p = 0.01$ and $p = 0.1$, respectively. For the Bayesian inference described in Section 3.3.2, we consider beta prior distributions for p with the mode equal to the true value of p , and variance $c = 0.005$ (denoted this kind of beta priors as π_1) and $c = 0.0002$ (denoted this kind of beta priors as π_2). Based on the maximum likelihood and Bayesian methods described in Section 3.3, the point estimates of $MTTF$, $MTTF_p$, and p , and the corresponding 95% confidence/credible intervals under different scenarios are computed. These results are presented in Table 3.2.

From Table 3.2, we observe that the maximum likelihood estimation method is unable to give useful point and interval estimates for the defective proportion $p = 0.01$ and the masking rate $w \geq 0.80$. Moreover, the MLEs and Bayes estimates of $MTTF$ and $MTTF_p$ can be very different when the masking rate $w \geq 0.80$. To further investigate the performance of the point and interval estimation methods discussed in this chapter, we conduct a comprehensive Monte Carlo simulation study in Section 3.5.

Table 3.1: One-shot device testing data simulated from $Gamma(4, 8)$ for $p = 0.01, 0.10$ with $w = 0.0, 0.5, 0.8, 0.9, 1.0$

p	w	Inspection Time	No. of Units Inspected	No. of Left-Censored	No. of Right-Censored	No. of Defectives	No. of Masked Units	No. of Masked Defectives
0.01	0.00	20	100	23	77	1	0	0
		35	100	54	46	0	0	0
		50	100	88	12	1	0	0
	0.50	20	100	25	75	2	10	1
		35	100	70	30	0	33	0
		50	100	85	15	0	39	0
	0.80	20	100	28	72	0	24	0
		35	100	60	40	2	44	1
		50	100	88	12	0	74	0
	0.90	20	100	31	69	3	29	3
		35	100	66	34	0	64	0
		50	100	92	8	0	79	0
	1.00	20	100	28	72	0	28	0
		35	100	70	30	0	70	0
		50	100	90	10	0	90	0
0.10	0.00	20	100	21	79	9	0	0
		35	100	78	22	6	0	0
		50	100	86	14	10	0	0
	0.50	20	100	33	67	5	16	1
		35	100	59	41	8	26	3
		50	100	93	7	8	48	3
	0.80	20	100	27	73	8	23	7
		35	100	68	32	13	54	10
		50	100	85	15	10	67	8
	0.90	20	100	33	67	11	29	9
		35	100	68	32	8	62	8
		50	100	86	14	11	79	9
	1.00	20	100	24	76	4	24	4
		35	100	71	29	12	71	12
		50	100	87	13	13	87	13

Table 3.2: Point estimates and 95% confidence/credible intervals of $MTTF$, $MTTF_p$ and p for the data presented in Table 3.1

$p = 0.01$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
	$MTTF$	$MTTF$	\hat{p}	$MTTF$	$MTTF$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}
0.00	32.2 (29.5, 35.1)	32.7 (30.3, 35.4)	0.003 (0.0005, 0.0233)	32.8 (30.5, 35.2)	30.4 (28.2, 32.8)	0.0100 (0.0037, 0.0283)	30.1 (27.9, 32.5)	30.1 (27.9, 32.5)	0.0100 (0.0037, 0.0283)	31.4 (29.2, 33.8)	31.2 (29.0, 33.6)	0.0049 (0.0015, 0.0179)	31.4 (29.2, 33.8)	31.2 (29.0, 33.6)	0.0049 (0.0015, 0.0179)
0.50	30.5 (28.5, 32.6)	33.7 (30.8, 36.8)	0.014 (0.0036, 0.0552)	33.5 (30.8, 36.3)	32.7 (30.2, 35.4)	0.0070 (0.0017, 0.0347)	32.5 (30.0, 35.2)	32.5 (30.0, 35.2)	0.0070 (0.0017, 0.0347)	32.3 (30.1, 34.6)	31.9 (29.8, 34.2)	0.0123 (0.0044, 0.0351)	32.3 (30.1, 34.6)	31.9 (29.8, 34.2)	0.0123 (0.0044, 0.0351)
0.80	30.6 (28.2, 33.2)	31.9 (29.5, 34.5)	0.000 (0.0000, 1.0000)	34.3 (31.1, 37.8)	34.0 (31.2, 36.9)	0.0017 (0.0005, 0.0473)	33.9 (31.2, 36.9)	33.9 (31.2, 36.9)	0.0017 (0.0005, 0.0473)	34.8 (32.3, 37.6)	34.6 (32.1, 37.4)	0.0061 (0.0014, 0.0328)	34.8 (32.3, 37.6)	34.6 (32.1, 37.4)	0.0061 (0.0014, 0.0328)
0.90	33.7 (31.1, 36.5)	32.8 (30.2, 35.6)	0.000 (0.0000, 1.0000)	29.3 (27.0, 31.8)	30.4 (28.2, 32.8)	0.0025 (0.0007, 0.0629)	30.3 (28.1, 32.7)	30.3 (28.1, 32.7)	0.0025 (0.0007, 0.0629)	34.1 (31.7, 37.3)	34.1 (31.4, 37.0)	0.0080 (0.0019, 0.0417)	34.1 (31.7, 37.3)	34.1 (31.4, 37.0)	0.0080 (0.0019, 0.0417)
1.00	32.2 (29.9, 34.7)	31.2 (28.6, 34.0)	0.100 (0.0051, 0.7075)	30.0 (25.9, 34.8)	32.4 (29.6, 35.5)	0.0082 (0.0017, 0.1080)	32.1 (29.3, 35.2)	32.1 (29.3, 35.2)	0.0082 (0.0017, 0.1080)	33.6 (30.8, 36.7)	33.3 (30.5, 36.3)	0.0100 (0.0023, 0.0504)	33.6 (30.8, 36.7)	33.3 (30.5, 36.3)	0.0100 (0.0023, 0.0504)
$p = 0.10$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
w	$MTTF$	$MTTF$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}	$MTTF$	$MTTF_p$	\hat{p}
0.00	29.4 (27.2, 31.9)	31.4 (28.9, 34.0)	0.120 (0.0878, 0.1619)	28.8 (25.9, 32.0)	32.8 (30.7, 35.1)	0.1094 (0.0798, 0.1483)	29.2 (27.3, 31.3)	29.2 (27.3, 31.3)	0.1094 (0.0798, 0.1483)	32.1 (29.2, 35.4)	29.2 (26.5, 32.2)	0.0907 (0.0723, 0.1134)	32.1 (29.2, 35.4)	29.2 (26.5, 32.2)	0.0907 (0.0723, 0.1134)
0.50	30.6 (28.4, 33.0)	33.2 (30.5, 36.1)	0.091 (0.0551, 0.1458)	27.3 (25.1, 29.7)	31.1 (28.5, 34.0)	0.0624 (0.0357, 0.1063)	29.2 (26.7, 31.9)	29.2 (26.7, 31.9)	0.0624 (0.0357, 0.1063)	31.1 (28.6, 33.8)	28.0 (25.8, 30.5)	0.0978 (0.0768, 0.1236)	31.1 (28.6, 33.8)	28.0 (25.8, 30.5)	0.0978 (0.0768, 0.1236)
0.80	28.6 (26.0, 31.3)	32.5 (29.3, 36.1)	0.125 (0.0639, 0.2292)	29.6 (26.7, 32.8)	34.1 (31.2, 37.1)	0.1004 (0.0538, 0.1693)	30.6 (28.1, 33.4)	30.6 (28.1, 33.4)	0.1004 (0.0538, 0.1693)	31.6 (29.2, 34.3)	28.7 (26.5, 31.1)	0.0922 (0.0707, 0.1193)	31.6 (29.2, 34.3)	28.7 (26.5, 31.1)	0.0922 (0.0707, 0.1193)
0.90	30.6 (28.3, 33.1)	32.4 (29.2, 36.0)	0.138 (0.0640, 0.2716)	31.3 (27.7, 35.3)	31.1 (28.4, 34.2)	0.0780 (0.0334, 0.1558)	28.7 (26.2, 31.5)	28.7 (26.2, 31.5)	0.0780 (0.0334, 0.1558)	30.0 (27.8, 32.4)	27.0 (25.0, 29.1)	0.1027 (0.0790, 0.1322)	30.0 (27.8, 32.4)	27.0 (25.0, 29.1)	0.1027 (0.0790, 0.1322)
1.00	30.3 (27.4, 33.4)	31.6 (29.1, 34.3)	0.291 (0.0630, 0.7139)	24.6 (19.2, 31.5)	31.8 (28.9, 35.1)	0.1065 (0.0392, 0.2039)	28.4 (25.8, 31.4)	28.4 (25.8, 31.4)	0.1065 (0.0392, 0.2039)	28.9 (26.3, 31.8)	26.0 (23.7, 28.6)	0.0999 (0.0760, 0.1299)	28.9 (26.3, 31.8)	26.0 (23.7, 28.6)	0.0999 (0.0760, 0.1299)

3.5. Monte Carlo Simulation Study

In this section, a Monte Carlo simulation study is used to evaluate the performance of the proposed estimation procedures under different settings when the lifetime distribution of the one-shot devices is assumed to be gamma.

3.5.1. Simulation Settings

Following the simulation settings used by [Balakrishnan and Ling \(2014b\)](#), we also consider different inspection time points in order to investigate the impact of early and late inspections on the performance of estimation procedures. The parameters of the gamma distributions for the non-defective units and the inspection time points considered in the simulation study are presented in [Table 3.3](#). We also plot the gamma distributions with the inspection time points in [Figure 3.2](#). We consider 100 units to be inspected at each inspection, with the defective proportion $p = 0.01, 0.02, 0.05, 0.10, 0.20$, and masking rate $w = 0.00, 0.25, 0.50, 0.60, 0.70, 0.80, 0.85, 0.90, 0.95, 1.00$ in the simulation study. For each simulated one-shot device testing data set, the point estimates and the 95% confidence/credible intervals of the model parameters, $MTTF$, and $MTTF_p$ based on the maximum likelihood estimation and Bayesian estimation method are computed. For the Bayesian estimation, we consider beta prior distributions for p discussed in [Section 3.3.2.3](#) with variance $c = 0.005$ (we denote this kind of beta priors as π_1) and $c = 0.0002$ (we denote this kind of beta priors as π_2). For each setting, we use 5000 simulations to estimate the biases and mean squared errors (MSEs) for the point estimation methods, as well as the coverage probabilities (CPs) and average widths (AWs) for the interval estimation methods. The results are presented in [Tables 3.4–3.17](#).

Table 3.3: Distributions and Inspection Time for Monte Carlo Simulation

Non-defective distribution	Inspection time (IT)
$Gamma(4, 8)$	(20, 35, 50)
	(12, 35, 50)
	(20, 35, 65)
	(12, 35, 65)
$Gamma(1.405, 40.477)$	(10, 30, 50)
$Gamma(1.405, 66.686)$	(15, 50, 85)
$Gamma(1.405, 24.533)$	(20, 80, 140)

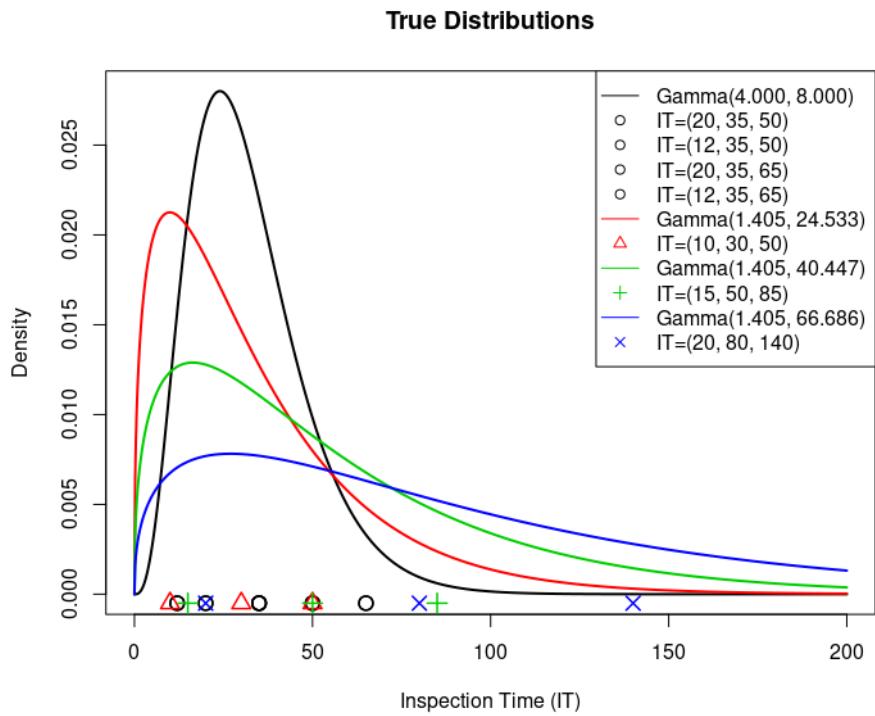


Figure 3.2: Gamma distributions and the inspection time points used in the Monte Carlo simulation study

3.5.2. Discussions

3.5.2.1. Point estimation

In this subsection, we compare the results for point estimation under the three different scenarios. Since masking is not considered in Scenario 1, the biases and MSEs are not affected by the masking rate w . In Scenario 2, we assume that the true value of the defective proportion p is known. Therefore, the biases and MSEs of the estimates under Scenario 2 are the smallest in general and can be considered as a benchmark for comparative purposes.

From Tables 3.4–3.7, when comparing the biases and MSEs of the MLEs of $MTTF$ under Scenarios 1 and 2, we observe that ignoring the existence of defectives in the analysis (i.e., Scenario 1) when $p = 0.01$ introduce slightly larger biases in estimating $MTTF$ while the MSEs are similar to those under Scenario 2. When $p \geq 0.02$, it is clear that the simulated biases and MSEs of the MLEs of $MTTF$ under Scenario 1 are larger than the case when defectives are considered in the analysis (i.e., the results under Scenarios 2 and 3). Moreover, the larger the true value of p , the larger the differences of the biases and MSEs of under Scenario 1 and those under Scenarios 2 and 3. From Tables 3.8–3.10, we observe that Scenario 1 gives smaller biases when $p \leq 0.02$. For $p \leq 0.02$, Scenario 1 has smaller MSEs than Scenario 3 when $w \geq 0.85$. These results indicate that unless the defective proportion is very small (says, ≤ 0.01), ignoring the existence of defectives may introduce serious errors in estimating $MTTF$ of the one-shot devices.

When comparing the biases and MSEs of the MLEs under Scenarios 2 and 3 (i.e., the defective proportion is known and unknown), from Tables 3.4 – 3.7, we observe that for $p \leq 0.02$ and the masking rate $w \leq 0.80$, the biases and MSEs of the MLEs of $MTTF$ under Scenario 3 are close to the corresponding values under Scenario 2. When $0.85 \leq w \leq 0.95$, the simulated biases of the MLEs of $MTTF$ under Scenario 3 are close to those under Scenario 2, but the simulated MSEs under Scenario 3 are slightly larger than those under Scenario 2. For example, in Table 3.4, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when the defective proportion $p = 0.01$ and masking rate $w = 0.9$, the simulated bias and

MSE for the MLEs of $MTTF$ under Scenario 3 with the maximum likelihood method are 0.046 and 1.812, respectively, while the simulated bias and MSE for the MLEs of $MTTF$ under Scenario 2 are 0.042 and 1.688 respectively. In the situation that whether a failed unit is defective or not is completely masked (i.e., $w = 1$), the simulated biases of the MLEs of $MTTF$ under Scenario 3 are significantly higher than those under Scenario 2. For example, in Table 3.4, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when $p = 0.01$ and $w = 1$, the simulated bias and MSE of the MLEs of $MTTF$ under Scenario 3 with the maximum likelihood method are 1.698 and 9.609, respectively, while the simulated bias and MSE of the MLEs of $MTTF$ under Scenario 2 are 0.058 and 1.768, respectively. Similar conclusion can be drawn from Table 3.8– 3.10.

For the effect of masking rate in the performance of the MLEs of $MTTF$, $MTTF_p$ and p , we observe that for $w \leq 0.80$, the masking rate only affects the performance of the MLEs of p in general. For $w \leq 0.80$, as we expected, the smaller the masking rate, the smaller the simulated biases and MSEs of the estimates of p in most cases. For $0.85 \leq w \leq 0.95$, the simulated MSEs of the MLEs of $MTTF$ and $MTTF_p$ get larger compared to those cases with $w \leq 0.80$. For the case of complete masking (i.e., $w = 1$), both the simulated biases and MSEs of the MLEs of $MTTF$, $MTTF_p$ and p are significantly higher. For example, in Table 3.4, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, for $p = 0.1$, when $w = 0.85$, the simulated bias and MSE of the MLEs of $MTTF$ under Scenario 3 with the maximum likelihood method are 0.039 and 2.773, respectively; when $w = 0.95$, the simulated bias and MSE of the MLEs of $MTTF$ under Scenario 3 with the maximum likelihood method are 0.124 and 4.793, respectively; when $w = 1$, the simulated bias and MSE of the MLEs of $MTTF$ under Scenario 3 with the maximum likelihood method are 1.100 and 12.594, respectively.

For the Bayesian estimation method, since informative priors are considered, the performance of the Bayesian estimates is better than the MLEs in general. The prior with a smaller variance c (i.e., π_2) gives better estimates for p and estimates of $MTTF$ compared to the prior π_1 in terms of MSEs. For prior π_1 , when the masking rate is high, larger biases are

introduced in the Bayesian estimates while the MSEs stay at a similar level. For example, in Table 3.8, for the case that the underlying lifetime distribution is $Gamma(1.405, 24.533)$, when $p = 0.01$ and $w = 1$, the simulated bias and MSE for $MTTF$ for the Bayesian estimates with prior π_1 are 0.405 and 9.359, respectively; the simulated bias and MSE for the Bayesian estimates with prior π_2 are 0.325 and 9.185, respectively; while the simulated bias and MSE under Scenario 2 (which can be considered as a benchmark as we mentioned before) are 0.325 and 9.186, respectively. One interesting observation is that for the estimation of the defective proportion p , the performance of the Bayesian estimates for prior π_2 is reasonably well, which may be due to the fact that the informative prior is dominant when there is a lack of information about defectives in the observed data.

When studying the effect of the inspection time points, we observe that an extremely early inspection time point will decrease biases and MSEs in the estimation of $MTTF$ and p when w is close to 1 for Scenario 3 with the maximum likelihood method. For example, in Tables 3.4 and 3.5, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when $p = 0.02$ and $w = 1$, the simulated bias and MSE of MLEs of $MTTF$ under Scenario 3 are 1.656 and 10.061 for inspection time points (20, 35, 50), while the simulated bias and MSE of MLEs of $MTTF$ under Scenario 3 are 0.659 and 4.019 for inspection time points (12, 35, 50); the simulated bias and MSE of MLEs of p under Scenario 3 are 0.068 and 0.015 for inspection time points (20, 35, 50), while the simulated bias and MSE of MLEs of p under Scenario 3 are 0.014 and 0.002 for inspection time points (12, 35, 50).

3.5.2.2. Interval estimation

In this subsection, we compare the results for interval estimation under the three different scenarios with different settings in terms of the coverage probabilities and average widths. Once again, the results under Scenario 2 can be treated as a benchmark for the comparisons.

From Tables 3.11–3.17, for Scenario 1, we observe that the simulated coverage probabilities of the confidence intervals for $MTTF$ are close to the nominal level when $p \leq 0.02$ and the simulated coverage probabilities decrease as p increases in most of the settings. The

average widths are consistent and close to that of Scenario 2. For Scenario 3 with the maximum likelihood estimation method, the simulated coverage probabilities of the confidence intervals for $MTTF$ are close to or above the nominal level except for the cases when the masking rate w is close to 1 in most of the settings.

For the effect of masking rate in the performance of the interval estimates, we observe under Scenario 3 that the simulated coverage probabilities of the confidence intervals of $MTTF$ based on the maximum likelihood method are close to the nominal level 95% and the simulated average width for the confidence intervals of $MTTF$ based on the maximum likelihood method are close to those values under Scenario 2 when $w \leq 0.80$. For masking rate $w \geq 0.85$, the simulated coverage probabilities drop below the nominal level and there is an increase in the average widths of the confidence intervals for $MTTF$ based on the maximum likelihood method. For the case with complete masking (i.e., $w = 1$), the simulated coverage probabilities of the confidence intervals for $MTTF$ based on the maximum likelihood method drop and the average widths increase significantly. Under the same setting, the simulated coverage probabilities and average widths for the confidence intervals of $MTTF_p$ based on the maximum likelihood method are usually better. For example, in Table 3.11, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, for $p = 0.05$, when $w = 0.95$, we have $CP = 0.919$ and $AW = 6.536$ for the confidence intervals of $MTTF$, respectively; while we have $CP = 0.948$ and $AW = 5.816$ for the confidence intervals of $MTTF_p$, respectively; when $w = 1$, we have $CP = 0.815$ and $AW = 12.413$ for the confidence intervals of $MTTF$, respectively; while we have $CP = 0.893$ and $AW = 8.996$ for the confidence intervals of $MTTF_p$, respectively.

Once again, since informative priors are used for the Bayesian interval estimation method, the simulated coverage probabilities and average widths for the credible intervals of $MTTF$ are consistent for different masking rates w , especially when the prior distribution π_2 which has a smaller variation is used. In contrast with the confidence intervals of $MTTF$, $MTTF_p$ and p based on the maximum likelihood estimation method, the corresponding interval estimation based on the Bayesian approach has advantages over the maximum likelihood

estimation method when the masking rate is closer to 1. For example, in Table 3.11, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when $p = 0.05$ and $w = 1$, the simulated coverage probability and average width of the confidence intervals of $MTTF$ based on the maximum likelihood method are $CP = 0.815$ and $AW = 12.413$, respectively; the simulated coverage probability and average width of the credible intervals of $MTTF$ based on Bayesian method with prior π_1 are $CP = 0.938$ and $AW = 5.381$, respectively; and the simulated coverage probability and average width of the credible intervals of $MTTF$ based on Bayesian method with prior π_2 are $CP = 0.948$ and $AW = 5.378$, respectively. The advantage of the Bayesian method in interval estimation over the maximum likelihood method is more obvious for estimating the defective proportion p as the coverage probabilities of the confidence intervals of p based on the maximum likelihood method can be lower than 80% in some cases. For example, in Table 3.11, for $p = 0.05$ and $w = 1$, the simulated coverage probability and average width of the confidence intervals of p based on the maximum likelihood method are $CP = 0.736$ and $AW = 0.779$, respectively; the simulated coverage probability and average width of the credible intervals of p based on Bayesian method with prior π_1 are $CP = 0.991$ and $AW = 0.135$, respectively; and the simulated coverage probability and average width of the credible intervals of $MTTF$ based on Bayesian method with prior π_2 are $CP = 1.000$ and $AW = 0.053$, respectively.

To investigate the effect of inspection time points on the performance of the interval estimation procedures, we observe that an extremely early inspection time point may decrease the average width in Scenario 3 with the maximum likelihood when w is close to 1. For example, in Tables 3.11 and 3.12, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when $p = 0.01$ and $w = 1$, the simulated average width of the confidence intervals of $MTTF$ based on the maximum likelihood method is 11.693 for inspection time points (20, 35, 50), while the simulated average width of the confidence intervals of $MTTF$ based on the maximum likelihood method is 7.562 for inspection time points (12, 35, 50). For the maximum likelihood estimation method, an extremely early or late inspection time may also increase the coverage probabilities when the masking rate is close to 1. For example,

in Tables 3.11–3.14, for the case that the underlying lifetime distribution is $Gamma(4, 8)$, when $p = 0.01$ and $w = 1$, the simulated coverage probability of the confidence intervals of $MTTF$ based on the maximum likelihood method is 0.846 for inspection time points (20, 35, 50), while the simulated coverage probabilities of the confidence intervals of $MTTF$ based on the maximum likelihood method are 0.898, 0.869, and 0.911 for inspection time points (12, 35, 50), (20, 35, 65) and (12, 35, 65), respectively.

3.6. Concluding Remarks

In this chapter, we study the analysis of one-shot device testing data when some of the devices are defectives. Three different scenarios based on the knowledge of the defective proportion are considered. Moreover, for the scenario that the defective proportion is unknown, we also consider different masking rates to reflect different levels of information we can obtain on whether a failed one-shot device is defective or not.

For point and interval estimation of the model parameters and those related reliability indices, the maximum likelihood estimation method and the Bayesian estimation method are proposed. The methodologies are illustrated by considering the lifetimes of the one-shot devices that are gamma-distributed. Based on our extensive simulation study, we observe that ignoring the existence of defectives when the defective proportion is larger than 2% will introduce errors in the estimation of the mean lifetime of those non-defective units. Therefore, the methods developed in this chapter can be useful in practice when the experimenter or data analyst suspects that there are defectives in the manufacturing process. In addition, we also observe that applying an informative prior with the Bayesian estimation method on the defective proportion can provide an advantage in the accuracy in both point and interval estimation, especially when the masking rate is close to or equal to 1.

Table 3.4: Simulated biases and MSEs of maximum likelihood estimates and Bayesian estimates of $MTTF$, $MTTF_p$ and p for the data simulated from $Gamma(4, 8)$ with inspection time points (20, 35, 50) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2			
	$MTTF$		p	$MTTF$		p	$MTTF_p$		p	$MTTF$		p	$MTTF$		p	
	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*	
$p = 0.01$																
0.00	-0.131	1.742	0.063	1.735	0.063	1.735	0.060	1.732	0.648	0.330	0.063	1.735	0.060	1.730	0.623	0.306
0.25	-0.127	1.784	0.063	1.779	0.063	1.779	0.067	1.784	-1.508	0.436	0.063	1.779	0.066	1.780	-1.426	0.394
0.50	-0.149	1.728	0.043	1.721	0.043	1.727	0.044	1.724	-0.502	0.666	0.043	1.726	0.044	1.719	-0.446	0.572
0.60	-0.144	1.743	0.049	1.739	0.049	1.755	0.049	1.732	-0.489	0.834	0.049	1.752	0.049	1.727	-0.371	0.690
0.70	-0.180	1.759	0.013	1.744	0.016	1.770	0.009	1.743	1.607	1.140	0.016	1.764	0.010	1.735	1.562	0.889
0.80	-0.163	1.736	0.029	1.728	0.030	1.770	0.030	1.740	-1.011	1.657	0.030	1.757	0.030	1.726	-0.672	1.143
0.85	-0.175	1.756	0.017	1.743	0.017	1.806	0.017	1.764	-1.349	2.242	0.017	1.781	0.017	1.743	-0.708	1.379
0.90	-0.150	1.690	0.042	1.688	0.046	1.812	0.041	1.736	-0.890	3.735	0.044	1.743	0.042	1.692	-0.584	1.750
0.95	-0.142	1.725	0.050	1.727	0.072	2.126	0.039	1.861	4.129	8.954	0.057	1.814	0.047	1.732	1.680	2.186
1.00	-0.135	1.762	0.058	1.768	1.698	9.609	-0.966	4.738	720.261	150.391	0.087	1.867	0.042	1.742	12.570	1.293
$p = 0.02$																
0.00	-0.370	1.913	0.017	1.771	0.017	1.771	0.020	1.791	-0.987	0.661	0.017	1.771	0.020	1.785	-0.946	0.609
0.25	-0.365	1.866	0.022	1.750	0.023	1.752	0.020	1.744	0.532	0.861	0.023	1.752	0.020	1.738	0.514	0.772
0.50	-0.354	1.842	0.029	1.742	0.034	1.738	0.034	1.738	-2.413	1.254	0.028	1.753	0.034	1.729	-2.169	1.067
0.60	-0.345	1.837	0.041	1.751	0.042	1.778	0.041	1.743	-0.487	1.616	0.042	1.774	0.041	1.731	-0.356	1.321
0.70	-0.328	1.819	0.058	1.748	0.061	1.798	0.057	1.747	-0.138	2.173	0.061	1.787	0.057	1.730	0.025	1.668
0.80	-0.349	1.833	0.037	1.749	0.037	1.840	0.038	1.763	-2.409	3.253	0.037	1.811	0.038	1.735	-1.562	2.198
0.85	-0.314	1.769	0.074	1.710	0.081	1.830	0.069	1.769	0.976	4.543	0.080	1.780	0.069	1.719	1.740	2.721
0.90	-0.328	1.848	0.059	1.783	0.078	2.049	0.047	1.857	4.978	7.193	0.072	1.901	0.051	1.775	4.229	3.277
0.95	-0.362	1.820	0.025	1.728	0.074	2.447	-0.002	1.982	12.560	16.831	0.052	1.919	0.010	1.730	9.899	4.431
1.00	-0.314	1.807	0.073	1.753	1.656	10.061	-0.903	4.790	677.779	154.986	0.121	1.931	0.045	1.692	21.138	2.158
$p = 0.05$																
0.00	-0.925	2.602	0.056	1.794	0.056	1.794	0.050	1.762	0.987	1.563	0.056	1.794	0.050	1.747	0.941	1.415
0.25	-0.943	2.616	0.034	1.789	0.036	1.797	0.026	1.756	2.087	2.134	0.036	1.797	0.027	1.738	1.990	1.873
0.50	-0.895	2.577	0.084	1.854	0.085	1.888	0.081	1.830	-0.936	3.144	0.086	1.883	0.081	1.802	-0.686	2.998
0.60	-0.959	2.693	0.018	1.850	0.024	1.911	0.012	1.828	1.760	3.852	0.024	1.899	0.013	1.794	1.843	3.046
0.70	-0.989	2.733	-0.014	1.838	-0.005	1.939	-0.020	1.850	5.247	5.247	-0.006	1.911	-0.019	1.800	2.565	3.850
0.80	-0.958	2.653	0.017	1.829	0.026	2.052	0.014	1.874	-2.363	7.757	0.026	1.957	0.014	1.793	0.626	4.933
0.85	-0.986	2.787	-0.009	1.904	0.009	2.186	-0.019	2.017	3.297	10.483	0.007	2.053	-0.017	1.893	4.517	5.861
0.90	-0.906	2.561	0.070	1.847	0.098	2.454	0.054	2.018	2.480	16.373	0.090	2.110	0.059	1.813	4.188	7.034
0.95	-0.926	2.581	0.052	1.837	0.124	3.520	0.016	2.301	7.704	36.080	0.094	2.265	0.029	1.776	12.861	8.615
1.00	-0.910	2.610	0.067	1.896	1.460	10.875	-0.754	5.000	550.107	162.963	0.141	2.165	0.026	1.724	30.607	3.256

* Biases and MSEs are in 10^{-4} .

Table 3.4: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2						
	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	
0.00	-1.942	5.626	0.049	1.948	0.049	1.948	0.045	1.872	2.993	0.049	1.948	0.045	1.835	-0.584	2.623	0.049	1.948	0.044	1.621
0.25	-1.934	5.550	0.059	1.931	0.060	1.943	0.054	1.855	3.942	0.060	1.942	0.054	1.808	-0.327	3.325	0.060	1.933	0.054	1.594
0.50	-1.917	5.486	0.071	1.948	0.074	1.995	0.070	1.920	5.843	0.074	1.984	0.069	1.848	-2.138	4.573	0.072	1.949	0.067	1.606
0.70	-1.927	5.532	0.067	2.007	0.081	2.124	0.048	1.919	4.924	0.080	2.096	0.050	1.840	4.800	5.163	0.071	2.015	0.059	1.636
0.80	-1.930	5.566	0.058	2.020	0.069	2.197	0.051	2.006	6.957	0.069	2.138	0.051	1.890	0.604	6.362	0.060	2.023	0.053	1.651
0.85	-1.978	5.660	0.037	2.061	0.057	2.483	0.027	2.056	13.811	0.055	2.310	0.028	1.888	2.392	7.963	0.039	2.071	0.034	1.674
0.90	-1.934	5.770	0.007	2.097	0.039	2.773	-0.004	2.136	1.137	0.032	2.441	-5.980	1.904	3.749	9.201	0.009	2.109	0.008	1.699
0.95	-1.948	5.523	0.053	2.017	0.084	3.186	0.042	2.226	28.432	0.079	2.466	0.042	1.872	2.120	10.479	0.054	2.027	0.049	1.634
1.00	-1.948	5.617	0.041	2.062	0.124	4.793	0.007	2.838	-7.051	0.098	2.631	0.015	1.872	14.759	11.615	0.043	2.066	0.038	1.671
	-1.924	5.536	0.065	2.082	1.100	12.594	-0.491	5.432	325.198	0.148	2.391	0.021	1.707	31.565	3.608	0.066	2.082	0.059	1.686

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2						
	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	
0.00	-4.099	18.693	0.048	2.078	0.048	2.078	0.031	1.897	5.402	0.048	2.078	0.032	1.797	2.085	4.435	0.048	2.078	0.037	1.374
0.25	-4.098	18.672	0.047	2.167	0.053	2.190	0.029	1.935	7.151	0.053	2.185	0.030	1.812	2.892	5.576	0.049	2.169	0.037	1.412
0.50	-4.107	18.825	0.046	2.327	0.063	2.431	0.019	2.057	9.911	0.062	2.400	0.022	1.892	6.412	7.012	0.050	2.330	0.035	1.507
0.60	-4.056	18.439	0.088	2.398	0.104	2.579	0.068	2.135	11.940	0.103	2.515	0.069	1.936	1.268	7.875	0.091	2.400	0.072	1.552
0.70	-4.076	18.568	0.064	2.428	0.085	2.840	0.052	2.168	16.094	0.084	2.675	0.053	1.913	-0.403	9.473	0.067	2.435	0.054	1.561
0.80	-4.087	18.644	0.052	2.453	0.102	3.226	0.025	2.289	22.802	0.093	2.828	0.031	1.916	7.224	10.975	0.056	2.458	0.043	1.576
0.85	-4.047	18.379	0.098	2.541	0.140	3.747	0.071	2.462	30.593	0.136	3.037	0.073	1.959	3.673	12.252	0.100	2.546	0.080	1.630
0.90	-4.120	18.902	0.015	2.468	0.109	4.543	-0.019	2.700	45.798	0.078	3.069	-0.005	1.904	12.479	13.183	0.017	2.470	0.013	1.583
0.95	-4.037	18.237	0.106	2.523	0.270	7.423	0.027	3.411	90.703	0.200	3.324	0.053	1.848	23.440	13.191	0.108	2.528	0.085	1.615
1.00	-4.058	18.359	0.083	2.489	0.250	18.683	0.066	6.682	285.967	0.168	2.871	0.040	1.642	25.874	4.077	0.083	2.488	0.067	1.594

* Biases and MSEs are in 10^{-4} .

Table 3.5: Simulated biases and MSEs of maximum likelihood estimates and Bayesian estimates of $MTTF$, $MTTF_p$ and p for the data simulated from $Gamma(4, 8)$ with inspection time points (12, 35, 50) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2			
	$MTTF$		p	$MTTF$		p	$MTTF_p$		p	$MTTF$		p	$MTTF_p$		p	
	Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		
$p = 0.01$																
0.00	-0.165	2.081	0.119	2.030	0.119	2.030	0.120	2.030	0.119	2.030	0.120	2.030	0.120	2.013	-0.571	0.197
0.25	-0.260	2.045	0.033	1.951	0.035	1.954	0.029	1.947	0.452	0.035	1.953	0.029	1.944	1.591	0.408	0.031
0.50	-0.178	2.046	0.110	2.015	0.111	2.036	0.114	2.004	-1.647	0.657	0.111	2.034	0.113	2.001	-1.228	0.564
0.60	-0.239	2.115	0.052	2.051	0.063	2.089	0.050	2.032	3.547	0.859	0.063	2.083	0.050	2.029	3.435	0.710
0.70	-0.212	2.095	0.074	2.063	0.081	2.138	0.080	2.031	-0.605	1.097	0.082	2.125	0.079	2.029	-1.143	0.852
0.80	-0.249	2.080	0.040	2.024	0.056	2.155	0.050	2.006	0.129	1.822	0.059	2.126	0.050	2.004	1.754	1.271
0.85	-0.232	2.113	0.057	2.065	0.083	2.264	0.072	2.038	1.055	2.457	0.090	2.214	0.070	2.037	4.302	1.588
0.90	-0.206	2.044	0.084	2.013	0.127	2.389	0.107	1.995	2.360	3.559	0.140	2.261	0.105	1.993	8.473	2.029
0.95	-0.237	2.026	0.056	1.984	0.235	2.803	0.104	1.947	32.943	6.621	0.193	2.461	0.094	1.961	26.806	2.977
1.00	-0.209	2.133	0.082	2.105	0.764	3.753	0.116	1.985	183.962	15.836	0.468	3.011	0.142	2.031	93.011	6.020
$p = 0.02$																
0.00	-0.486	2.376	0.070	2.075	0.070	2.075	0.075	2.076	-1.914	0.655	0.070	2.075	0.075	2.070	-1.837	0.603
0.25	-0.463	2.324	0.105	2.076	0.106	2.081	0.103	2.055	0.203	0.831	0.106	2.081	0.103	2.050	0.234	0.746
0.50	-0.523	2.346	0.051	2.060	0.061	2.100	0.046	2.022	3.588	1.288	0.061	2.096	0.047	2.016	3.507	1.101
0.60	-0.513	2.324	0.058	2.057	0.070	2.112	0.059	2.012	2.147	1.581	0.070	2.104	0.059	2.006	2.285	1.289
0.70	-0.475	2.322	0.097	2.102	0.119	2.216	0.101	2.052	3.413	2.163	0.119	2.196	0.101	2.045	3.845	1.680
0.80	-0.470	2.384	0.099	2.178	0.111	2.362	0.114	2.144	-3.966	3.256	0.118	2.316	0.112	2.134	-4.515	2.251
0.85	-0.495	2.294	0.078	2.083	0.113	2.468	0.101	2.017	-1.103	4.369	0.125	2.370	0.099	2.014	4.587	2.787
0.90	-0.456	2.278	0.118	2.111	0.137	2.650	0.149	2.061	-9.952	5.963	0.179	2.486	0.143	2.051	6.532	3.366
0.95	-0.478	2.369	0.096	2.182	0.167	3.266	0.149	2.123	-5.087	9.505	0.226	2.825	0.134	2.119	22.069	4.409
1.00	-0.492	2.365	0.082	2.160	0.659	4.019	0.131	2.032	142.957	17.463	0.484	3.205	0.146	2.051	94.439	6.741
$p = 0.05$																
0.00	-1.291	3.952	0.088	2.164	0.088	2.164	0.079	2.124	1.493	1.579	0.088	2.164	0.080	2.108	1.421	1.429
0.25	-1.296	3.834	0.079	2.074	0.079	2.085	0.082	2.018	-2.441	2.021	0.080	2.084	0.082	2.001	-2.218	1.780
0.50	-1.301	3.919	0.078	2.236	0.081	2.293	0.083	2.126	-3.137	2.903	0.082	2.285	0.083	2.107	-2.515	1.423
0.60	-1.340	4.075	0.052	2.273	0.067	2.370	0.053	2.161	1.786	3.592	0.069	2.352	0.053	2.140	2.354	2.886
0.70	-1.287	3.859	0.110	2.255	0.137	2.489	0.110	2.105	3.118	4.693	0.139	2.443	0.111	2.087	4.355	3.547
0.80	-1.316	3.887	0.086	2.207	0.135	2.611	0.096	2.070	4.823	7.053	0.142	2.505	0.096	2.046	8.648	4.770
0.85	-1.308	3.885	0.087	2.232	0.129	2.844	0.113	2.079	-4.366	9.139	0.149	2.646	0.109	2.051	5.718	5.539
0.90	-1.298	3.917	0.096	2.312	0.118	3.230	0.131	2.141	-16.602	12.638	0.174	2.855	0.121	2.109	17.878	6.388
0.95	-1.305	3.915	0.094	2.306	0.041	3.926	0.171	2.149	-58.787	19.735	0.228	3.058	0.137	2.093	17.573	7.323
1.00	-1.355	4.044	0.046	2.325	0.316	4.695	0.128	2.115	28.277	27.749	0.406	3.369	0.102	2.051	79.367	8.406

* Biases and MSEs are in 10^{-4} .

Table 3.5: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE							
	Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE								
0.00	-2.614	9.210	0.070	2.205	0.057	2.109	1.892	2.969	0.070	2.205	0.058	2.070	1.774	2.603	0.070	2.205	0.061	1.841	1.841	0.754	0.470	
0.25	-2.629	9.227	0.063	2.217	0.064	2.230	0.062	2.079	0.064	2.230	0.062	2.036	-1.584	3.214	0.065	2.219	0.060	1.830	1.830	-0.487	0.450	
0.50	-2.646	9.410	0.066	2.442	0.075	2.541	0.057	2.183	0.075	2.541	0.058	2.138	1.294	4.215	0.072	2.456	0.062	1.986	1.986	0.625	0.400	
0.60	-2.610	9.241	0.100	2.458	0.114	2.629	0.094	2.199	0.114	2.629	0.094	2.147	0.790	4.981	0.109	2.475	0.095	2.000	2.000	0.537	0.376	
0.70	-2.597	9.117	0.113	2.474	0.132	2.758	0.112	2.169	0.132	2.758	0.112	2.117	0.315	5.597	0.124	2.496	0.110	2.006	2.006	0.315	0.318	
0.80	-2.583	9.084	0.132	2.589	0.154	3.170	0.140	2.274	0.154	3.170	0.140	2.274	-8.420	7.243	0.146	2.619	0.129	2.106	2.106	0.286	0.269	
0.85	-2.614	9.300	0.106	2.655	0.131	3.398	0.130	2.329	0.131	3.398	0.130	2.248	1.173	8.113	0.122	2.682	0.109	2.162	2.162	0.286	0.224	
0.90	-2.589	9.034	0.156	2.589	0.180	3.978	0.179	2.229	0.180	3.978	0.179	2.134	12.286	9.642	0.175	2.629	0.152	2.101	2.101	1.537	0.188	
0.95	-2.599	9.151	0.137	2.673	-0.012	5.118	0.224	2.356	-0.012	5.118	0.224	2.186	8.925	10.244	0.154	2.711	0.137	2.173	2.173	0.291	0.129	
1.00	-2.629	9.331	0.100	2.749	-0.071	6.633	0.245	2.397	-0.071	6.633	0.245	2.178	46.269	9.152	0.114	2.804	0.102	2.244	2.244	-0.197	0.058	
$p = 0.20$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
w	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	$MTTF$		MSE	
	Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		Bias	MSE		
0.00	-5.192	29.566	0.057	2.502	0.057	2.503	0.077	2.132	-9.820	5.165	0.057	2.503	0.074	2.037	-8.895	4.240	0.057	2.503	0.054	1.641	-2.681	0.385
0.25	-5.189	29.376	0.102	2.602	0.106	2.619	0.081	2.165	-0.009	6.455	0.106	2.616	0.082	2.058	5.105	5.077	0.104	2.603	0.083	1.691	0.060	0.353
0.50	-5.206	29.813	0.118	2.847	0.136	2.985	0.078	2.264	6.091	8.952	0.136	2.951	0.082	2.139	5.976	6.327	0.124	2.856	0.094	1.831	1.445	0.310
0.60	-5.187	29.522	0.134	2.833	0.155	3.072	0.102	2.230	1.629	10.602	0.157	3.002	0.105	2.095	2.825	7.331	0.141	2.843	0.110	1.817	0.640	0.281
0.70	-5.165	29.376	0.167	3.082	0.198	3.522	0.135	2.362	-0.363	12.988	0.203	3.373	0.138	2.215	2.837	8.291	0.177	3.098	0.139	1.976	0.456	0.240
0.80	-5.162	29.292	0.184	3.163	0.193	4.063	0.169	2.325	-17.173	17.394	0.219	3.693	0.167	2.168	-4.817	9.591	0.193	3.191	0.156	2.018	-0.707	0.193
0.85	-5.203	29.610	0.148	3.123	0.174	4.377	0.139	2.294	-16.554	21.973	0.215	3.771	0.136	2.115	2.777	10.628	0.161	3.151	0.127	1.995	0.096	0.166
0.90	-5.202	29.670	0.154	3.222	0.172	4.999	0.157	2.370	-28.756	28.291	0.251	3.990	0.148	2.157	7.036	11.195	0.167	3.247	0.133	2.062	0.110	0.130
0.95	-5.164	29.439	0.185	3.431	0.034	7.081	0.240	2.598	-110.721	52.961	0.313	4.378	0.188	2.263	8.678	12.055	0.198	3.459	0.159	2.203	-0.474	0.093
1.00	-5.187	29.534	0.166	3.455	-0.693	12.825	0.427	2.813	-419.207	133.862	0.315	4.487	0.190	2.183	9.293	9.573	0.174	3.496	0.144	2.217	-1.708	0.053

* Biases and MSEs are in 10^{-4} .

Table 3.6: Simulated biases and MSEs of maximum likelihood estimates and Bayesian estimates of $MTTF$, $MTTF_p$ and p for the data simulated from $Gamma(4, 8)$ with inspection time points (20, 35, 65) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2			
	$MTTF$		p	$MTTF$		p	$MTTF_p$		p	$MTTF$		p	$MTTF$		p	
	Bias	MSE	Bias*	MSE	Bias	MSE	Bias	MSE	Bias*	MSE*	Bias	MSE	Bias	MSE	Bias*	MSE*
$p = 0.01$																
0.00	-0.253	1.995	-0.050	1.949	-0.050	1.949	-0.052	1.948	0.854	0.345	-0.050	1.949	-0.052	1.945	0.822	0.319
0.25	-0.222	2.006	-0.018	1.982	-0.020	1.972	-0.020	1.972	0.971	0.449	-0.017	1.984	-0.020	1.969	0.929	0.405
0.50	-0.212	1.950	-0.009	1.933	-0.010	1.928	-0.010	1.928	0.373	0.650	-0.008	1.938	-0.010	1.924	0.372	0.559
0.60	-0.222	2.025	-0.016	2.008	-0.013	2.026	-0.021	1.994	2.353	0.828	-0.013	2.023	-0.021	1.989	2.188	0.686
0.70	-0.184	1.971	-0.001	1.966	0.021	1.991	0.016	1.963	0.847	1.127	0.021	1.985	0.016	1.955	0.850	0.877
0.80	-0.205	1.961	-0.001	1.954	0.004	1.954	-0.005	1.947	1.765	1.704	0.003	1.998	-0.004	1.935	1.667	1.173
0.85	-0.227	2.022	-0.025	2.005	-0.019	2.073	-0.027	2.022	2.209	2.350	-0.020	2.046	-0.026	2.001	1.321	1.439
0.90	-0.208	2.047	-0.007	2.039	-0.006	2.171	-0.004	2.072	-2.522	3.596	-0.005	2.104	-0.005	2.035	-0.983	1.775
0.95	-0.188	2.023	0.014	2.025	0.072	2.622	-0.016	2.179	19.213	11.359	0.032	2.149	0.005	2.026	6.566	6.614
1.00	-0.234	2.043	-0.032	2.027	1.650	9.881	-0.932	4.667	701.373	138.268	0.036	2.321	-0.067	2.013	29.025	3.363
$p = 0.02$																
0.00	-0.410	2.098	-0.002	1.972	0.002	1.972	5.333	1.951	-0.841	0.651	-0.002	1.972	4.227	1.946	-0.806	0.600
0.25	-0.390	2.096	0.012	2.004	0.013	2.010	0.015	1.970	-1.081	0.852	0.013	2.010	0.015	1.964	-1.012	0.764
0.50	-0.426	2.200	-0.020	2.074	-0.019	2.085	-0.019	2.062	-0.204	1.341	-0.019	2.083	-0.019	2.052	-0.144	1.140
0.60	-0.418	2.072	-0.030	1.949	-0.031	1.980	-0.025	1.930	-2.394	1.651	-0.030	1.975	-0.025	1.918	-2.080	1.352
0.70	-0.428	2.215	-0.011	2.089	-0.006	2.127	-0.014	2.092	1.568	2.216	-0.006	2.117	-0.014	2.073	1.605	1.698
0.80	-0.428	2.215	-0.023	2.101	-0.015	2.187	-0.027	2.110	2.347	3.335	-0.016	2.158	-0.026	2.081	2.334	2.249
0.85	-0.438	2.117	-0.031	1.992	-0.018	2.125	-0.039	2.030	4.184	4.632	-0.020	2.069	-0.037	1.984	3.998	2.766
0.90	-0.441	2.130	-0.036	2.007	-0.010	2.329	-0.047	2.059	6.797	7.652	-0.018	2.164	-0.044	1.985	5.908	3.615
0.95	-0.449	2.152	-0.044	2.024	9.856	2.765	-0.062	2.271	8.832	16.819	-0.024	2.182	-0.051	2.026	6.128	4.153
1.00	-0.432	2.077	-0.027	1.962	1.552	9.730	-0.857	4.590	643.495	137.291	0.068	2.293	-0.077	1.951	41.038	4.748
$p = 0.05$																
0.00	-1.024	3.049	-0.004	2.084	-0.003	2.084	0.004	2.070	-2.027	1.602	-0.003	2.084	0.003	2.053	-1.927	1.450
0.25	-1.043	3.057	-0.016	2.116	-0.014	2.127	-0.020	2.035	0.356	2.166	-0.014	2.126	-0.020	2.017	1.407	1.901
0.50	-1.045	3.065	-0.026	2.105	-0.022	2.140	-0.025	2.059	3.153	3.153	-0.022	2.134	-0.025	2.032	0.460	2.606
0.60	-1.027	3.023	-0.007	2.108	-0.003	2.150	-0.005	2.088	-0.698	3.709	-0.003	2.139	-0.005	2.052	-0.399	3.002
0.70	-1.028	3.041	-0.004	2.140	0.007	2.237	-0.009	2.119	2.640	5.112	0.006	2.209	-0.008	2.072	2.767	3.760
0.80	-1.048	3.057	-0.025	2.135	-0.007	2.405	-0.033	2.105	3.822	7.792	-0.009	2.311	-0.032	2.038	4.361	4.971
0.85	-1.025	3.047	-0.004	2.173	0.011	2.512	-0.008	2.204	0.072	10.328	0.010	2.359	-0.008	2.097	2.001	5.731
0.90	-1.022	3.041	2.031	2.179	0.047	2.846	-0.020	2.296	9.757	16.641	0.033	2.466	-0.013	2.108	9.684	7.186
0.95	-1.046	3.070	-0.026	2.161	0.056	3.875	-0.054	2.602	9.125	36.283	0.027	2.585	-0.046	2.092	16.386	8.805
1.00	-1.035	3.037	-0.014	2.152	1.417	11.131	-0.725	4.910	534.624	150.756	0.138	2.736	-0.087	2.024	61.775	7.321

* Biases and MSEs are in 10^{-4} .

Table 3.6: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
	\widehat{MTTF}		\widehat{p}	\widehat{MTTF}		\widehat{p}	\widehat{MTTF}		\widehat{p}	\widehat{MTTF}		\widehat{p}	\widehat{MTTF}		\widehat{p}
	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*	Bias	MSE	MSE*
$p = 0.10$															
0.00	-2.085	6.327	2.197	-0.007	2.197	3.134	-0.007	2.197	3.100	-0.017	2.043	3.100	-0.007	2.197	3.100
0.25	-2.104	6.437	2.242	-0.024	2.252	3.832	-0.025	2.250	5.577	-0.041	2.072	5.557	-0.027	2.243	5.557
0.50	-2.068	6.308	2.301	0.007	2.356	5.673	0.007	2.344	2.081	-0.004	2.109	2.081	0.002	2.304	4.442
0.60	-2.089	6.466	2.420	-0.030	2.528	7.094	-0.018	2.277	-4.440	-0.019	2.196	-4.440	-0.031	2.425	5.257
0.70	-2.080	6.253	2.249	-0.014	2.439	5.717	-0.013	2.376	-3.561	-0.009	2.060	-3.561	-0.017	2.252	4.222
0.80	-2.080	6.308	2.343	0.019	2.835	13.931	0.014	2.637	6.230	-0.022	2.081	6.230	-0.010	2.357	8.015
0.85	-2.054	6.142	2.343	0.019	2.835	13.931	0.014	2.637	6.230	-0.022	2.081	6.230	-0.010	2.357	8.015
0.90	-2.079	6.255	2.291	0.059	3.014	18.901	0.050	2.652	9.277	0.005	2.068	9.277	0.018	2.300	10.559
0.95	-2.105	6.439	2.407	0.071	3.607	29.079	0.044	2.784	13.972	-0.027	2.068	13.972	-0.009	2.308	12.179
1.00	-2.101	6.370	2.356	1.054	13.008	327.335	1.054	13.008	60.185	-0.103	1.984	60.185	-0.036	2.356	7.146
$p = 0.20$															
0.00	-4.276	20.362	2.543	-0.025	2.543	5.230	-0.025	2.543	-0.370	-0.019	2.058	-0.370	-0.025	2.543	4.294
0.25	-4.282	20.403	2.569	-0.034	2.588	6.830	-0.034	2.584	5.327	-0.032	2.059	5.327	-0.037	2.570	5.327
0.50	-4.268	20.228	2.694	-0.009	2.824	9.629	-0.010	2.788	3.416	-0.027	2.076	3.416	-0.021	2.701	6.809
0.60	-4.255	20.139	2.704	0.007	2.924	12.046	0.007	2.859	-2.038	0.001	2.098	-2.038	-0.003	2.710	7.931
0.70	-4.225	19.828	2.653	0.050	3.008	15.018	0.047	2.859	2.575	0.016	2.062	2.575	0.026	2.655	8.885
0.80	-4.268	20.234	2.794	0.061	3.699	23.894	0.043	3.233	15.347	-0.037	2.116	15.347	-0.013	2.801	11.443
0.85	-4.306	20.552	2.805	0.022	4.127	30.788	-0.042	3.543	15.419	-0.075	2.112	15.419	-0.063	2.811	12.390
0.90	-4.279	20.281	2.816	0.065	5.134	44.593	0.030	3.572	20.717	-0.053	2.051	20.717	-0.042	2.828	14.454
0.95	-4.244	20.031	2.885	0.130	8.094	89.235	0.092	3.771	20.717	-0.026	2.075	20.717	-0.003	2.890	13.368
1.00	-4.282	20.307	2.841	0.520	19.059	257.899	0.129	3.487	56.570	-0.098	1.910	56.570	-0.038	2.843	6.811

* Biases and MSEs are in 10^{-4} .

Table 3.7: Simulated biases and MSEs of maximum likelihood estimates and Bayesian estimates of $MTTF$, $MTTF_p$ and p for the data simulated from $Gamma(4, 8)$ with inspection time points (12, 35, 65) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	$MTTF$ Bias	$MTTF$ MSE	$MTTF_p$ MSE	$MTTF$ Bias	$MTTF$ MSE	$MTTF_p$ MSE	$MTTF$ Bias	$MTTF$ MSE	$MTTF_p$ MSE	$MTTF$ Bias	$MTTF$ MSE	$MTTF_p$ MSE	$MTTF$ Bias	$MTTF$ MSE	$MTTF_p$ MSE							
$p = 0.01$																						
0.00	-0.265	2.456	0.046	2.394	0.046	2.394	0.047	2.379	-0.427	0.322	0.046	2.394	0.047	2.377	-0.411	0.298	0.046	2.394	0.047	2.365	-0.323	0.186
0.25	-0.277	2.391	0.032	2.325	0.032	2.328	0.033	2.310	-0.330	0.430	0.033	2.328	0.033	2.308	-0.293	0.388	0.033	2.327	0.033	2.296	-0.173	0.213
0.50	-0.302	2.497	0.014	2.438	0.016	2.452	0.014	2.402	0.271	0.641	0.016	2.449	0.014	2.399	0.381	0.551	0.016	2.441	0.014	2.391	0.468	0.241
0.70	-0.266	2.448	0.048	2.423	0.024	2.468	0.014	2.410	2.293	0.850	0.023	2.466	0.015	2.409	2.259	0.702	0.021	2.449	0.015	2.401	1.535	0.264
0.80	-0.322	2.397	-0.011	2.330	0.062	2.510	-0.001	2.292	2.049	1.149	0.008	2.496	0.052	2.383	2.328	0.896	-0.002	2.453	0.051	2.380	1.392	0.268
0.85	-0.315	2.466	2.386	2.409	0.021	2.614	0.015	2.364	-0.463	1.793	0.002	2.480	-5.280	2.294	1.509	1.248	-0.002	2.359	-0.005	2.288	0.702	0.260
0.90	-0.278	2.417	0.036	2.389	0.096	2.848	0.061	2.342	6.611	3.745	0.105	2.552	0.015	2.366	3.707	1.539	0.012	2.437	0.007	2.369	1.375	0.223
0.95	-0.273	2.499	0.042	2.476	0.240	3.306	0.080	2.418	41.672	6.621	0.208	2.994	0.080	2.427	35.254	3.340	0.064	2.523	0.043	2.353	2.263	0.223
1.00	-0.277	2.477	0.037	2.451	0.720	4.031	0.093	2.350	178.933	14.428	0.473	3.387	0.108	2.368	104.836	3.616	0.062	2.501	0.049	2.422	3.660	0.057
$p = 0.02$																						
0.00	-0.573	2.728	0.041	2.409	0.041	2.409	0.043	2.370	-0.040	0.647	0.041	2.409	0.043	2.365	-0.902	0.595	0.041	2.409	0.042	2.339	-0.656	0.315
0.25	-0.632	2.706	-0.013	2.428	0.078	2.434	-0.010	2.385	-0.758	0.843	-0.012	2.433	-0.010	2.380	-0.678	0.757	-0.011	2.430	-0.010	2.354	-0.368	0.343
0.50	-0.538	2.658	0.079	2.448	0.081	2.499	0.078	2.380	-0.152	1.304	0.082	2.493	0.078	2.375	0.090	1.113	0.083	2.468	0.079	2.358	0.382	0.386
0.60	-0.575	2.721	0.046	2.447	0.052	2.509	0.049	2.380	-0.374	1.637	0.053	2.500	0.049	2.375	0.116	1.947	0.053	2.467	0.050	2.360	0.355	0.394
0.70	-0.560	2.664	0.059	2.426	0.071	2.534	0.066	2.360	-0.310	2.053	0.073	2.515	0.066	2.357	0.525	1.586	0.069	2.454	0.065	2.342	0.604	0.363
0.80	-0.575	2.802	0.044	2.570	0.059	2.821	0.060	2.472	-3.416	3.336	0.070	2.766	0.060	2.473	0.353	2.313	0.057	2.619	0.053	2.475	0.436	0.357
0.85	-0.617	2.774	0.005	2.486	0.037	2.892	0.022	2.397	0.310	4.334	0.054	2.783	0.024	2.397	6.491	2.763	0.024	2.540	0.015	2.397	2.317	0.326
0.90	-0.576	2.746	0.047	2.519	0.092	3.174	0.073	2.426	-0.696	6.167	0.134	2.949	0.074	2.425	13.944	3.493	0.071	2.585	0.058	2.434	3.252	0.281
0.95	-0.576	2.781	0.046	2.553	0.138	3.570	0.096	2.428	2.828	9.169	0.195	3.154	0.092	2.445	25.712	4.442	0.071	2.617	0.061	2.470	2.392	0.213
1.00	-0.581	2.657	0.032	2.489	0.601	4.094	0.094	2.258	138.185	16.183	0.456	3.328	0.109	2.272	97.697	6.840	0.069	2.494	0.055	2.355	3.677	0.095
$p = 0.05$																						
0.00	-1.476	4.747	0.015	2.595	0.015	2.595	0.014	2.503	0.259	1.556	0.015	2.595	0.014	2.488	0.245	1.409	0.015	2.595	0.014	2.390	0.142	0.462
0.25	-1.443	4.522	0.066	2.522	0.069	2.527	0.057	2.428	2.288	1.999	0.069	2.526	0.057	2.411	2.229	1.760	0.069	2.523	0.061	2.316	1.314	0.466
0.50	-1.480	4.523	0.033	2.451	0.046	2.506	0.028	2.320	3.719	2.997	0.047	2.497	0.029	2.303	3.763	2.505	0.042	2.461	0.034	2.230	1.893	0.469
0.60	-1.486	4.649	0.018	2.575	0.024	2.663	0.023	2.427	-1.805	3.615	0.027	2.645	0.023	2.408	-0.760	2.925	0.027	2.584	0.024	2.339	0.262	0.442
0.70	-1.477	4.602	0.034	2.603	0.047	2.894	0.044	2.418	-3.075	4.806	0.052	2.784	0.045	2.401	-0.936	3.630	0.047	2.684	0.043	2.357	0.170	0.414
0.80	-1.501	4.758	0.012	2.735	0.038	3.126	0.023	2.528	-3.028	6.939	0.051	3.012	0.024	2.510	4.203	4.672	0.030	2.765	0.022	2.477	1.719	0.350
0.85	-1.441	4.609	0.075	2.802	0.111	3.468	0.090	2.555	-3.065	9.566	0.142	3.255	0.093	2.543	8.239	5.799	0.096	2.850	0.084	2.536	1.859	0.319
0.90	-1.474	4.624	0.041	2.700	0.044	3.736	0.070	2.450	-20.134	12.729	0.126	3.300	0.070	2.433	9.202	6.498	0.067	2.762	0.056	2.451	1.977	0.259
0.95	-1.460	4.582	0.064	2.733	0.072	4.466	0.107	2.444	-30.545	19.282	0.236	3.517	0.101	2.427	30.900	7.620	0.090	2.787	0.075	2.477	3.138	0.183
1.00	-1.471	4.701	0.048	2.848	0.341	5.142	0.107	2.493	43.588	24.949	0.429	3.854	0.110	2.486	83.640	8.836	0.074	2.898	0.062	2.583	2.351	0.085

* Biases and MSEs are in 10^{-4} .

Table 3.7: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2								
	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}						
0.00	-2.888	10.948	0.015	2.710	0.021	2.509	-2.320	3.077	0.015	2.711	0.020	2.470	-2.170	2.696	0.015	2.711	0.016	2.244	-0.922	0.487	
0.25	-2.853	10.754	0.065	2.840	0.052	2.560	2.093	3.722	0.069	2.855	0.053	2.519	2.032	3.158	0.068	2.843	0.058	2.328	0.859	0.443	
0.50	-2.887	10.919	0.045	2.924	0.032	2.588	4.292	5.132	0.058	2.990	0.034	2.543	4.354	4.097	0.052	2.932	0.042	2.383	1.545	0.390	
0.60	-2.913	10.934	0.009	2.797	0.019	2.447	-1.367	6.420	0.023	2.930	0.012	2.401	-0.092	4.914	0.017	2.811	0.013	2.270	0.273	0.376	
0.70	-2.870	10.827	0.060	2.984	0.073	2.536	-3.831	7.907	0.081	3.222	0.063	2.512	-0.723	5.659	0.070	3.009	0.062	2.418	0.168	0.322	
0.80	-2.849	10.633	0.077	2.971	0.090	2.536	-11.121	11.388	0.112	3.323	0.092	2.481	-2.350	7.213	0.089	2.993	0.080	2.411	-0.210	0.275	
0.85	-2.861	10.677	0.086	3.010	0.119	2.518	-5.358	13.921	0.150	3.566	0.090	2.470	7.374	7.940	0.102	3.046	0.086	2.436	1.458	0.230	
0.90	-2.897	11.045	0.047	3.200	0.044	2.616	-25.210	20.309	0.133	3.843	0.065	2.601	9.260	9.309	0.064	3.220	0.053	2.588	1.287	0.189	
0.95	-2.881	10.883	0.073	3.181	0.023	2.616	-57.514	33.065	0.240	4.045	0.097	2.546	27.641	10.311	0.096	3.220	0.079	2.580	1.929	0.140	
1.00	-2.856	10.702	0.102	3.192	0.022	2.576	-90.172	46.235	0.387	4.258	0.138	2.512	55.277	10.294	0.123	3.237	0.106	2.592	1.252	0.079	
$p = 0.20$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2								
w	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}	MTTF	MSE	\hat{p}
0.00	-5.577	33.673	0.054	2.880	0.054	2.393	-3.433	5.316	0.054	2.880	0.054	2.295	-3.111	4.364	0.054	2.880	0.046	1.885	-0.936	0.306	
0.25	-5.591	33.821	0.082	3.135	0.088	2.448	3.062	6.270	0.088	3.159	0.057	2.348	2.834	4.932	0.084	3.139	0.064	2.021	0.786	0.343	
0.50	-5.637	34.337	0.036	3.287	0.032	2.547	3.340	8.627	0.053	3.373	0.024	2.426	3.452	6.284	0.042	3.291	0.030	2.117	0.769	0.297	
0.60	-5.612	34.066	0.050	3.344	0.067	2.566	-2.773	10.492	0.069	3.497	0.048	2.433	-1.078	7.269	0.057	3.350	0.045	2.148	-0.215	0.280	
0.70	-5.642	34.431	0.034	3.419	0.055	2.581	-3.416	12.284	0.062	3.639	0.036	2.441	-0.040	7.864	0.042	3.422	0.033	2.195	-0.005	0.231	
0.80	-5.590	33.744	0.111	3.540	0.149	2.543	-4.467	17.487	0.168	4.088	0.097	2.401	4.259	9.774	0.123	3.562	0.096	2.258	0.561	0.200	
0.85	-5.610	34.135	0.098	3.752	0.135	2.698	-8.299	21.865	0.175	4.354	0.087	2.533	8.030	10.887	0.113	3.756	0.087	2.393	0.971	0.177	
0.90	-5.617	34.148	0.093	3.769	0.097	2.646	-26.637	28.613	0.182	4.632	0.087	2.481	7.949	11.649	0.104	3.798	0.080	2.408	0.606	0.145	
0.95	-5.601	33.975	0.125	3.906	-0.014	2.694	-91.502	48.066	0.244	4.945	0.125	2.496	10.039	12.303	0.137	3.927	0.108	2.487	0.266	0.111	
1.00	-5.612	34.105	0.105	3.965	-0.419	2.744	-266.894	100.871	0.310	5.115	0.129	2.477	25.078	11.572	0.115	3.984	0.093	2.530	-0.334	0.067	

* Biases and MSEs are in 10^{-4} .

Table 3.8: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{MTTF}	Bias	MSE	\widehat{p}	MSE*			
$p = 0.10$	-1.922	14.200	0.354	9.905	0.354	9.904	0.315	8.337	0.755	3.073	0.354	9.905	0.315	8.295	0.712	2.694	0.354	9.905	0.317	8.059	0.302	0.487
0.25	-1.953	14.172	0.343	9.944	0.341	9.963	0.325	8.348	-5.935	3.803	0.341	9.961	0.323	8.300	-5.374	3.213	0.342	9.946	0.314	8.079	-1.947	0.437
0.50	-1.958	14.988	0.335	10.534	0.345	10.608	0.288	8.839	4.327	5.956	0.344	10.590	0.290	8.770	4.193	4.404	0.337	10.533	0.298	8.547	1.284	0.385
0.60	-1.858	14.298	0.447	10.712	0.462	10.881	0.387	8.957	5.191	6.957	0.461	10.838	0.388	8.876	5.159	5.205	0.450	10.719	0.400	8.678	1.350	0.351
0.70	-1.934	14.671	0.368	10.679	0.374	10.855	0.329	8.945	-2.561	9.088	0.374	10.875	0.328	8.837	-0.713	6.234	0.369	10.688	0.332	8.648	-1.036	0.295
0.80	-2.030	14.746	0.281	10.321	0.316	10.850	0.217	8.697	11.526	13.886	0.313	10.626	0.223	8.528	12.096	8.134	0.285	10.326	0.250	8.352	1.790	0.232
0.85	-1.958	14.429	0.354	10.462	0.377	11.236	0.292	8.890	2.815	17.853	0.377	10.854	0.297	8.651	6.741	8.987	0.356	10.469	0.318	8.469	0.358	0.178
0.90	-1.935	14.429	0.381	10.656	0.397	12.032	0.291	9.036	0.069	27.913	0.407	11.219	0.306	8.682	9.759	10.848	0.381	10.669	0.342	8.614	0.032	0.131
0.95	-1.900	14.787	0.421	11.177	0.420	13.945	0.305	10.026	-16.566	55.421	0.481	11.854	0.321	9.151	23.230	12.665	0.421	11.184	0.378	9.046	0.288	0.077
1.00	-1.957	14.644	0.369	10.852	0.649	18.029	-0.141	10.268	113.549	130.781	0.489	11.098	0.237	8.750	53.802	7.459	0.368	10.847	0.331	8.786	0.009	0.005
$p = 0.20$																						
w																						
0.00	-4.453	32.322	0.379	11.361	0.379	11.359	0.318	7.895	-4.478	5.983	0.379	11.361	0.317	7.782	-4.061	4.337	0.379	11.362	0.308	7.313	-1.227	0.393
0.25	-4.498	32.260	0.379	11.457	0.384	11.506	0.306	7.905	-1.221	6.617	0.383	11.498	0.306	7.779	-1.625	5.176	0.381	11.460	0.305	7.359	-0.199	0.349
0.50	-4.476	33.063	0.394	12.434	0.401	12.545	0.319	8.590	-2.699	9.561	0.400	12.500	0.318	8.415	-1.625	6.825	0.395	12.420	0.317	7.986	-0.351	0.294
0.60	-4.437	33.455	0.459	13.342	0.473	13.687	0.363	9.007	-1.126	11.225	0.471	13.572	0.363	8.836	0.299	7.522	0.460	13.346	0.368	8.537	0.005	0.253
0.70	-4.392	32.278	0.496	12.837	0.506	13.444	0.392	8.670	-5.555	15.361	0.507	13.201	0.391	8.449	-1.540	9.254	0.497	12.842	0.398	8.201	-0.324	0.222
0.80	-4.426	32.703	0.500	13.538	0.518	14.632	0.383	9.185	-5.202	21.476	0.520	14.092	0.384	8.865	1.765	10.704	0.500	13.544	0.400	8.651	-0.217	0.160
0.85	-4.530	33.206	0.396	12.914	0.434	14.479	0.275	8.966	0.808	28.759	0.436	13.558	0.284	8.516	10.237	12.121	0.398	12.914	0.316	8.260	0.457	0.129
0.90	-4.352	32.651	0.574	14.141	0.570	16.318	0.419	9.935	-18.898	42.066	0.600	14.754	0.429	9.252	5.337	13.185	0.572	14.127	0.460	9.046	-0.672	0.093
0.95	-4.494	32.613	0.455	13.056	0.481	18.322	0.236	9.402	-19.514	80.072	0.545	13.999	0.289	8.337	31.098	13.726	0.456	13.060	0.362	8.341	0.491	0.053
1.00	-4.441	32.327	0.504	13.306	-0.056	27.783	0.197	11.374	-252.270	237.808	0.611	13.653	0.315	8.359	44.529	6.570	0.504	13.306	0.403	8.510	0.028	0.007

* Biases and MSEs are in 10^{-4} .

Table 3.9: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$							
0.00	-3.278	40.010	0.529	26.680	0.528	26.677	0.476	22.638	-0.054	3.018	0.528	26.678	0.476	22.514	-0.052	2.646	0.529	26.681	0.476	21.783	-0.026	0.478
0.25	-3.326	38.656	0.515	25.671	0.519	25.699	0.461	21.705	0.407	3.813	0.519	25.695	0.462	21.567	0.459	3.223	0.517	25.668	0.463	20.901	0.220	0.440
0.50	-3.294	39.950	0.553	27.226	0.563	27.426	0.499	23.029	-0.372	5.521	0.563	27.378	0.499	22.831	0.043	4.348	0.555	27.221	0.499	22.136	0.049	0.382
0.60	-3.392	39.747	0.449	26.967	0.457	27.367	0.401	22.752	-1.108	6.825	0.457	27.258	0.400	22.515	-0.186	5.110	0.451	26.973	0.404	21.884	0.050	0.347
0.70	-3.313	37.286	0.563	25.663	0.580	26.557	0.492	21.514	0.164	9.106	0.580	26.287	0.492	21.246	-1.631	6.267	0.565	25.697	0.506	20.778	0.268	0.298
0.80	-3.396	38.856	0.480	26.537	0.480	27.460	0.430	22.832	-6.400	13.532	0.488	26.980	0.426	22.297	-1.542	8.001	0.479	26.483	0.435	21.542	-0.602	0.234
0.85	-3.374	38.173	0.501	26.155	0.552	28.133	0.403	22.365	5.408	18.061	0.553	27.112	0.411	21.725	9.566	9.229	0.505	26.149	0.450	21.188	0.741	0.187
0.90	-3.306	38.875	0.579	27.410	0.602	30.487	0.482	23.880	-6.100	26.570	0.631	28.508	0.480	22.775	9.063	10.559	0.580	27.387	0.521	22.210	0.164	0.134
0.95	-3.411	38.750	0.479	26.571	0.378	33.969	0.395	23.971	-45.701	52.105	0.549	28.177	0.367	21.794	13.938	11.727	0.477	26.567	0.431	21.511	-0.369	0.076
1.00	-3.243	39.714	0.651	28.933	0.990	45.637	0.043	27.129	66.544	112.827	0.879	29.765	0.435	23.321	56.752	7.904	0.649	28.918	0.585	23.433	-0.132	0.007
$p = 0.20$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
w	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$	MTTF	MSE	$\hat{\rho}$							
0.00	-7.583	90.324	0.497	28.776	0.497	28.773	0.422	19.970	-4.669	5.367	0.497	28.778	0.420	19.676	-4.236	4.406	0.497	28.776	0.405	18.487	-1.277	0.400
0.25	-7.697	92.317	0.413	29.980	0.414	30.024	0.353	20.837	-4.896	6.546	0.414	30.012	0.351	20.490	-4.266	5.124	0.413	29.974	0.337	19.292	-1.052	0.348
0.50	-7.541	90.552	0.586	31.412	0.592	31.888	0.486	21.484	-6.025	9.341	0.592	31.740	0.485	21.073	-4.409	6.185	0.587	31.406	0.474	20.128	-0.920	0.292
0.60	-7.555	90.481	0.628	32.673	0.664	33.734	0.483	22.250	2.421	11.861	0.660	33.384	0.482	21.538	3.324	7.967	0.633	32.688	0.502	20.907	0.551	0.270
0.70	-7.613	92.332	0.552	32.658	0.611	34.307	0.395	22.096	7.844	14.631	0.604	33.659	0.402	21.538	8.993	8.912	0.559	32.675	0.438	20.871	1.371	0.218
0.80	-7.455	88.736	0.767	33.304	0.811	36.200	0.592	23.019	-3.995	21.831	0.815	34.691	0.591	22.093	3.547	11.037	0.770	33.289	0.615	21.311	-0.019	0.167
0.85	-7.575	89.598	0.644	32.305	0.724	36.495	0.429	22.025	5.704	28.699	0.727	34.046	0.449	21.021	14.106	12.321	0.649	32.294	0.514	20.635	0.769	0.136
0.90	-7.442	88.689	0.771	32.973	0.778	39.145	0.553	22.811	-17.432	39.785	0.833	34.919	0.560	21.269	8.658	12.814	0.770	32.960	0.617	21.067	-0.345	0.093
0.95	-7.639	91.565	0.612	33.918	0.535	47.458	0.400	24.844	-55.084	78.513	0.762	36.445	0.384	21.704	27.636	14.218	0.613	33.911	0.489	21.673	0.168	0.058
1.00	-7.437	87.624	0.835	33.457	-0.325	69.137	0.600	28.912	-315.253	216.788	1.046	34.541	0.527	21.021	47.711	7.691	0.834	33.436	0.668	21.396	-0.086	0.009

* Biases and MSEs are in 10^{-4} .

Table 3.10: Simulated biases and MSEs of maximum likelihood estimates and Bayesian estimates of $MTTF$, $MTTF_p$ and p for the data simulated from $Gamma(1.405, 66.686)$ with inspection time points (20, 80, 140) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*
0.00	-0.009	63.498	0.539	62.358	0.544	61.580	0.544	62.359	0.544	61.550	0.540	62.365	0.542	61.418	-0.785
0.25	0.045	61.437	0.612	60.747	0.616	59.872	0.616	60.764	0.616	59.843	-0.199	0.395	0.612	60.760	0.613
0.50	0.325	65.056	0.891	64.596	0.879	63.228	0.879	64.784	0.879	63.211	-0.112	0.564	0.891	64.705	0.879
0.60	-0.067	62.334	0.507	61.752	0.508	60.950	0.507	60.950	0.508	60.885	-0.470	0.678	0.508	61.741	0.504
0.70	0.036	61.927	0.605	61.518	0.607	61.707	0.607	61.518	0.607	61.549	0.370	0.879	0.607	61.549	0.597
0.80	0.105	63.168	0.674	62.736	0.674	62.939	0.672	62.041	0.674	61.869	-0.696	1.212	0.674	62.727	0.670
0.85	0.091	64.133	0.665	63.601	0.665	64.046	0.665	62.912	0.665	62.644	0.058	1.474	0.665	63.621	0.658
0.90	0.152	66.508	0.723	66.078	0.737	66.920	0.699	65.498	0.737	66.920	1.149	4.003	0.737	66.123	0.714
0.95	-0.010	61.134	0.567	60.726	0.705	63.698	0.402	59.796	0.567	60.726	25.067	9.374	0.567	60.794	0.566
1.00	0.118	63.565	0.693	63.231	2.811	74.906	-0.981	58.973	370.846	45.418	1.128	64.218	0.328	60.285	78.857
$p = 0.02$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
w	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*
0.00	-0.605	64.470	0.526	61.876	0.527	60.139	0.527	61.881	0.527	60.094	-1.043	0.570	0.527	61.883	0.524
0.25	-0.447	63.841	0.708	61.855	0.711	61.883	0.682	59.764	0.682	59.724	1.201	0.812	0.710	61.867	0.686
0.50	-0.315	64.209	0.833	62.807	0.844	62.992	0.803	60.811	0.804	60.730	1.970	1.124	0.839	62.874	0.809
0.60	-0.407	64.872	0.735	63.375	0.772	63.732	0.718	61.571	0.719	61.462	3.352	1.325	0.764	63.592	0.728
0.70	-0.553	66.150	0.599	64.310	0.606	64.507	0.581	62.513	0.581	62.339	0.888	1.084	0.602	64.296	0.584
0.80	-0.544	64.093	0.609	62.486	0.610	63.058	0.595	60.942	0.592	60.630	-0.027	2.169	0.610	62.511	0.596
0.85	-0.505	65.253	0.646	63.611	0.635	64.435	0.643	62.198	0.643	61.718	-1.636	2.674	0.643	63.620	0.636
0.90	-0.448	63.337	0.714	61.964	0.730	63.405	0.687	61.031	0.674	60.215	4.902	3.476	0.719	61.945	0.697
0.95	-0.544	64.800	0.613	63.211	0.728	67.666	0.444	62.073	0.488	61.014	22.221	5.354	0.622	63.273	0.592
1.00	-0.481	64.562	0.680	63.212	2.492	75.244	-0.789	59.733	313.542	47.002	1.143	64.237	0.293	59.450	82.853
$p = 0.05$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
w	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*	\widehat{MTTF} Bias	\widehat{MTTF} MSE	\widehat{MTTF} MSE*
0.00	-2.178	77.269	0.709	65.406	0.709	65.403	0.683	60.779	0.683	60.639	-0.770	1.423	0.709	65.415	0.679
0.25	-2.195	75.805	0.710	66.578	0.712	66.749	0.675	60.622	0.675	60.516	-0.552	1.787	0.711	66.640	0.675
0.50	-2.125	76.982	0.805	66.540	0.820	66.354	0.755	62.169	0.755	61.867	2.428	2.505	0.811	66.379	0.761
0.60	-2.259	77.777	0.690	67.855	0.681	68.070	0.675	62.876	0.684	67.903	-3.169	2.943	0.688	67.527	0.661
0.70	-2.174	71.971	0.756	63.383	0.764	64.421	0.704	58.257	0.706	64.154	0.718	3.729	0.718	63.456	0.714
0.80	-2.120	74.753	0.818	66.873	0.814	68.566	0.765	61.715	0.765	61.060	-0.241	4.896	0.816	66.928	0.776
0.85	-2.219	75.409	0.729	67.043	0.719	69.623	0.679	62.247	0.679	62.247	0.324	5.915	0.726	67.075	0.692
0.90	-2.139	75.694	0.827	67.653	0.858	71.087	0.731	64.391	0.731	62.317	11.255	7.525	0.830	67.570	0.782
0.95	-2.275	73.770	0.691	65.195	0.623	74.390	0.583	62.331	0.583	62.331	24.664	9.240	0.694	65.247	0.650
1.00	-2.172	77.164	0.804	69.477	1.728	86.407	-0.210	63.948	159.302	55.110	1.263	70.853	0.387	61.583	80.351

* Biases and MSEs are in 10^{-4} .

Table 3.11: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}							
0.00	0.701	5.323	0.947	5.423	0.947	5.423	0.948	5.348	0.959	0.068	0.947	5.423	0.931	4.880	0.959	0.066	0.947	5.423	0.942	4.880	0.999	0.043
0.25	0.697	5.318	0.952	5.473	0.952	5.485	0.954	5.379	0.954	0.078	0.952	5.472	0.936	4.925	0.963	0.074	0.952	5.473	0.951	4.925	0.999	0.045
0.50	0.708	5.315	0.953	5.524	0.954	5.601	0.951	5.428	0.950	0.095	0.951	5.523	0.933	4.970	0.956	0.086	0.952	5.523	0.952	4.971	1.000	0.047
0.60	0.695	5.302	0.951	5.536	0.950	5.679	0.955	5.457	0.956	0.106	0.945	5.536	0.937	4.978	0.963	0.093	0.951	5.536	0.949	4.981	1.000	0.048
0.70	0.693	5.288	0.954	5.543	0.953	5.801	0.951	5.500	0.955	0.122	0.944	5.543	0.932	4.986	0.958	0.101	0.955	5.543	0.951	4.988	1.000	0.049
0.80	0.689	5.316	0.953	5.599	0.951	6.103	0.955	5.632	0.960	0.153	0.935	5.599	0.936	5.035	0.957	0.119	0.950	5.599	0.952	5.039	1.000	0.051
0.85	0.688	5.297	0.953	5.594	0.940	6.330	0.953	5.712	0.961	0.186	0.926	5.593	0.934	5.028	0.961	0.119	0.950	5.594	0.953	5.034	1.000	0.051
0.90	0.697	5.280	0.951	5.598	0.937	6.781	0.952	5.903	0.961	0.263	0.928	5.596	0.937	5.031	0.965	0.127	0.950	5.597	0.951	5.038	1.000	0.052
0.95	0.691	5.305	0.950	5.628	0.993	7.711	0.937	6.321	0.932	0.473	0.923	5.628	0.939	5.053	0.962	0.139	0.955	5.628	0.958	5.065	1.000	0.053
1.00	0.696	5.296	0.954	5.631	0.745	14.377	0.867	9.781	0.770	0.762	0.938	5.634	0.955	5.053	0.991	0.135	0.953	5.631	0.953	5.068	1.000	0.053
<hr/>																						
w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}				
0.00	0.198	5.473	0.956	5.758	0.956	5.758	0.956	5.452	0.950	0.090	0.956	5.758	0.915	4.603	0.965	0.086	0.956	5.758	0.952	4.605	1.000	0.047
0.25	0.199	5.469	0.954	5.853	0.956	5.876	0.954	5.504	0.942	0.102	0.955	5.853	0.916	4.679	0.954	0.095	0.955	5.853	0.955	4.682	1.000	0.049
0.50	0.208	5.468	0.955	5.962	0.955	6.098	0.949	5.592	0.949	0.122	0.950	5.962	0.918	4.763	0.960	0.108	0.955	5.961	0.951	4.768	1.000	0.050
0.60	0.214	5.475	0.950	6.013	0.949	6.258	0.950	5.654	0.953	0.134	0.945	6.012	0.917	4.805	0.965	0.115	0.948	6.013	0.948	4.810	1.000	0.051
0.70	0.211	5.481	0.952	6.069	0.956	6.515	0.951	5.746	0.944	0.153	0.942	6.067	0.922	4.849	0.955	0.123	0.954	6.069	0.951	4.855	1.000	0.051
0.80	0.205	5.468	0.951	6.117	0.949	6.987	0.949	5.909	0.950	0.185	0.934	6.116	0.920	4.883	0.958	0.134	0.952	6.116	0.951	4.893	1.000	0.052
0.85	0.215	5.481	0.947	6.159	0.943	7.419	0.947	6.070	0.949	0.211	0.926	6.157	0.920	4.918	0.956	0.140	0.948	6.159	0.947	4.927	1.000	0.053
0.90	0.197	5.453	0.949	6.160	0.938	8.173	0.949	6.332	0.945	0.257	0.923	6.161	0.925	4.915	0.959	0.148	0.949	6.160	0.949	4.927	1.000	0.053
0.95	0.216	5.463	0.953	6.202	0.879	9.747	0.923	6.962	0.922	0.377	0.910	6.205	0.931	4.944	0.966	0.156	0.952	6.202	0.953	4.961	1.000	0.053
1.00	0.211	5.471	0.953	6.245	0.580	18.159	0.790	10.948	0.810	0.748	0.939	6.247	0.946	4.981	0.988	0.168	0.953	6.245	0.953	4.996	1.000	0.054

Table 3.12: Simulated coverage probabilities and average widths of 95% confidence/credible intervals of $MTTF$, $MTTF_p$ and p based on maximum likelihood and Bayesian methods for the data simulated from $Gamma(4, 8)$ with inspection time points (12, 35, 50) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

p	S3 ML											
	S1				S2				S3 ML			
	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}
$p = 0.01$	S3 Bayes with π_1											
	S1				S2				S3 ML			
	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}
	0.948	5.586	0.946	5.536	0.945	5.532	0.947	5.524	0.963	0.081	0.946	5.536
	0.944	5.570	0.947	5.534	0.947	5.536	0.949	5.516	0.963	0.133	0.947	5.533
	0.950	5.575	0.952	5.556	0.949	5.574	0.952	5.538	0.962	0.266	0.948	5.555
	0.60	0.940	5.583	0.945	5.572	0.944	5.608	5.538	0.955	0.327	0.942	5.572
	0.70	0.942	5.574	0.945	5.568	0.944	5.634	0.944	0.956	0.446	0.940	5.568
	0.80	0.949	5.583	0.956	5.584	0.948	5.711	0.957	0.955	0.598	0.945	5.582
	0.85	0.949	5.577	0.950	5.581	0.943	5.740	0.950	0.954	0.674	0.940	5.582
0.90	0.950	5.588	0.950	5.596	0.936	5.798	0.952	0.920	0.758	0.934	5.599	
0.95	0.952	5.565	0.953	5.580	0.919	5.999	0.956	0.859	0.792	0.920	5.585	
1.00	0.944	5.576	0.946	5.592	0.898	7.562	0.956	0.753	0.703	0.888	5.614	
S3 Bayes with π_2												
S1				S2				S3 ML				
CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	
0.930	5.650	0.948	5.558	0.948	5.558	0.942	5.541	0.961	0.036	0.948	5.558	
0.25	0.936	5.658	0.951	5.593	0.950	5.593	5.554	0.968	0.050	0.949	5.592	
0.50	0.935	5.651	0.949	5.615	0.947	5.650	5.562	0.960	0.096	0.946	5.613	
0.60	0.933	5.653	0.948	5.629	0.948	5.690	5.564	0.960	0.145	0.949	5.628	
0.70	0.934	5.658	0.947	5.650	0.945	5.763	5.581	0.951	0.223	0.941	5.648	
0.80	0.933	5.661	0.945	5.667	0.935	5.876	5.598	0.944	0.379	0.936	5.669	
0.85	0.935	5.641	0.948	5.656	0.935	5.930	5.582	0.937	0.472	0.929	5.655	
0.90	0.940	5.650	0.950	5.672	0.930	5.998	5.601	0.921	0.598	0.925	5.672	
0.95	0.932	5.653	0.941	5.682	0.899	6.137	0.941	0.876	0.722	0.902	5.680	
1.00	0.934	5.657	0.948	5.692	0.889	7.936	0.950	0.647	0.679	0.880	5.706	
S3 Bayes with π_1												
S1				S2				S3 ML				
CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	
0.930	5.650	0.948	5.558	0.948	5.558	0.940	5.447	0.942	0.032	0.948	5.558	
0.25	0.936	5.658	0.951	5.593	0.950	5.593	5.554	0.968	0.050	0.949	5.592	
0.50	0.935	5.651	0.949	5.615	0.947	5.650	5.562	0.960	0.096	0.946	5.613	
0.60	0.933	5.653	0.948	5.629	0.948	5.690	5.564	0.960	0.145	0.949	5.628	
0.70	0.934	5.658	0.947	5.650	0.945	5.763	5.581	0.951	0.223	0.941	5.648	
0.80	0.933	5.661	0.945	5.667	0.935	5.876	5.598	0.944	0.379	0.936	5.669	
0.85	0.935	5.641	0.948	5.656	0.935	5.930	5.582	0.937	0.472	0.929	5.655	
0.90	0.940	5.650	0.950	5.672	0.930	5.998	5.601	0.921	0.598	0.925	5.672	
0.95	0.932	5.653	0.941	5.682	0.899	6.137	0.941	0.876	0.722	0.902	5.680	
1.00	0.934	5.657	0.948	5.692	0.889	7.936	0.950	0.647	0.679	0.880	5.706	
S3 Bayes with π_2												
S1				S2				S3 ML				
CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	
0.930	5.650	0.948	5.558	0.948	5.558	0.940	5.447	0.942	0.032	0.948	5.558	
0.25	0.936	5.658	0.951	5.593	0.950	5.593	5.554	0.968	0.050	0.949	5.592	
0.50	0.935	5.651	0.949	5.615	0.947	5.650	5.562	0.960	0.096	0.946	5.613	
0.60	0.933	5.653	0.948	5.629	0.948	5.690	5.564	0.960	0.145	0.949	5.628	
0.70	0.934	5.658	0.947	5.650	0.945	5.763	5.581	0.951	0.223	0.941	5.648	
0.80	0.933	5.661	0.945	5.667	0.935	5.876	5.598	0.944	0.379	0.936	5.669	
0.85	0.935	5.641	0.948	5.656	0.935	5.930	5.582	0.937	0.472	0.929	5.655	
0.90	0.940	5.650	0.950	5.672	0.930	5.998	5.601	0.921	0.598	0.925	5.672	
0.95	0.932	5.653	0.941	5.682	0.899	6.137	0.941	0.876	0.722	0.902	5.680	
1.00	0.934	5.657	0.948	5.692	0.889	7.936	0.950	0.647	0.679	0.880	5.706	
S3 Bayes with π_2												
S1				S2				S3 ML				
CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	CP	AW	$MTTF$	\hat{p}	
0.855	5.843	0.944	5.647	0.944	5.647	0.946	5.594	0.959	0.051	0.944	5.647	
0.25	0.860	5.828	0.952	5.690	0.951	5.700	5.605	0.957	0.057	0.951	5.689	
0.50	0.857	5.839	0.948	5.771	0.944	5.834	5.643	0.959	0.071	0.941	5.767	
0.60	0.842	5.833	0.949	5.800	0.946	5.915	5.652	0.949	0.062	0.943	5.794	
0.70	0.862	5.837	0.945	5.833	0.937	6.042	5.666	0.956	0.101	0.930	5.823	
0.80	0.861	5.836	0.951	5.874	0.945	6.267	5.682	0.943	0.157	0.934	5.864	
0.85	0.857	5.842	0.949	5.898	0.933	6.436	5.707	0.939	0.223	0.925	5.885	
0.90	0.858	5.845	0.947	5.915	0.922	6.643	5.730	0.927	0.336	0.915	5.907	
0.95	0.854	5.830	0.944	5.933	0.882	6.739	5.738	0.902	0.527	0.903	5.918	
1.00	0.854	5.836	0.947	5.963	0.864	9.049	5.864	0.843	0.639	0.882	5.952	

Table 3.12: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2														
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}												
0.00	0.618	6.077	0.949	5.788	0.949	5.788	0.947	5.648	0.959	0.068	0.949	5.788	0.930	5.207	0.959	0.072	0.949	5.788	0.943	5.208	0.999	0.043					
0.25	0.611	6.077	0.951	5.872	0.952	5.886	0.947	5.680	0.955	0.077	0.952	5.871	0.931	5.284	0.959	0.076	0.951	5.871	0.946	5.284	0.999	0.045					
0.50	0.610	6.076	0.942	5.986	0.941	6.074	0.944	5.722	0.954	0.091	0.939	5.979	0.931	5.379	0.959	0.082	0.942	5.983	0.944	5.384	1.000	0.047					
0.60	0.624	6.090	0.942	6.063	0.943	6.223	0.943	5.793	0.950	0.099	0.937	6.051	0.931	5.444	0.952	0.087	0.943	6.060	0.942	5.453	1.000	0.047					
0.70	0.626	6.087	0.945	6.129	0.941	6.424	0.946	5.830	0.955	0.112	0.935	6.112	0.941	5.499	0.953	0.093	0.944	6.125	0.947	5.512	1.000	0.048					
0.80	0.621	6.092	0.945	6.209	0.931	6.761	0.945	5.830	0.955	0.137	0.921	6.180	0.939	5.560	0.944	0.100	0.943	6.205	0.945	5.584	1.000	0.049					
0.85	0.623	6.087	0.943	6.252	0.931	7.034	0.945	5.853	0.951	0.160	0.916	6.216	0.935	5.591	0.942	0.104	0.942	6.249	0.941	5.623	1.000	0.050					
0.90	0.625	6.095	0.945	6.303	0.908	7.401	0.952	5.886	0.930	0.218	0.902	6.252	0.941	5.616	0.926	0.109	0.942	6.299	0.944	5.668	1.000	0.050					
0.95	0.627	6.094	0.942	6.355	0.855	7.730	0.949	5.943	0.920	0.377	0.890	6.298	0.939	5.660	0.929	0.115	0.938	6.354	0.940	5.718	1.000	0.051					
1.00	0.614	6.085	0.942	6.392	0.806	10.853	0.953	6.242	0.895	0.599	0.884	6.339	0.940	5.673	0.924	0.123	0.940	6.384	0.939	5.745	1.000	0.052					
$p = 0.20$																S3 Bayes with π_1						S3 Bayes with π_2					
w	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}
0.00	0.167	6.421	0.946	6.152	0.946	6.152	0.950	5.721	0.952	0.090	0.946	6.152	0.914	4.925	0.964	0.086	0.946	6.152	0.943	4.923	1.000	0.047					
0.25	0.172	6.442	0.949	6.298	0.948	6.318	0.953	5.780	0.952	0.100	0.948	6.297	0.922	5.035	0.963	0.094	0.948	6.298	0.948	5.037	1.000	0.048					
0.50	0.177	6.452	0.944	6.495	0.942	6.615	0.951	5.857	0.949	0.116	0.938	6.485	0.921	5.182	0.960	0.104	0.944	6.492	0.941	5.192	1.000	0.050					
0.60	0.170	6.444	0.944	6.588	0.943	6.807	0.948	5.893	0.950	0.125	0.936	6.572	0.924	5.252	0.957	0.109	0.944	6.584	0.944	5.266	1.000	0.050					
0.70	0.179	6.456	0.939	6.715	0.934	7.111	0.947	5.947	0.950	0.139	0.924	6.686	0.924	5.343	0.957	0.115	0.938	6.710	0.937	5.367	1.000	0.051					
0.80	0.175	6.441	0.940	6.847	0.928	7.598	0.954	6.002	0.949	0.161	0.913	6.797	0.935	5.437	0.951	0.122	0.938	6.839	0.938	5.471	1.000	0.051					
0.85	0.162	6.426	0.942	6.922	0.920	8.001	0.952	6.037	0.946	0.178	0.908	6.867	0.934	5.487	0.951	0.126	0.940	6.918	0.942	5.534	1.000	0.052					
0.90	0.171	6.435	0.942	7.022	0.919	8.617	0.952	6.102	0.946	0.207	0.904	6.955	0.938	5.554	0.951	0.131	0.940	7.016	0.942	5.612	1.000	0.052					
0.95	0.177	6.457	0.936	7.140	0.877	9.630	0.945	6.214	0.941	0.287	0.896	7.050	0.932	5.628	0.950	0.136	0.934	7.127	0.935	5.701	1.000	0.052					
1.00	0.170	6.427	0.938	7.206	0.725	14.355	0.946	6.861	0.924	0.564	0.886	7.102	0.938	5.671	0.958	0.142	0.937	7.199	0.936	5.760	1.000	0.053					

Table 3.13: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2											
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}									
0.00	0.694	5.524	0.945	5.765	0.945	5.765	0.949	5.626	0.951	0.068	0.945	5.765	0.930	5.185	0.951	0.066	0.945	5.765	0.944	5.187	0.998	0.043		
0.25	0.684	5.528	0.948	5.825	0.949	5.839	0.948	5.662	0.957	0.078	0.949	5.826	0.928	5.239	0.964	0.074	0.948	5.826	0.944	5.241	0.999	0.045		
0.50	0.692	5.532	0.949	5.895	0.950	5.975	0.948	5.720	0.956	0.095	0.947	5.895	0.934	5.303	0.961	0.086	0.948	5.895	0.946	5.305	1.000	0.047		
0.60	0.683	5.516	0.938	5.898	0.942	6.040	0.939	5.733	0.957	0.105	0.938	5.897	0.925	5.308	0.961	0.092	0.939	5.898	0.935	5.308	1.000	0.048		
0.70	0.698	5.530	0.950	5.942	0.951	6.203	0.948	5.796	0.958	0.122	0.944	5.942	0.934	5.347	0.961	0.101	0.952	5.942	0.948	5.348	1.000	0.049		
0.80	0.690	5.520	0.946	5.961	0.944	6.482	0.954	5.883	0.962	0.153	0.934	5.965	0.934	5.361	0.959	0.113	0.945	5.962	0.945	5.365	1.000	0.051		
0.85	0.705	5.527	0.951	5.982	0.949	6.757	0.951	5.973	0.961	0.185	0.934	5.987	0.938	5.380	0.961	0.120	0.949	5.982	0.951	5.383	1.000	0.051		
0.90	0.701	5.532	0.949	6.002	0.942	7.264	0.952	6.141	0.953	0.261	0.930	6.012	0.936	5.396	0.963	0.129	0.949	6.003	0.949	5.402	1.000	0.052		
0.95	0.687	5.515	0.945	6.000	0.884	8.194	0.934	6.459	0.927	0.463	0.915	6.018	0.933	5.393	0.959	0.140	0.943	6.000	0.945	5.399	1.000	0.053		
1.00	0.691	5.522	0.950	6.022	0.793	14.241	0.903	8.366	0.787	0.751	0.929	6.053	0.940	5.409	0.980	0.157	0.950	6.022	0.949	5.420	1.000	0.054		
$p = 0.20$																S3 Bayes with π_2								
w	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}
0.00	0.202	5.594	0.944	6.127	0.944	6.127	0.946	5.698	0.952	0.090	0.944	6.127	0.910	4.900	0.965	0.086	0.944	6.127	0.941	4.901	1.000	0.047		
0.25	0.190	5.580	0.946	6.221	0.946	6.244	0.949	5.746	0.946	0.102	0.945	6.222	0.915	4.975	0.957	0.095	0.946	6.222	0.944	4.977	1.000	0.049		
0.50	0.193	5.580	0.944	6.346	0.944	6.484	0.949	5.836	0.952	0.122	0.941	6.348	0.921	5.073	0.963	0.108	0.944	6.347	0.946	5.077	1.000	0.050		
0.60	0.194	5.591	0.948	6.408	0.947	6.656	0.950	5.897	0.949	0.134	0.939	6.409	0.921	5.125	0.962	0.115	0.946	6.409	0.948	5.127	1.000	0.051		
0.70	0.195	5.594	0.950	6.474	0.954	6.928	0.948	5.983	0.956	0.153	0.943	6.478	0.926	5.175	0.964	0.124	0.950	6.475	0.948	5.179	1.000	0.052		
0.80	0.189	5.592	0.948	6.536	0.945	7.439	0.949	6.128	0.946	0.185	0.929	6.547	0.923	5.220	0.955	0.135	0.947	6.537	0.947	5.228	1.000	0.052		
0.85	0.182	5.572	0.946	6.541	0.943	7.844	0.950	6.285	0.944	0.211	0.923	6.550	0.923	5.222	0.956	0.141	0.945	6.542	0.946	5.232	1.000	0.053		
0.90	0.185	5.559	0.946	6.558	0.927	8.586	0.955	6.455	0.949	0.254	0.916	6.572	0.928	5.240	0.961	0.149	0.946	6.559	0.946	5.247	1.000	0.053		
0.95	0.198	5.581	0.948	6.619	0.888	10.158	0.928	6.959	0.922	0.374	0.915	6.637	0.928	5.287	0.965	0.158	0.948	6.620	0.948	5.295	1.000	0.053		
1.00	0.184	5.586	0.949	6.660	0.667	17.920	0.840	9.978	0.811	0.715	0.929	6.699	0.941	5.317	0.979	0.170	0.948	6.660	0.949	5.328	1.000	0.054		

Table 3.14: Simulated coverage probabilities and average widths of 95% confidence/credible intervals of $MTTF$, $MTTF_p$ and p based on maximum likelihood and Bayesian methods for the data simulated from $Gamma(4, 8)$ with inspection time points (12, 35, 65) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

p	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}				
	CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW	CP	AW
0.00	0.943	6.049	0.942	6.032	0.941	6.015	0.968	0.075	0.943	6.032	0.939	5.972	0.968	0.024	0.943	6.032	0.940	5.972	0.968	0.021		
0.25	0.949	6.048	0.951	6.048	0.950	6.017	0.969	0.132	0.949	6.048	0.951	5.988	0.965	0.028	0.949	6.048	0.952	5.987	0.978	0.024		
0.50	0.944	6.045	0.947	6.060	0.946	6.073	0.947	0.254	0.947	6.073	0.946	6.059	0.946	0.034	0.947	6.059	0.946	5.998	0.980	0.027		
0.60	0.945	6.048	0.950	6.073	0.949	6.105	0.957	0.333	0.950	6.070	0.948	6.008	0.953	0.038	0.950	6.065	0.948	6.003	0.977	0.029		
0.70	0.948	6.048	0.950	6.078	0.947	6.159	0.950	0.440	0.944	6.076	0.949	6.013	0.949	0.042	0.946	6.078	0.949	6.016	0.981	0.031		
0.80	0.946	6.047	0.950	6.084	0.949	6.200	0.949	0.598	0.943	6.088	0.949	6.025	0.936	0.049	0.948	6.088	0.950	6.026	0.984	0.034		
0.85	0.945	6.041	0.951	6.080	0.944	6.237	0.939	0.675	0.939	6.089	0.949	6.016	0.932	0.053	0.949	6.088	0.949	6.022	0.988	0.036		
0.90	0.946	6.067	0.950	6.111	0.932	6.344	0.950	0.835	0.930	6.119	0.949	6.051	0.914	0.059	0.948	6.111	0.948	6.048	0.990	0.038		
0.95	0.945	6.047	0.946	6.095	0.922	6.769	0.946	0.859	0.920	6.109	0.945	6.025	0.883	0.066	0.944	6.092	0.946	6.029	0.993	0.040		
1.00	0.945	6.054	0.945	6.106	0.911	7.880	0.948	0.690	0.899	6.178	0.946	6.049	0.839	0.078	0.942	6.104	0.945	6.041	0.993	0.043		
$p = 0.02$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
w	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}				
	CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW	CP	AW
0.00	0.931	6.100	0.946	6.069	0.946	6.035	0.959	0.038	0.947	6.069	0.944	5.948	0.940	0.032	0.946	6.069	0.946	5.948	0.980	0.027		
0.25	0.930	6.084	0.944	6.084	0.945	6.022	0.965	0.050	0.943	6.087	0.939	5.965	0.955	0.037	0.943	6.083	0.942	5.961	0.984	0.030		
0.50	0.938	6.101	0.948	6.130	0.946	6.074	0.961	0.100	0.945	6.134	0.948	6.011	0.959	0.043	0.947	6.125	0.948	6.002	0.983	0.034		
0.60	0.934	6.094	0.946	6.142	0.946	6.071	0.957	0.152	0.944	6.143	0.944	6.020	0.957	0.047	0.945	6.137	0.945	6.013	0.987	0.035		
0.70	0.934	6.099	0.952	6.173	0.950	6.071	0.960	0.222	0.948	6.157	0.949	6.033	0.960	0.052	0.950	6.156	0.949	6.033	0.991	0.037		
0.80	0.930	6.100	0.939	6.170	0.935	6.378	0.949	0.383	0.928	6.169	0.940	6.044	0.948	0.058	0.937	6.175	0.940	6.051	0.991	0.040		
0.85	0.938	6.090	0.947	6.167	0.939	6.467	0.949	0.468	0.935	6.171	0.947	6.042	0.936	0.063	0.944	6.176	0.945	6.051	0.993	0.041		
0.90	0.934	6.092	0.948	6.178	0.928	6.528	0.947	0.588	0.923	6.237	0.948	6.098	0.917	0.068	0.945	6.182	0.946	6.057	0.995	0.043		
0.95	0.928	6.097	0.944	6.194	0.906	6.854	0.882	0.710	0.908	6.214	0.945	6.072	0.901	0.074	0.941	6.203	0.943	6.078	0.995	0.044		
1.00	0.940	6.101	0.949	6.208	0.916	8.196	0.804	0.667	0.909	6.609	0.955	6.394	0.858	0.086	0.949	6.207	0.950	6.081	0.996	0.046		
$p = 0.05$	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2									
w	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}	$MTTF$		\hat{p}	$MTTF_p$		\hat{p}				
	CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW		CP	AW	CP	AW
0.00	0.842	6.205	0.944	6.165	0.944	6.164	0.942	6.066	0.957	0.051	0.944	6.164	0.935	5.855	0.957	0.049	0.944	6.164	0.940	5.855	0.991	0.037
0.25	0.856	6.209	0.947	6.224	0.949	6.234	0.947	6.091	0.957	0.058	0.949	6.223	0.941	5.910	0.959	0.054	0.948	6.224	0.947	5.912	0.995	0.039
0.50	0.857	6.205	0.953	6.299	0.953	6.363	0.953	6.117	0.956	0.071	0.950	6.298	0.948	5.980	0.953	0.067	0.952	6.297	0.951	5.981	0.997	0.042
0.60	0.849	6.193	0.947	6.319	0.947	6.436	0.946	6.120	0.946	0.080	0.940	6.310	0.942	5.994	0.951	0.067	0.947	6.315	0.945	5.999	0.997	0.043
0.70	0.848	6.196	0.950	6.359	0.946	6.586	0.950	6.153	0.950	0.102	0.940	6.356	0.946	6.037	0.947	0.072	0.948	6.360	0.948	6.041	0.997	0.045
0.80	0.843	6.192	0.947	6.386	0.943	6.778	0.947	6.144	0.950	0.162	0.932	6.383	0.944	6.059	0.942	0.079	0.945	6.389	0.944	6.068	0.998	0.046
0.85	0.850	6.206	0.944	6.422	0.928	6.951	0.946	6.172	0.935	0.228	0.918	6.414	0.942	6.085	0.934	0.084	0.940	6.427	0.942	6.104	1.000	0.047
0.90	0.850	6.205	0.946	6.436	0.919	7.125	0.948	6.195	0.927	0.340	0.920	6.411	0.946	6.110	0.929	0.089	0.943	6.448	0.944	6.124	0.999	0.048
0.95	0.854	6.210	0.946	6.466	0.890	7.475	0.950	6.215	0.901	0.499	0.912	6.472	0.948	6.126	0.912	0.095	0.944	6.475	0.944	6.149	0.999	0.049
1.00	0.842	6.196	0.941	6.485	0.886	9.240	0.949	6.254	0.854	0.594	0.896	6.540	0.945	6.153	0.883	0.104	0.939	6.526	0.941	6.198	0.999	0.050

Table 3.14: (Continued)

w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}
0.00	0.589	6.330	0.946	6.326	0.946	6.097	0.954	0.068	0.946	6.325	0.932	5.693	0.941	5.694	0.999
0.25	0.602	6.348	0.947	6.440	0.949	6.151	0.955	0.077	0.946	6.438	0.936	5.792	0.941	5.793	0.999
0.50	0.588	6.332	0.941	6.542	0.943	6.633	0.945	0.091	0.941	6.537	0.936	5.879	0.941	5.885	1.000
0.60	0.588	6.328	0.948	6.597	0.946	6.760	0.947	0.099	0.941	6.586	0.940	5.925	0.941	5.930	1.000
0.70	0.590	6.331	0.941	6.673	0.942	6.965	0.946	0.112	0.934	6.660	0.939	5.992	0.940	6.004	1.000
0.80	0.603	6.331	0.948	6.733	0.942	7.309	0.948	0.135	0.932	6.751	0.945	6.073	0.941	6.071	1.000
0.85	0.596	6.346	0.946	6.817	0.936	7.589	0.954	0.158	0.925	6.774	0.944	6.088	0.941	6.130	1.000
0.90	0.586	6.334	0.942	6.839	0.912	7.957	0.949	0.222	0.913	6.826	0.941	6.131	0.937	6.170	1.000
0.95	0.586	6.337	0.938	6.892	0.869	8.390	0.949	0.344	0.903	6.942	0.942	6.216	0.933	6.199	1.000
1.00	0.602	6.340	0.943	6.946	0.843	10.819	0.954	0.539	0.893	6.926	0.945	6.190	0.923	6.347	1.000
w	S1			S2			S3 ML			S3 Bayes with π_1			S3 Bayes with π_2		
	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}	MTTF CP	MTTF AW	\hat{p}
0.00	0.118	6.481	0.950	6.706	0.950	6.706	0.951	0.100	0.951	6.715	0.921	5.372	0.951	5.372	1.000
0.25	0.120	6.483	0.946	6.852	0.944	6.883	0.948	0.116	0.943	6.862	0.922	5.485	0.944	5.490	1.000
0.50	0.110	6.479	0.946	7.069	0.947	7.189	0.949	0.246	0.943	7.056	0.927	5.638	0.946	5.650	1.000
0.60	0.114	6.474	0.946	7.161	0.944	7.394	0.949	0.291	0.939	7.160	0.929	5.725	0.946	5.723	1.000
0.70	0.114	6.471	0.947	7.289	0.947	7.683	0.946	0.320	0.939	7.262	0.931	5.805	0.946	5.828	1.000
0.80	0.107	6.489	0.944	7.440	0.937	8.201	0.952	0.393	0.928	7.424	0.937	5.930	0.943	5.949	1.000
0.85	0.114	6.476	0.938	7.505	0.930	8.572	0.949	0.423	0.914	7.492	0.936	5.980	0.936	6.017	1.000
0.90	0.118	6.478	0.945	7.623	0.921	9.141	0.953	0.471	0.910	7.562	0.943	6.037	0.947	6.062	1.000
0.95	0.116	6.476	0.935	7.722	0.893	10.069	0.946	0.544	0.901	7.656	0.938	6.109	0.944	6.137	1.000
1.00	0.119	6.470	0.940	7.781	0.817	13.610	0.954	0.767	0.893	7.748	0.944	6.174	0.937	6.216	1.000

Table 3.15: (Continued)

$p = 0.10$	S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2	
	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW
w										
0.00	0.852	12.830	0.958	12.264	0.960	11.298	0.955	11.034	0.958	12.264
0.25	0.852	12.715	0.960	12.308	0.960	11.319	0.955	11.078	0.958	12.306
0.50	0.846	12.832	0.951	12.450	0.953	11.421	0.952	11.191	0.952	12.445
0.60	0.856	12.894	0.958	12.578	0.956	11.536	0.952	11.302	0.956	12.573
0.70	0.848	12.842	0.958	12.573	0.957	11.547	0.952	11.301	0.958	12.568
0.80	0.850	12.714	0.957	12.483	0.956	11.479	0.952	11.189	0.944	12.476
0.85	0.849	12.821	0.957	12.627	0.956	11.633	0.952	11.319	0.951	12.621
0.90	0.849	12.844	0.961	12.676	0.953	11.694	0.945	11.330	0.961	12.670
0.95	0.854	12.859	0.953	12.717	0.931	13.810	0.914	11.334	0.952	12.713
1.00	0.852	12.839	0.956	12.731	0.865	18.456	0.793	11.276	0.956	12.728
$p = 0.20$										
w										
0.00	0.654	13.899	0.958	13.102	0.958	13.101	0.948	10.483	0.958	13.102
0.25	0.659	13.766	0.956	13.244	0.956	13.264	0.954	10.483	0.958	13.243
0.50	0.658	13.894	0.956	13.513	0.958	13.614	0.950	10.592	0.956	13.243
0.60	0.651	13.937	0.953	13.657	0.954	13.837	0.951	10.799	0.956	13.509
0.70	0.659	13.979	0.956	13.774	0.957	14.077	0.949	10.907	0.953	13.652
0.80	0.662	13.934	0.960	13.910	0.955	14.463	0.951	10.989	0.957	13.766
0.85	0.649	13.882	0.960	13.900	0.962	14.690	0.944	11.077	0.954	13.902
0.90	0.662	14.074	0.960	14.120	0.957	15.277	0.943	11.045	0.947	13.883
0.95	0.655	13.919	0.960	14.089	0.929	16.066	0.914	11.199	0.959	14.112
1.00	0.661	13.923	0.964	14.117	0.801	23.455	0.705	11.107	0.960	14.081

Table 3.16: Simulated coverage probabilities and average widths of 95% confidence/credible intervals of $MTTF$, $MTTF_p$ and p based on maximum likelihood and Bayesian methods for the data simulated from $Gamma(1.405, 40.447)$ with inspection time points (15, 50, 85) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW
0.00	0.949	18.728	0.957	18.603	0.955	18.542	0.948	18.540	0.949	18.395	18.405	0.965	0.069	0.957	18.603	0.957	18.419	0.965	18.604	0.957	18.604	0.958	18.419	0.965	18.419	0.958	18.389	0.971	0.024	
0.25	0.945	18.677	0.950	18.576	0.948	18.540	0.949	18.540	0.949	18.395	18.405	0.965	0.137	0.950	18.575	0.949	18.389	0.960	18.576	0.950	18.576	0.950	18.389	0.971	0.024	0.950	18.389	0.971	0.024	
0.50	0.942	18.735	0.954	18.641	0.954	18.596	0.953	18.441	0.954	18.497	18.441	0.965	0.253	0.957	18.640	0.956	18.451	0.952	18.640	0.957	18.640	0.955	18.452	0.982	0.028	0.952	18.452	0.982	0.028	
0.60	0.948	18.758	0.954	18.669	0.954	18.656	0.950	18.497	0.954	18.497	18.497	0.965	0.339	0.954	18.668	0.952	18.482	0.961	18.668	0.952	18.668	0.952	18.482	0.979	0.030	0.952	18.482	0.979	0.030	
0.70	0.945	18.834	0.951	18.715	0.952	18.705	0.951	18.594	0.953	18.534	18.534	0.965	0.452	0.949	18.712	0.950	18.528	0.947	18.712	0.945	18.712	0.950	18.528	0.982	0.032	0.950	18.528	0.982	0.032	
0.80	0.945	18.834	0.951	18.715	0.952	18.705	0.951	18.594	0.953	18.534	18.534	0.965	0.452	0.949	18.712	0.950	18.528	0.947	18.712	0.945	18.712	0.950	18.528	0.982	0.032	0.950	18.528	0.982	0.032	
0.85	0.951	18.804	0.956	18.741	0.953	18.728	0.954	18.567	0.953	18.507	18.507	0.947	0.673	0.954	18.733	0.956	18.544	0.941	18.733	0.956	18.733	0.956	18.544	0.991	0.038	0.956	18.544	0.991	0.038	
0.90	0.951	18.804	0.956	18.741	0.953	18.728	0.954	18.567	0.953	18.507	18.507	0.947	0.673	0.954	18.733	0.956	18.544	0.941	18.733	0.956	18.733	0.956	18.544	0.991	0.038	0.956	18.544	0.991	0.038	
0.95	0.948	18.736	0.954	18.663	0.946	18.784	0.946	18.784	0.946	18.463	18.463	0.920	0.886	0.953	18.636	0.952	18.443	0.934	18.636	0.952	18.636	0.952	18.443	0.997	0.043	0.952	18.443	0.997	0.043	
1.00	0.944	18.762	0.950	18.691	0.910	22.344	0.914	19.026	0.892	0.760	0.692	0.692	0.760	0.941	18.534	0.943	18.282	0.967	18.534	0.943	18.534	0.943	18.282	1.000	0.046	0.943	18.282	1.000	0.046	
$p = 0.02$	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
w	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
0.00	0.945	19.054	0.956	18.783	0.956	18.784	0.957	18.499	0.957	18.499	0.957	0.037	0.955	18.784	0.956	18.406	0.941	0.032	18.784	0.956	18.784	0.956	18.406	0.978	0.027	0.956	18.406	0.978	0.027	
0.25	0.945	19.022	0.956	18.746	0.954	18.703	0.954	18.410	0.952	18.410	0.965	0.051	0.957	18.781	0.955	18.405	0.955	0.037	18.781	0.957	18.781	0.957	18.405	0.983	0.030	0.957	18.405	0.983	0.030	
0.50	0.945	18.937	0.957	18.746	0.953	18.740	0.955	18.432	0.952	18.432	0.962	0.101	0.956	18.744	0.954	18.372	0.961	0.044	18.744	0.956	18.744	0.956	18.370	0.987	0.034	0.956	18.370	0.987	0.034	
0.60	0.941	18.956	0.953	18.763	0.954	18.762	0.954	18.443	0.953	18.443	0.963	0.149	0.956	18.762	0.953	18.393	0.962	0.049	18.762	0.953	18.762	0.953	18.389	0.991	0.036	0.953	18.389	0.991	0.036	
0.70	0.943	18.940	0.956	18.758	0.958	18.796	0.954	18.442	0.954	18.442	0.960	0.219	0.956	18.749	0.954	18.368	0.958	0.055	18.753	0.956	18.753	0.956	18.374	0.993	0.038	0.956	18.374	0.993	0.038	
0.80	0.942	18.958	0.955	18.800	0.955	18.846	0.950	18.451	0.954	18.451	0.968	0.378	0.954	18.785	0.952	18.408	0.954	0.064	18.795	0.954	18.795	0.954	18.418	0.995	0.041	0.954	18.418	0.995	0.041	
0.85	0.942	19.138	0.953	18.979	0.953	19.086	0.953	18.660	0.953	18.660	0.958	0.490	0.951	18.966	0.952	18.592	0.958	0.069	18.976	0.954	18.976	0.954	18.597	0.997	0.043	0.954	18.597	0.997	0.043	
0.90	0.944	18.946	0.954	18.801	0.952	19.016	0.954	18.490	0.944	18.490	0.944	0.622	0.952	18.767	0.954	18.381	0.948	0.077	18.796	0.954	18.796	0.954	18.417	0.998	0.045	0.954	18.417	0.998	0.045	
0.95	0.948	18.983	0.958	18.845	0.948	19.166	0.952	18.550	0.921	0.791	0.921	0.791	0.953	18.789	0.954	18.398	0.946	0.086	18.840	0.958	18.840	0.957	18.461	1.000	0.047	0.957	18.461	1.000	0.047	
1.00	0.940	18.919	0.952	18.791	0.910	23.034	0.921	19.351	0.725	0.759	0.759	0.759	0.945	18.624	0.943	18.176	0.966	0.101	18.788	0.953	18.788	0.953	18.412	1.000	0.049	0.953	18.412	1.000	0.049	
$p = 0.05$	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
w	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
0.00	0.922	19.574	0.957	18.996	0.957	19.067	0.954	18.271	0.952	18.271	0.952	0.051	0.957	18.996	0.956	18.041	0.952	0.049	18.996	0.957	18.996	0.957	18.044	0.990	0.037	0.957	18.044	0.990	0.037	
0.25	0.920	19.589	0.956	19.059	0.955	19.067	0.954	18.310	0.961	18.310	0.961	0.058	0.955	19.058	0.951	18.099	0.961	0.055	19.058	0.956	19.058	0.956	18.101	0.996	0.039	0.956	18.101	0.996	0.039	
0.50	0.915	19.669	0.951	19.226	0.952	19.270	0.954	18.462	0.964	18.462	0.964	0.072	0.951	19.219	0.952	18.260	0.960	0.064	19.221	0.952	19.221	0.952	18.260	0.998	0.042	0.952	18.260	0.998	0.042	
0.60	0.924	19.729	0.956	19.342	0.958	19.421	0.959	18.568	0.965	18.568	0.965	0.083	0.957	19.332	0.956	18.362	0.960	0.070	19.332	0.956	19.332	0.956	18.367	0.998	0.044	0.956	18.367	0.998	0.044	
0.70	0.921	19.753	0.955	19.380	0.953	19.492	0.956	18.591	0.965	18.591	0.965	0.107	0.951	19.363	0.954	18.397	0.956	0.076	19.372	0.956	19.372	0.956	18.403	0.999	0.045	0.956	18.403	0.999	0.045	
0.80	0.915	19.729	0.956	19.406	0.957	19.625	0.956	18.614	0.959	18.614	0.959	0.165	0.955	19.381	0.952	18.412	0.959	0.085	19.399	0.955	19.399	0.955	18.428	0.999	0.047	0.955	18.428	0.999	0.047	
0.85	0.917	19.726	0.954	19.408	0.952	19.742	0.953	18.636	0.957	18.636	0.957	0.234	0.950	19.358	0.952	18.390	0.959	0.091	19.398	0.954	19.398	0.954	18.428	1.000	0.048	0.954	18.428	1.000	0.048	
0.90	0.911	19.605	0.954	19.298	0.949	19.829	0.947	18.575	0.950	18.575	0.950	0.380	0.947	19.227	0.951	18.268	0.954	0.097	19.291	0.954	19.291	0.954	18.327	1.000	0.049	0.954	18.327	1.000	0.049	
0.95	0.919	19.729	0.957	19.442	0.931	20.321	0.948	18.814	0.915	18.814	0.915	0.600	0.952	19.328	0.950	18.335	0.940	0.106	19.436	0.957	19.436	0.957	18.463	1.000	0.051	0.957	18.463	1.000	0.051	
1.00	0.916	19.674	0.954	19.402	0.902	25.032	0.924	20.145	0.771	0.752	0.752	0.943	19.140	0.945	18.086	0.948	0.121	19.454	0.954	19.454	0.954	18.429	1.000	0.052	0.954	18.429	1.000	0.052		

Table 3.16: (Continued)

w	$p = 0.10$						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2																	
	S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2		S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2		S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2							
	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	\hat{p} CP	\hat{p} AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	\hat{p} CP	\hat{p} AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	\hat{p} CP	\hat{p} AW						
0.00	0.848	20.805	0.952	19.746	0.952	19.745	0.950	18.209	0.957	0.068	0.952	19.745	0.952	19.745	0.946	17.767	0.957	0.066	0.952	19.745	0.952	19.745	0.949	17.770	0.949	17.745	0.952	19.745	0.952	19.745	0.949	17.770	0.949	17.770	0.999	0.043
0.25	0.850	20.643	0.957	19.786	0.958	19.802	0.958	18.202	0.954	0.078	0.958	19.785	0.953	19.785	0.953	17.801	0.961	0.073	0.957	19.784	0.957	19.784	0.957	17.801	0.957	19.784	0.957	19.784	0.957	19.784	0.951	17.804	0.951	17.804	0.999	0.045
0.50	0.844	20.701	0.952	19.983	0.952	20.074	0.950	18.354	0.953	0.093	0.951	19.975	0.946	19.975	0.946	17.976	0.957	0.084	0.952	19.977	0.951	19.977	0.951	17.976	0.951	19.977	0.951	19.977	0.951	17.978	0.951	17.978	1.000	0.047		
0.60	0.846	20.702	0.955	20.043	0.955	20.192	0.957	18.357	0.954	0.104	0.953	20.022	0.952	20.022	0.952	18.019	0.957	0.090	0.955	20.033	0.955	20.033	0.955	18.019	0.955	20.033	0.955	20.033	0.955	18.030	0.955	18.030	1.000	0.048		
0.70	0.857	20.627	0.960	20.078	0.959	20.344	0.959	18.430	0.957	0.119	0.958	20.041	0.956	20.041	0.956	18.031	0.954	0.097	0.960	20.067	0.960	20.067	0.960	18.031	0.954	20.067	0.960	20.067	0.960	18.059	0.960	18.059	1.000	0.049		
0.80	0.850	20.618	0.957	20.152	0.956	20.616	0.954	18.511	0.959	0.148	0.954	20.093	0.951	20.093	0.951	18.092	0.947	0.106	0.956	20.140	0.956	20.140	0.956	18.092	0.947	20.140	0.956	20.140	0.956	18.128	0.956	18.128	1.000	0.050		
0.85	0.851	20.672	0.954	20.192	0.957	20.930	0.956	18.604	0.951	0.181	0.950	20.108	0.953	20.108	0.953	18.083	0.942	0.112	0.953	20.181	0.953	20.181	0.953	18.083	0.942	20.181	0.953	20.181	0.954	18.162	0.954	18.162	1.000	0.051		
0.90	0.852	20.685	0.957	20.261	0.951	21.357	0.952	18.721	0.947	0.252	0.951	20.156	0.954	20.156	0.954	18.133	0.946	0.119	0.958	20.252	0.958	20.252	0.958	18.133	0.946	20.252	0.958	20.252	0.956	18.227	0.956	18.227	1.000	0.051		
0.95	0.848	20.603	0.959	20.228	0.931	22.012	0.948	18.917	0.917	0.450	0.948	20.074	0.952	20.074	0.952	18.048	0.945	0.127	0.958	20.222	0.958	20.222	0.958	18.048	0.945	20.222	0.958	20.222	0.958	18.200	0.958	18.200	1.000	0.052		
1.00	0.853	20.756	0.955	20.409	0.878	28.812	0.927	21.184	0.810	0.728	0.940	20.141	0.949	20.141	0.949	18.044	0.957	0.139	0.954	20.405	0.954	20.405	0.954	18.044	0.957	20.405	0.954	20.405	0.955	18.365	0.955	18.365	1.000	0.053		
w	$p = 0.20$						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2																	
0.00	0.647	22.441	0.960	20.993	0.960	20.992	0.962	17.615	0.946	0.090	0.960	20.993	0.952	20.993	0.952	16.794	0.961	0.086	0.960	20.993	0.960	20.993	0.959	16.794	0.959	20.993	0.960	20.993	0.959	16.794	0.959	16.794	1.000	0.047		
0.25	0.638	22.305	0.953	21.222	0.953	21.253	0.956	17.705	0.953	0.101	0.952	21.219	0.948	21.219	0.948	16.978	0.964	0.094	0.954	21.220	0.954	21.220	0.955	16.978	0.955	21.220	0.954	21.220	0.955	16.977	0.955	16.977	1.000	0.049		
0.50	0.649	22.373	0.954	21.525	0.955	21.698	0.952	17.869	0.956	0.119	0.954	21.508	0.946	21.508	0.946	17.204	0.965	0.106	0.954	21.516	0.954	21.516	0.955	17.204	0.955	21.516	0.954	21.516	0.955	17.213	0.955	17.213	1.000	0.050		
0.60	0.657	22.521	0.954	21.877	0.954	22.178	0.955	18.117	0.948	0.131	0.951	21.849	0.950	21.849	0.950	17.462	0.965	0.112	0.955	21.866	0.954	21.866	0.954	17.462	0.954	21.866	0.954	21.866	0.954	17.490	0.954	17.490	1.000	0.051		
0.70	0.646	22.529	0.953	21.933	0.956	22.450	0.954	18.154	0.949	0.148	0.950	21.885	0.946	21.885	0.946	17.479	0.955	0.119	0.953	21.920	0.953	21.920	0.953	17.479	0.955	21.920	0.953	21.920	0.953	17.532	0.953	17.532	1.000	0.051		
0.80	0.667	22.587	0.959	22.262	0.961	23.200	0.954	18.478	0.946	0.177	0.953	22.168	0.951	22.168	0.951	17.725	0.946	0.127	0.958	22.248	0.958	22.248	0.958	17.725	0.946	22.248	0.958	22.248	0.958	17.798	0.958	17.798	1.000	0.052		
0.85	0.653	22.380	0.954	22.108	0.955	23.403	0.953	18.348	0.943	0.200	0.948	21.950	0.948	21.950	0.948	17.527	0.944	0.132	0.955	22.091	0.955	22.091	0.954	17.527	0.944	22.091	0.954	22.091	0.954	17.671	0.954	17.671	1.000	0.052		
0.90	0.652	22.578	0.968	22.346	0.954	24.355	0.960	18.714	0.941	0.241	0.952	22.140	0.960	22.140	0.960	17.701	0.947	0.138	0.967	22.332	0.967	22.332	0.968	17.701	0.947	22.332	0.967	22.332	0.968	17.866	0.968	17.866	1.000	0.052		
0.95	0.650	22.308	0.957	22.264	0.923	25.611	0.946	18.871	0.908	0.350	0.937	21.969	0.951	21.969	0.951	17.528	0.941	0.144	0.957	22.250	0.957	22.250	0.956	17.528	0.941	22.250	0.956	22.250	0.956	17.799	0.956	17.799	1.000	0.053		
1.00	0.663	22.555	0.962	22.607	0.818	36.828	0.943	22.501	0.836	0.699	0.947	22.247	0.957	22.247	0.957	17.723	0.959	0.152	0.962	22.599	0.962	22.599	0.962	17.723	0.959	22.599	0.962	22.599	0.962	18.079	0.962	18.079	1.000	0.053		

Table 3.17: Simulated coverage probabilities and average widths of 95% confidence/credible intervals of $MTTF$, $MTTF_p$ and p based on maximum likelihood and Bayesian methods for the data simulated from $Gamma(1.405, 66.686)$ with inspection time points (20, 80, 140) under Scenario 1 (S1), Scenario 2 (S2), and Scenario 3 (S3).

w	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW
0.00	0.948	30.982	0.951	30.630	0.951	30.631	0.951	30.631	0.951	30.404	0.964	0.077	0.951	30.631	0.951	30.328	0.964	0.024	0.951	30.631	0.951	30.328	0.964	0.024	0.951	30.631	0.951	30.328	0.964	0.021
0.25	0.950	31.043	0.957	30.750	0.954	30.698	0.953	30.464	0.967	0.143	0.967	0.143	0.957	30.752	0.956	30.448	0.963	0.028	0.957	30.751	0.957	30.446	0.973	0.024	0.957	30.751	0.957	30.446	0.973	0.024
0.50	0.949	31.279	0.952	30.984	0.952	30.953	0.953	30.700	0.960	0.255	0.960	0.255	0.952	30.982	0.952	30.670	0.954	0.035	0.952	30.982	0.952	30.670	0.980	0.028	0.952	30.982	0.952	30.670	0.980	0.028
0.60	0.949	30.878	0.956	30.610	0.953	30.543	0.952	30.288	0.962	0.341	0.955	0.307	0.955	30.607	0.955	30.304	0.959	0.035	0.955	30.608	0.955	30.303	0.981	0.029	0.955	30.608	0.955	30.303	0.981	0.029
0.70	0.949	31.030	0.954	30.763	0.951	30.661	0.952	30.390	0.952	0.446	0.954	0.446	0.954	30.749	0.954	30.440	0.946	0.044	0.954	30.755	0.954	30.446	0.983	0.032	0.954	30.755	0.954	30.446	0.983	0.032
0.80	0.951	31.122	0.954	30.872	0.951	30.870	0.952	30.573	0.952	0.604	0.955	0.558	0.955	30.858	0.952	30.455	0.947	0.058	0.955	30.866	0.954	30.559	0.985	0.035	0.955	30.866	0.954	30.559	0.985	0.035
0.85	0.946	31.023	0.950	30.793	0.948	30.808	0.950	30.493	0.942	0.694	0.948	0.694	0.947	30.767	0.952	30.463	0.937	0.058	0.950	30.785	0.951	30.478	0.989	0.037	0.950	30.785	0.951	30.478	0.989	0.037
0.90	0.943	31.212	0.950	30.968	0.945	31.016	0.946	30.648	0.945	0.781	0.947	0.781	0.947	30.925	0.947	30.615	0.946	0.064	0.950	30.961	0.949	30.650	0.995	0.039	0.950	30.961	0.949	30.650	0.995	0.039
0.95	0.956	31.007	0.958	30.773	0.950	30.834	0.952	30.325	0.899	0.861	0.954	0.826	0.957	30.826	0.957	30.292	0.924	0.073	0.958	30.763	0.959	30.453	0.996	0.042	0.958	30.763	0.959	30.453	0.996	0.042
1.00	0.948	31.087	0.953	30.854	0.933	34.261	0.929	30.654	0.716	0.743	0.942	0.743	0.942	30.827	0.947	29.803	0.931	0.087	0.953	30.849	0.953	30.539	1.000	0.045	0.953	30.849	0.953	30.539	1.000	0.045
$p = 0.02$	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
w	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
0.00	0.946	31.506	0.956	30.866	0.956	30.866	0.956	30.866	0.956	30.403	0.964	0.037	0.956	30.866	0.956	30.252	0.947	0.032	0.956	30.866	0.956	30.251	0.982	0.027	0.956	30.866	0.956	30.251	0.982	0.027
0.25	0.945	31.525	0.960	30.940	0.960	30.942	0.957	30.458	0.962	0.050	0.962	0.050	0.960	30.937	0.956	30.313	0.951	0.037	0.960	30.936	0.958	30.314	0.982	0.030	0.960	30.936	0.958	30.314	0.982	0.030
0.50	0.945	31.574	0.955	31.006	0.953	30.962	0.956	30.450	0.963	0.101	0.951	0.101	0.951	30.999	0.954	30.372	0.963	0.044	0.953	30.999	0.954	30.375	0.986	0.034	0.953	30.999	0.954	30.375	0.986	0.034
0.60	0.945	31.573	0.956	31.042	0.956	31.015	0.953	30.480	0.961	0.144	0.956	0.144	0.956	31.031	0.953	30.400	0.960	0.049	0.956	31.033	0.955	30.406	0.989	0.036	0.956	31.033	0.955	30.406	0.989	0.036
0.70	0.947	31.477	0.955	30.965	0.954	30.945	0.954	30.392	0.958	0.226	0.954	0.226	0.954	30.946	0.955	30.326	0.957	0.054	0.955	30.954	0.955	30.333	0.990	0.038	0.955	30.954	0.955	30.333	0.990	0.038
0.80	0.946	31.473	0.956	30.988	0.957	31.062	0.956	30.460	0.958	0.369	0.956	0.369	0.956	30.956	0.955	30.342	0.958	0.062	0.956	30.977	0.956	30.358	0.996	0.041	0.956	30.977	0.956	30.358	0.996	0.041
0.85	0.946	31.648	0.954	31.178	0.956	31.299	0.955	30.656	0.954	0.495	0.952	0.495	0.952	31.136	0.952	30.525	0.954	0.073	0.954	31.167	0.954	30.547	0.998	0.042	0.954	31.167	0.954	30.547	0.998	0.042
0.90	0.951	31.608	0.961	31.149	0.959	31.380	0.957	30.648	0.940	0.621	0.959	0.621	0.959	30.876	0.948	30.450	0.945	0.082	0.952	31.137	0.961	30.513	0.998	0.044	0.952	31.137	0.961	30.513	0.998	0.044
0.95	0.942	31.544	0.953	31.081	0.938	31.317	0.948	30.453	0.899	0.770	0.947	0.770	0.947	30.876	0.948	30.212	0.929	0.094	0.952	31.066	0.952	30.439	1.000	0.046	0.952	31.066	0.952	30.439	1.000	0.046
1.00	0.945	31.611	0.955	31.170	0.934	35.113	0.927	31.183	0.747	0.742	0.947	0.742	0.947	30.976	0.947	29.769	0.927	0.095	0.956	31.163	0.955	30.538	1.000	0.048	0.955	31.163	0.955	30.538	1.000	0.048
$p = 0.05$	S1						S2						S3 ML						S3 Bayes with π_1						S3 Bayes with π_2					
w	$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}		$MTTF$		$MTTF_p$		\hat{p}	
0.00	0.921	33.062	0.954	31.507	0.954	31.507	0.953	30.307	0.958	0.050	0.955	0.050	0.955	31.508	0.951	29.933	0.958	0.049	0.954	31.508	0.953	29.933	0.991	0.037	0.954	31.508	0.953	29.933	0.991	0.037
0.25	0.917	33.079	0.956	31.641	0.956	31.653	0.957	30.400	0.959	0.058	0.956	0.058	0.956	31.639	0.955	30.054	0.960	0.055	0.956	31.637	0.956	30.053	0.995	0.039	0.956	31.637	0.956	30.053	0.995	0.039
0.50	0.922	32.987	0.955	31.694	0.954	31.758	0.955	30.434	0.964	0.071	0.959	0.071	0.959	31.679	0.953	30.092	0.959	0.064	0.955	31.682	0.955	30.096	0.997	0.042	0.955	31.682	0.955	30.096	0.997	0.042
0.60	0.923	32.871	0.955	31.704	0.954	31.821	0.955	30.462	0.966	0.082	0.962	0.082	0.962	31.684	0.953	30.113	0.958	0.068	0.956	31.689	0.955	30.108	0.998	0.044	0.956	31.689	0.955	30.108	0.998	0.044
0.70	0.923	33.027	0.956	31.840	0.956	31.990	0.956	30.539	0.961	0.104	0.960	0.104	0.960	31.798	0.955	30.206	0.954	0.075	0.957	31.821	0.956	30.227	0.999	0.045	0.957	31.821	0.956	30.227	0.999	0.045
0.80	0.921	33.023	0.953	31.906	0.952	32.109	0.950	30.542	0.956	0.163	0.949	0.163	0.949	31.820	0.950	30.237	0.953	0.083	0.953	31.882	0.954	30.289	0.999	0.047	0.954	31.882	0.954	30.289	0.999	0.047
0.85	0.921	33.024	0.955	31.953	0.952	32.386	0.953	30.688	0.949	0.234	0.950	0.234	0.950	31.835	0.955	30.251	0.949	0.088	0.955	31.930	0.956	30.334	0.999	0.048	0.955	31.930	0.956	30.334	0.999	0.048
0.90	0.927	33.038	0.959	32.009	0.954	32.791	0.952	30.877	0.934	0.371	0.954	0.371	0.954	31.806	0.953	30.207	0.937	0.094	0.960	31.985	0.959	30.385	1.000	0.049	0.960	31.985	0.959	30.385	1.000	0.049
0.95	0.920	32.826	0.960	31.810	0.938	32.827	0.951	30.736	0.903	0.586	0.948	0.586	0.948	31.485	0.952	29.863	0.923	0.114	0.952	31.789	0.959	30.196	1.000	0.050	0.952	31.789	0.959	30.196	1.000	0.050
1.00	0.923	33.085	0.957	32.124	0.925	38.215	0.936	32.553	0.806	0.728	0.945	0.728	0.945	31.453	0.950	29.690	0.926	0.114	0.957	32.113	0.957	30.508	1.000	0.051	0.957	32.113	0.957	30.508	1.000	0.051

Table 3.17: (Continued)

w	S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2											
	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW										
0.00	0.859	35.123	0.958	32.441	0.958	29.913	0.956	0.068	0.958	32.443	0.958	29.187	0.956	0.066	0.958	32.443	0.957	29.194	0.998	0.043
0.25	0.860	35.038	0.951	32.524	0.951	29.911	0.946	0.077	0.951	32.521	0.946	29.252	0.952	0.073	0.952	32.519	0.950	29.261	0.999	0.045
0.50	0.859	35.032	0.958	32.822	0.958	32.940	0.957	0.093	0.957	32.789	0.954	29.492	0.959	0.083	0.957	32.804	0.957	29.518	0.999	0.047
0.60	0.863	34.957	0.955	32.847	0.958	33.065	0.957	0.102	0.956	32.801	0.954	29.535	0.953	0.089	0.957	32.825	0.956	29.546	1.000	0.048
0.70	0.847	34.914	0.951	32.941	0.954	33.322	0.951	0.117	0.950	32.863	0.946	29.598	0.946	0.095	0.951	32.915	0.950	29.628	1.000	0.049
0.80	0.859	35.017	0.954	33.168	0.948	33.792	0.954	0.145	0.947	33.022	0.952	29.731	0.942	0.104	0.954	33.138	0.954	29.825	1.000	0.050
0.85	0.868	35.367	0.960	33.602	0.962	34.577	0.961	0.176	0.943	33.887	0.959	30.032	0.938	0.109	0.960	33.570	0.961	30.211	1.000	0.051
0.90	0.854	34.856	0.952	33.163	0.945	34.655	0.951	0.245	0.943	32.880	0.950	29.589	0.937	0.115	0.953	33.135	0.953	29.821	1.000	0.051
0.95	0.864	35.295	0.958	33.688	0.938	35.907	0.946	0.422	0.947	33.208	0.951	29.817	0.923	0.122	0.958	33.663	0.959	30.294	1.000	0.052
1.00	0.853	35.062	0.955	33.492	0.900	43.460	0.932	0.673	0.940	32.637	0.944	29.157	0.918	0.133	0.956	33.472	0.955	30.122	1.000	0.052

w	S1		S2		S3 ML		S3 Bayes with π_1		S3 Bayes with π_2											
	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW	MTTF CP	MTTF AW										
0.00	0.690	38.811	0.957	34.664	0.957	29.072	0.945	0.090	0.956	34.665	0.948	27.718	0.962	0.086	0.957	34.665	0.956	27.728	1.000	0.047
0.25	0.670	38.635	0.959	34.850	0.958	34.903	0.957	0.101	0.958	34.848	0.947	27.857	0.962	0.094	0.959	34.846	0.959	27.872	1.000	0.048
0.50	0.689	38.928	0.956	35.719	0.956	35.982	0.958	0.118	0.954	35.686	0.953	28.546	0.964	0.105	0.956	35.703	0.956	28.562	1.000	0.050
0.60	0.677	38.851	0.957	35.848	0.957	36.288	0.958	0.129	0.954	35.781	0.950	28.617	0.957	0.110	0.957	35.823	0.958	28.657	1.000	0.050
0.70	0.696	39.386	0.957	36.461	0.955	37.199	0.956	0.145	0.951	36.338	0.951	29.054	0.959	0.117	0.956	36.428	0.956	29.140	1.000	0.051
0.80	0.682	38.994	0.960	36.490	0.961	37.856	0.956	0.172	0.950	36.289	0.952	28.992	0.951	0.125	0.961	36.451	0.959	29.156	1.000	0.052
0.85	0.698	39.265	0.958	36.944	0.956	38.775	0.956	0.193	0.950	36.626	0.952	29.271	0.939	0.129	0.959	36.809	0.958	29.516	1.000	0.052
0.90	0.686	38.714	0.961	36.746	0.956	39.541	0.956	0.231	0.952	36.309	0.953	29.010	0.940	0.134	0.961	36.706	0.961	29.363	1.000	0.052
0.95	0.686	38.372	0.959	36.481	0.937	41.103	0.950	0.330	0.945	35.869	0.951	28.674	0.937	0.139	0.958	36.442	0.958	29.155	1.000	0.052
1.00	0.691	38.566	0.957	36.915	0.861	55.573	0.936	0.653	0.940	36.022	0.950	28.681	0.945	0.146	0.957	36.889	0.957	29.511	1.000	0.053

Chapter 4

Proposed Research Directions

4.1. Reparameterization of the generalized gamma distribution

In the model fitting of the three-parameter generalized gamma distribution, we observe that even though the parameter estimates obtained from different methods are quite different, the fitted density based on different sets of parameter estimates give similar fitted probability density functions. We illustrate this point by using a simulated data set with $n = 200$ observations from generalized gamma distribution $GG(15, 3, 5)$ with a censoring proportion 54%. The parameter estimates obtained from direct optimization method, SEM algorithm, and EM algorithm are $(0.74, 0.988, 32.2)$, $(2.54, 1.30, 19.3)$ and $(20.2, 3.86, 2.52)$, respectively. The fitted probability density functions based on the parameter estimates obtained from these three methods are plotted in Figure 4.1. From Figure 4.1, we observe that although the values of parameter estimates are quite different, the fitted probability density functions are similar.

For this reason, we propose to further investigate the estimation of parameters for generalized gamma distribution with the reparameterization proposed by [Prentice \(1974\)](#) and revisited by [Cox et al. \(2007\)](#), which has the probability density function as

$$f(x; \lambda, \sigma, \beta) = \frac{|\lambda|}{\sigma x \Gamma(\lambda^{-2})} \left[\lambda^{-2} (e^{-\beta} x)^{\frac{\lambda}{\sigma}} \right] \exp \left\{ -\lambda^{-2} (e^{-\beta} x)^{\frac{\lambda}{\sigma}} \right\}, \quad x > 0 \quad (4.1)$$

This parameterization may also relieve the difficulty in computing MLEs of the model parameters.

GG(15, 3, 5) pdf: 54% censoring

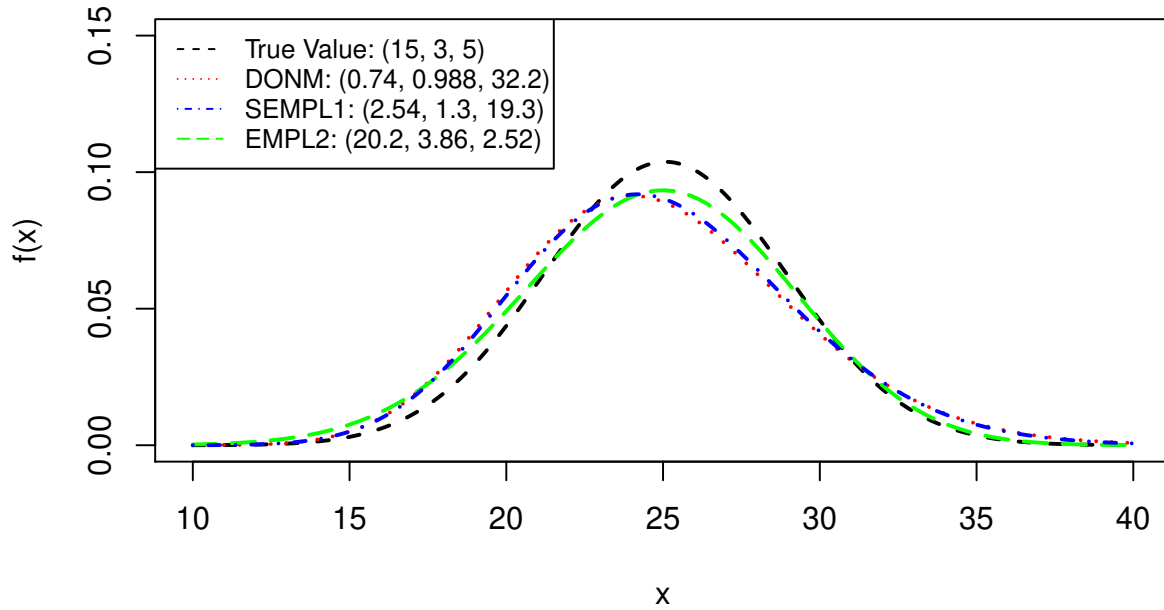


Figure 4.1: Fitted probability density functions (pdfs) based on a simulated data from generalized gamma distribution $GG(15, 3, 5)$ with a censoring proportion 54%

4.2. Accelerated life tests for one-shot device testing with defectives

Accelerated life tests (ALTs) are often used to obtain lifetime information of highly reliable products within a reasonable testing time. The test units are run some higher-than-usual levels of stress (e.g., temperature, voltage, pressure, etc.) to induce early failures (El-Din et al., 2016). For data obtained from an ALT, we can apply the estimation methods proposed in this thesis to study the lifetime distribution and the reliability of the one-shot devices testing with defectives.

4.3. Optimal testing planning for one-shot devices when defectives are involved

Based on the results obtained in Chapter 3, we observe that when some of the one-shot devices are defectives, the performance of statistical point and interval estimation procedures depends on the defective rate and the masking rate. If we consider that it is possible to find out if a one-shot device that failed at the inspection is defective or not (e.g., through an autopsy of the failed unit) and there is a cost involved in this process, then it will be of interest to determinate an optimal testing planning for one-shot devices with defectives when there is a cost constraint. Specifically, suppose the experimenter wants to control the variance of $MTTF$ estimates below a prefixed value V , the cost of inspecting a unit is C_I , the cost of finding out a one-shot device failed at the inspection is defective or not is C_D , when there are N one-shot devices available for the experiment with inspection time points (t_1, t_2, \dots, t_K) and masking rate w , then the expected cost of this experiment can be expressed as

$$C_D(1 - w) \sum_{k=1}^K \Pr(X < t_i) + C_I N.$$

For a fixed value of N , V and the cost constraint C , one can determine the optimal inspection time points and masking rate w by solving the following optimization problem:

$$\begin{aligned} \min C_D(1 - w) \sum_{k=1}^K \Pr(X < t_i) + C_I N < C \\ \text{such that } \widehat{Var}(MTTF) < V. \end{aligned}$$

Further investigation of the optimal experimental planning problem will be of interest.

Appendix A

Appendix

A.1. Derivatives related to the cdf of a gamma distribution

$$\frac{\partial^2}{\partial \alpha^2} \ln F_X(t_i; \boldsymbol{\theta}) = \frac{\partial^2}{\partial \alpha^2} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) - \frac{\partial^2}{\partial \alpha^2} \ln \Gamma(\alpha)$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} \ln F_X(t_i; \boldsymbol{\theta}) = \frac{\partial^2}{\partial \alpha \partial \beta} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) - \frac{\partial^2}{\partial \alpha \partial \beta} \ln \Gamma(\alpha)$$

$$\frac{\partial^2}{\partial \beta^2} \ln F_X(t_i; \boldsymbol{\theta}) = \frac{\partial^2}{\partial \beta^2} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) - \frac{\partial^2}{\partial \beta^2} \ln \Gamma(\alpha)$$

$$\frac{\partial^2}{\partial \alpha^2} \ln (1 - F_X(t_i; \boldsymbol{\theta})) = \frac{\partial^2}{\partial \alpha^2} \ln \Gamma \left(\alpha, \frac{t_i}{\beta} \right) - \frac{\partial^2}{\partial \alpha^2} \ln \Gamma(\alpha)$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} \ln (1 - F_X(t_i; \boldsymbol{\theta})) = \frac{\partial^2}{\partial \alpha \partial \beta} \ln \Gamma \left(\alpha, \frac{t_i}{\beta} \right) - \frac{\partial^2}{\partial \alpha \partial \beta} \ln \Gamma(\alpha)$$

$$\frac{\partial^2}{\partial \beta^2} \ln(1 - F_X(t_i; \boldsymbol{\theta})) = \frac{\partial^2}{\partial \beta^2} \ln \Gamma\left(\alpha, \frac{t_i}{\beta}\right) - \frac{\partial^2}{\partial \beta^2} \ln \Gamma(\alpha)$$

$$\frac{\partial}{\partial \alpha} F_X(t_i; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)^2} \left[\Gamma(\alpha) \frac{\partial}{\partial \alpha} \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \gamma\left(\alpha, \frac{t_i}{\beta}\right) \frac{d}{d\alpha} \Gamma(\alpha) \right]$$

$$\frac{\partial}{\partial \beta} F_X(t_i; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)} \left[\frac{\partial}{\partial \beta} \gamma\left(\alpha, \frac{t_i}{\beta}\right) \right]$$

$$\begin{aligned} \frac{\partial^2}{\partial \alpha^2} F_X(t_i; \boldsymbol{\theta}) &= \frac{1}{\Gamma(\alpha)^3} \left[\Gamma(\alpha)^2 \frac{\partial^2}{\partial \alpha^2} \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \Gamma(\alpha) \gamma\left(\alpha, \frac{t_i}{\beta}\right) \frac{d^2}{d\alpha^2} \Gamma(\alpha) \right. \\ &\quad \left. - 2\Gamma(\alpha) \frac{\partial}{\partial \alpha} \gamma\left(\alpha, \frac{t_i}{\beta}\right) \frac{d}{d\alpha} \Gamma(\alpha) + 2\gamma\left(\alpha, \frac{t_i}{\beta}\right) \left(\frac{d}{d\alpha} \Gamma(\alpha)\right)^2 \right] \end{aligned}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} F_X(t_i; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)^2} \left[\Gamma(\alpha) \frac{\partial^2}{\partial \alpha \partial \beta} \gamma\left(\alpha, \frac{t_i}{\beta}\right) - \frac{d}{d\alpha} \Gamma(\alpha) \frac{\partial}{\partial \beta} \gamma\left(\alpha, \frac{t_i}{\beta}\right) \right]$$

$$\frac{\partial^2}{\partial \beta^2} F_X(t_i; \boldsymbol{\theta}) = \frac{1}{\Gamma(\alpha)} \left[\frac{\partial^2}{\partial \beta^2} \gamma\left(\alpha, \frac{t_i}{\beta}\right) \right]$$

A.2. Derivatives related to the incomplete gamma functions

A.2.1. Lower incomplete gamma function

$$\begin{aligned} \frac{\partial^2}{\partial \alpha^2} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) &= -\gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-2} \left[\frac{\partial}{\partial \alpha} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \right]^2 \\ &\quad + \gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-1} \frac{\partial^2}{\partial \alpha^2} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial \beta^2} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) &= -\gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-2} \left[\frac{\partial}{\partial \beta} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \right]^2 \\ &\quad + \gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-1} \frac{\partial^2}{\partial \beta^2} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2}{\partial \alpha \partial \beta} \ln \gamma \left(\alpha, \frac{t_i}{\beta} \right) &= -\gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-2} \frac{\partial}{\partial \alpha} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \frac{\partial}{\partial \beta} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \\ &\quad + \gamma \left(\alpha, \frac{t_i}{\beta} \right)^{-1} \frac{\partial^2}{\partial \alpha \partial \beta} \gamma \left(\alpha, \frac{t_i}{\beta} \right) \end{aligned}$$

$$\frac{\partial^2}{\partial \alpha^2} \gamma \left(\alpha, \frac{t_i}{\beta} \right) = \int_0^{\frac{t_i}{\beta}} u^{\alpha-1} (\ln u)^2 e^{-u} du$$

$$\begin{aligned} \frac{\partial^2}{\partial \beta^2} \gamma \left(\alpha, \frac{t_i}{\beta} \right) &= (\alpha + 1) t_i^\alpha \beta^{-(\alpha+2)} e^{-\frac{t_i}{\beta}} \\ &\quad - t_i^{\alpha+1} \beta^{-(\alpha+3)} e^{-\frac{t_i}{\beta}} \\ &= (\alpha + 1 - t_i \beta^{-1}) t_i^\alpha \beta^{-(\alpha+2)} e^{-\frac{t_i}{\beta}} \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial\alpha\partial\beta}\gamma\left(\alpha, \frac{t_i}{\beta}\right) &= -\frac{t_i}{\beta^2}\left(\frac{t_i}{\beta}\right)^{\alpha-1}\ln\frac{t_i}{\beta}e^{-\frac{t_i}{\beta}} \\
&= -t_i^\alpha\beta^{-(\alpha+1)}\ln\frac{t_i}{\beta}e^{-\frac{t_i}{\beta}}
\end{aligned}$$

A.2.2. Upper incomplete gamma function

$$\begin{aligned}
\frac{\partial^2}{\partial\alpha^2}\ln\Gamma\left(\alpha, \frac{t_i}{\beta}\right) &= -\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-2}\left[\frac{\partial}{\partial\alpha}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)\right]^2 \\
&\quad +\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-1}\frac{\partial^2}{\partial\alpha^2}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial\beta^2}\ln\Gamma\left(\alpha, \frac{t_i}{\beta}\right) &= -\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-2}\left[\frac{\partial}{\partial\beta}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)\right]^2 \\
&\quad +\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-1}\frac{\partial^2}{\partial\beta^2}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial\alpha\partial\beta}\ln\Gamma\left(\alpha, \frac{t_i}{\beta}\right) &= -\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-2}\frac{\partial}{\partial\alpha}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)\frac{\partial}{\partial\beta}\Gamma\left(\alpha, \frac{t_i}{\beta}\right) \\
&\quad +\Gamma\left(\alpha, \frac{t_i}{\beta}\right)^{-1}\frac{\partial^2}{\partial\alpha\partial\beta}\Gamma\left(\alpha, \frac{t_i}{\beta}\right)
\end{aligned}$$

$$\frac{\partial^2}{\partial\alpha^2}\Gamma\left(\alpha, \frac{t_i}{\beta}\right) = \int_{\frac{t_i}{\beta}}^{\infty} u^{\alpha-1}(\ln u)^2 e^{-u} du$$

$$\begin{aligned}
\frac{\partial^2}{\partial \beta^2} \Gamma \left(\alpha, \frac{t_i}{\beta} \right) &= -(\alpha + 1) t_i^\alpha \beta^{-(\alpha+2)} e^{-\frac{t_i}{\beta}} \\
&\quad + t_i^{\alpha+1} \beta^{-(\alpha+3)} e^{-\frac{t_i}{\beta}} \\
&= -(\alpha + 1 - t_i \beta^{-1}) t_i^\alpha \beta^{-(\alpha+2)} e^{-\frac{t_i}{\beta}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial \alpha \partial \beta} \Gamma \left(\alpha, \frac{t_i}{\beta} \right) &= \frac{t_i}{\beta^2} \left(\frac{t_i}{\beta} \right)^{\alpha-1} \ln \frac{t_i}{\beta} e^{-\frac{t_i}{\beta}} \\
&= t_i^\alpha \beta^{-(\alpha+1)} \ln \frac{t_i}{\beta} e^{-\frac{t_i}{\beta}}
\end{aligned}$$

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