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Empirical Likelihood Ratio Tests for Homogeneity of Distributions of Component Lifetimes from System Lifetime Data with Known System Structures

Jingjing Qu
Southern Methodist University, jingjingq@smu.edu

EMPIRICAL LIKELIHOOD RATIO TESTS FOR HOMOGENEITY OF DISTRIBUTIONS OF COMPONENT LIFETIMES FROM SYSTEM LIFETIME DATA WITH KNOWN SYSTEM STRUCTURES

Approved by:

Dr. Hon Keung Tony Ng
Professor, Department of Mathematical Sciences, Bentley University

Dr. Jing Cao
Professor, Department of Statistical Science, SMU

Dr. Ian Richard Harris
Associate Professor, Department of Statistical Science, SMU

Dr. Chul Moon
Assistant Professor, Department of Statistical Science, SMU

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Jingjing Qu

B.A., Mathematics, Shandong University, China
M.S., Mathematics, Shandong University, China
Ph.D., Mathematics, University of CAS, China

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Qu, Jingjing

B.A., Mathematics, Shandong University, China
M.S., Mathematics, Shandong University, China
Ph.D., Mathematics, University of CAS, China

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In system reliability, practitioners may be interested in testing the homogeneity of the component lifetime distributions based on system lifetimes from multiple data sources for various reasons, such as identifying the component supplier that provides the most reliable components.

In the first part of the dissertation, we develop distribution-free hypothesis testing procedures for the homogeneity of the component lifetime distributions based on system lifetime data when the system structures are known. Several nonparametric testing statistics based on the empirical likelihood method are proposed for testing the homogeneity of two or more component lifetime distributions. The computational approaches to obtain the critical values of the proposed test procedures are provided. The performances of the proposed empirical likelihood ratio test procedures are evaluated and compared to the nonparametric Mann-Whitney U test and some parametric test procedures. The simulation results show that the proposed test procedures provide comparable power performance under different sample sizes and underlying component lifetime distributions, and they are powerful in detecting changes in the shape of the distributions.

In collecting system lifetime data, censoring is often adopted due to time and budget constraints. In the second part of the dissertation, we consider the situation where

the system lifetime data from two different kinds of systems are subjected to Type-II censoring, and we are interested in testing the homogeneity of distributions of component lifetimes from Type-II censored system lifetime data with known system structures. Based on the Mann-Whitney U test and empirical likelihood ratio tests developed for testing the homogeneity of distributions of component lifetimes with complete system lifetime data, we propose different non-parametric test procedures using the idea of permutation of the censored system lifetimes. We consider a restricted assumption on the equality of the censored lifetimes to reduce the permutations required in the computation. The computational approaches to obtain the critical values of the proposed test procedures are provided using the Monte Carlo method. A practical example is used to illustrate the proposed test procedures. Then, the power performances of the proposed test procedures are evaluated and compared using a Monte Carlo simulation study. The simulation results show that the proposed test procedures provide good power performance for Type-II censored system lifetime data under different scenarios.

Finally, summaries of the major contributions of the thesis and concluding remarks are provided. Some possible future research directions are also discussed.

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I dedicate this dissertation to my family.

CHAPTER 1

Introduction

1.1. Problem of interest

In evaluating the reliability of systems, life-testing experiments are usually conducted to gather information on the lifetime characteristics of the system of interest. In addition to assessing the reliability of systems, in many circumstances, practitioners may also be interested in evaluating and comparing the lifetime characteristics of the components that made up those systems. For example, when there are multiple suppliers of the components in a system, the manufacturer of the system is interested in comparing the lifetime performance of components from different suppliers and select the one with the best reliability characteristics.

Comparing the lifetime characteristics of the components that made up the systems becomes challenging when the system structures are different, especially when the comparisons can only be made based on system lifetime data. There are many situations where the comparison of component lifetime characteristics can only be made based on system lifetime data, such as when the performance of components varies in different systems. For example, the lifetimes of single-cell cylindrical dry batteries can be very different when used in high-drain devices (e.g., digital cameras, radio-controlled toys, etc.) compared to using in low-drain devices (e.g., clocks, remote controls). Suppose one wants to compare the lifetime characteristics of single-cell cylindrical dry batteries when they are used in high-drain devices and low-drain devices. In this case, the life testing experiments can be done when the batteries are used in the devices (systems) and based on the system lifetimes. Another example is that if the life test involves fielded systems, the information on which component leads to the system failure cannot usually be accessed because the experimenters often do not have the need or capability to measure the failed component one by one or the whole system may be discarded after failure. Therefore, the development of statistical inference for the lifetime distribution of components based on system-level lifetime data is of interest.

In the past decades, numerous research papers have been published in developing statistical inference of the component lifetime characteristics based on system lifetime data. For example, early literature involving this topic is based on “masked data," which assumes that only partial information is available on the component failures that lead to the failure of the system. Under this framework, Meilijson (1981) and Bueno (1988) estimated the component lifetime distribution based on system failure times together with autopsy information on the components in the system. Miyakawa (1984) discussed parametric and nonparametric estimation methods for component reliability in two-component series system under competing risks with incomplete data. Boyles and Samaniego (1987) derived the nonparametric maximum likelihood estimator of component reliability based on nomination sampling in parallel systems. Usher and Hodgson (1988) explored a general method for estimating component reliability from a J-component series system lifetime data. Guess et al. (1991) extended Miyakawa's work and treated a broader class of estimation problems based on masked data. More recent work includes the paper by Bhattacharya and Samaniego (2010) in which the authors estimated component reliability from system failure data using the known system signature. Given the coherent systems with known system structures described by system signature, statistical inferences are developed for component lifetime distributions based on complete or censored system lifetime data. Eryilmaz et al. (2011) discussed reliability properties of m-consecutive-k-out-of-n systems with exchangeable components. Balakrishnan et al. (2011a,b) developed the exact nonparametric method to measure some characteristics of the component lifetime distribution based on complete and censored system lifetime data, respectively. A general method was derived by Navarro et al. (2012) for inference on the scale parameter of the component lifetime distribution from system lifetime data. Ng et al. (2012) discussed parametric statistical inference for the component lifetime distributions from the system lifetimes when the component lifetimes follow a proportional hazard rate model. While most previous works dealt with system lifetimes from systems with the same system signature, recently Hall et al. (2015) developed a novel nonparametric estimator of component reliability function by maximizing the combined system likelihood function when the systems have different known system signatures. Then, Jin et al. (2017) extended the work to the situation that the system signatures are unknown.

In this dissertation, we study the problem of testing the homogeneity of component lifetime distributions based on system-level lifetime data from multiple data sources. Based on the empirical likelihood method, we develop distribution-free statistical inferential method for complete data and then generalize the inferential method for Type-II censored data. The proposed test procedures are compared to those existing parametric and nonparametric tests for homogeneity of component lifetime distributions based on complete and Type-II censored system-level lifetime data.

1.2. Coherent System and System Signature

In reliability theory, an n-component system is said to be coherent if each component is relevant and its structure function is non-decreasing in each vector argument. In this dissertation, we assume that the systems are coherent and contain n independent and identically distributed (i.i.d.) components. For the structure of system, we consider the system signature proposed by Samaniego (1985) as an index that characterizes a system under the assumption of i.i.d. components. The system signature characterizes a system without using a complex structure function.

Suppose T is the system lifetime and $X_{\ell:n}$ is the ℓ-th order statistic out of n component lifetimes in an n-component system. The system signature of an n-component system is an n-dimensional probability vector defined as $\mathrm{s} = (s_1, s_2, \cdots, s_n)$, where

$$s_\ell = \Pr(\text{system fails upon the failure of the } \ell\text{-th component})) = \Pr(T = X_{\ell:n}) \qquad (1.1)$$

where s_ℓ, $\ell = 1, 2, \ldots, n$ are non-negative real numbers in $[0, 1]$ that do not depend on the component lifetime distribution F_X with $\sum_{\ell=1}^{n} s_\ell = 1$.

System signatures can be calculated using a combinatorial method which is a well-organized method. Suppose the random variables $X_1, X_2, \cdots, X_n$ represent the failure times of the n components in a system. Since the n components are i.i.d., the permutations of these n failure times are equally likely. The i-th element of signature s, $s_i = \Pr(T = X_{i:n})$, can be obtained as the ratio of the number of orderings for which the i-th component failure caused the system failure, to the total possible orderings of the n component lifetimes $n!$. For example, consider a 3-component series-parallel system presented in Figure 1.1. There are six possible orderings of the three component lifetimes (see Table 1.1). In the first two cases, the system lifetime $T = X_{1:3}$, which gives $s_1 = \Pr(T = X_{1:3}) = 2/6$. In the last four cases, the system lifetime $T = X_{2:3}$, hence, $s_2 = \Pr(T = X_{2:3}) = 4/6$, and then we have $s_3 = \Pr(T = X_{3:3}) = 0$. Therefore, the signature for the 3-component series-parallel system is $(2/6, 4/6, 0) = (1/3, 2/3, 0)$.

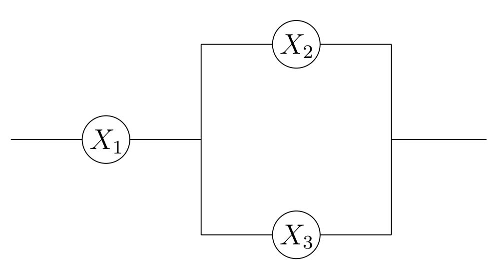

Figure 1.1: A 3-component parallel-series system

Table 1.1: Six possible orderings of the three component lifetimes in a 3-component series-parallel system

Orderings	System Lifetime T
$X_1 \leq X_2 \leq X_3$	$X_{1:3}$
$X_1 \leq X_3 \leq X_2$	$X_{1:3}$
$X_2 \leq X_1 \leq X_3$	$X_{2:3}$
$X_2 \leq X_3 \leq X_1$	$X_{2:3}$
$X_3 \leq X_1 \leq X_2$	$X_{2:3}$
$X_3 \leq X_2 \leq X_1$	$X_{2:3}$

Another example is the 3-component series system presented in Figure 1.2. Since the series system fails when the first component failure occurs, the lifetime of the system $T = \min\{X_1, X_2, X_3\} = X_{1:3}$ and the signature is $\mathbf{s} = (1, 0, 0)$. In practice, many real n component systems contain n i.i.d. components that can be described by system signature. For example, Bhattacharya and Samaniego (2010) pointed out that the batteries in a lighting device, chips or wafers in a digital computer, heating elements in a broiler, and the subsystem of spark plugs are some practical systems that contain n i.i.d. components. Hence, the methodology developed in this thesis can be used in many cases.

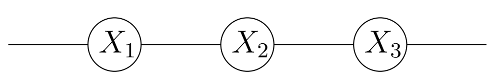

Figure 1.2: A 3-component series system

We denote the cumulative distribution function (CDF), survival function (SF), and probability density function (PDF) of the lifetimes of System i (i.e., T_{ij}) by F_{T_i}, $\overline{F}_{T_i}$, and f_{T_i}, for $i = 1, 2$, and the CDF, SF, and PDF of the lifetimes of the components in System i (i.e., X_{ijk}) by F_{X_i}, $\overline{F}_{X_i}$, and f_{X_i}, for $i = 1, 2$. Based on the notion of system signature, the SF of the system lifetime T can be written in terms of the SF of the component lifetime X as

Kochar et al. (1999); Samaniego (2007)

$$\begin{aligned}\overline{F}_T(t) &= \sum_{\ell=1}^{n} s_\ell \sum_{k=0}^{\ell-1} \binom{n}{k} [1-\overline{F}_X(t)]^k [\overline{F}_X(t)]^{n-k} \\ &\triangleq h(p(t)), \end{aligned} \tag{1.2}$$

where $p(t) = \overline{F}_X(t)$ and $h(p(t))$ is a polynomial function in terms of component SF $p(t)$. We further denote the inverse function of h as

$$h^{-1}(q(t)) = p(t),$$

where $q(t) = \overline{F}_T(t)$. Navarro et al. (2011) stated that the polynomial $h(p)$ is strictly increasing for $p \in (0,1)$, with $h(0) = 0$ and $h(1) = 1$, hence, its inverse function h^{-1} in $(0,1)$ exists and is also strictly increasing in $(0,1)$ with $h^{-1}(0) = 0$ and $h^{-1}(1) = 1$. The reliability polynomial $h(p)$ works as a bridge between system and component survival functions, which means that given the system SF or the component SF, the other SF can be obtained from solving Eq. (1.2) when the system structure is known. Take the 3-component series system in Figure 1.2 as an example. Suppose the three components are i.i.d. with SF $p(t)$, then from Eq. (1.2), the reliability function of a 3-component series system is $h(p) = p^3$. If the component in this 3-component series system follow $Exp(1)$ i.i.d., then the SF of this system is $h(p) = p^3 = (\exp(-t))^3 = \exp(-3t)$.

Moreover, the PDF of the system lifetime T_j can be expressed in terms of $f_{X_i}(t)$ and $p_i(t) = \overline{F}_{X_i}(t)$ as

$$f_{T_i}(t) = \sum_{k=1}^{n_i} s_{ik} \binom{n_i}{k} k f_{X_i}(t) [1-p_i(t)]^{k-1} [p_i(t)]^{n_i-k}. \tag{1.3}$$

1.3. Censoring in Life Testing Experiments

In survival and reliability analysis, censoring occurs when incomplete information is available about the lifetime of some experimental units. It usually arises when the experimental units are not observed for the full duration of time to failure. The possible reason for censoring data may be a loss, early termination, death, or other causes. Censoring can be pre-planned to save time and the cost of the life-testing experiment.

There are different types of censored data in survival and reliability analysis, such as left-censored, right-censored, and interval-censored data. For right-censored data, there are various censoring schemes have been proposed for different purposes. Among these censoring schemes, Type-I and Type-II censorings are two commonly used censoring schemes in life testing procedures.

For Type-I censoring, the life-testing experiment will be terminated at a prefixed time c. Then, only the failures until time c will be observed, and the units that are not failed at time c are right-censored. The data obtained from such a restrained life test will be referred to as a *Type-I censored sample*. Note that the number of failures observed here is random and has a binomial distribution with parameters M and probability of “success" $F(c)$, where M is the number of experimental units placed on the life-testing experiment and $F(\cdot)$ is the CDF of the experimental units.

For Type-II censoring, the life-testing experiment with M experimental units will be terminated as soon as the r-th failure is observed, where $r < M$ is a prefixed number. Then, only the first r failures out of M units under test will be observed. The data obtained from such a restrained life test will be referred to as a *Type-II censored sample*. In contrast to Type-I censoring, the number of failures observed is fixed (i.e., r) while the duration of the experiment is a random variable ($X_{r:M}$).

1.4. Parametric Lifetime Distributions and Likelihood Inference

1.4.1. Some useful lifetime distributions

In this section, we review several commonly used lifetime distributions which will be used to evaluate the performance of test statistics in this thesis.

1. Exponential distribution ($Exp(\theta)$)

   The PDF and SF of exponential distribution with scale parameter $\theta > 0$, denoted as $Exp(\theta)$, are

   $$f(t;\theta) = \frac{1}{\theta}e^{-\frac{t}{\theta}}, \quad \text{for } t \geq 0;$$
   $$\overline{F}(t;\theta) = e^{-\frac{t}{\theta}}, \quad \text{for } t \geq 0,$$

   respectively. The mean and variance of a random variable follows $Exp(\theta)$ are θ and θ^2, respectively.

2. Gamma distribution ($Gamma(\alpha,\beta)$)

   The PDF of gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, denoted as $Gamma(\alpha,\beta)$, is

   $$f(t;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}, \text{ for } t \geq 0.$$

   When $\alpha = 1$, the gamma distribution becomes $Exp(1/\beta)$. The mean and variance of a random variable follows $Gamma(\alpha,\beta)$ are α/β and α/β^2, respectively.

3. Weibull distribution ($Weibull(\lambda, k)$)

   The PDF of Weibull distribution with scale parameter $\lambda > 0$ and shape parameter

$k > 0$, denoted as $Weibull(\lambda, k)$, is

$$f(t; \lambda, k) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} \exp\left[] - \left(\frac{t}{\lambda}\right)^{k}\right], \quad \text{for } t > 0. \tag{1.4}$$

The mean and variance of a random variable follows $Weibull(\lambda, k)$ are $\lambda\Gamma(1 + 1/k)$ and $\lambda^2[\Gamma(1 + 2/k) - \Gamma(1 + 1/k)^2]$, respectively, where $\Gamma(k) = \int_0^\infty x^k \exp(-x)dx$ is the gamma function.

4. Lognormal distribution ($Lognormal(\mu, \sigma)$)

   The PDF of lognormal distribution with scale parameter $\exp(\mu) > 0$ and shape parameter $\sigma > 0$, denoted as $Lognormal(\mu, \sigma^2)$, is

$$f(t; \mu, \sigma) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log t - \mu)^2}{2\sigma^2}\right], \quad \text{for } t > 0.$$

   The mean and variance of a random variable follows $Lognormal(\mu, \sigma)$ are $\exp(\mu + \frac{\sigma^2}{2})$ and $[\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2)$, respectively.

1.4.2. Maximum likelihood estimation

In statistics, maximum likelihood estimation method is a commonly used estimation method for the parameters in a probability model given the observed data. The likelihood function is the function of the model parameters based on the observed data. Suppose $\boldsymbol{T} = (T_1, \cdots, T_M)$ is a random sample of size M and T_j follows a probability distribution with PDF $f(t; \boldsymbol{\theta})$ and CDF $F(t; \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the parameter vector, then the likelihood function based on $\boldsymbol{T}$ can be expressed as

$$L(\boldsymbol{\theta}|\boldsymbol{T}) = \prod_{j=1}^{M} L_j(\boldsymbol{\theta}|T_j),$$

where $L_j(\boldsymbol{\theta}|T_j)$ is the contribution of the j-th observation to the total likelihood. For completely observed failure times, the contribution of the j-th observation to the likelihood is $L_j(\boldsymbol{\theta}|T_j) = f(T_j;\boldsymbol{\theta})$. For right-censored observations, the contribution of the j-th censored observation to the likelihood is

$$L_j(\boldsymbol{\theta}|T_j) = \int_{T_j}^{\infty} f(t;\boldsymbol{\theta})dt = 1 - F(T_j;\boldsymbol{\theta}).$$

Suppose we have a data set $\boldsymbol{T} = ((T_1, \delta_1), \cdots (T_M, \delta_M))$, where T_j is the observed time and δ_j is the indicator of censoring defined in the following:

$$\delta_j = \begin{cases} 0, \text{if } T_j \text{ is the censoring time;} \\ 1, \text{if } T_j \text{ is the failure time.} \end{cases}$$

The likelihood function for the whole data set is

$$L(\boldsymbol{\theta}|\boldsymbol{T}) = \prod_{j=1}^{M} (f(T_j;\boldsymbol{\theta}))^{\delta_j} (1 - F(T_j;\boldsymbol{\theta}))^{1-\delta_j}. \tag{1.5}$$

The maximum likelihood estimate (MLE) of the parameter vector $\boldsymbol{\theta}$ can be obtained by maximizing the likelihood function $L(\boldsymbol{\theta}|\boldsymbol{T})$ with respect to $\boldsymbol{\theta}$. The logic of maximum likelihood is both intuitive and flexible, and as such the method has become a very important means of statistical inference.

In some cases, the MLEs of the model parameters can be obtained in a closed form. For example, the distribution of T_j is assumed to be $Exp(\theta)$. Based on the sample $\boldsymbol{T} = ((T_1, \delta_1), \cdots (T_M, \delta_M))$, the likelihood function in Eq. (1.5) can be expressed as

$$
\begin{aligned}
L(\theta|\boldsymbol{T}) &= \prod_{j=1}^{M}[f(T_j;\theta)]^{\delta_j}[1-F(T_j;\theta)]^{1-\delta_j} \\
&= \prod_{j=1}^{M}\left[\frac{1}{\theta}\exp\left(-\frac{T_j}{\theta}\right)\right]^{\delta_j}\left[\exp\left(-\frac{T_j}{\theta}\right)\right]^{1-\delta_j}.
\end{aligned}
$$

Maximizing the above likelihood function with respect to θ, the MLE of θ based on right-censored data is

$$
\hat{\theta} = \sum_{j=1}^{M} T_j \Big/ \sum_{j=1}^{M} \delta_j.
$$

For complete data without any censoring, we have $\sum_{j=1}^{M}\delta_j = M$, which yields $\hat{\theta} = \frac{1}{M}\sum_{j=1}^{M} T_j$. However, for distributions with more than one parameter (e.g., Gamma and Weibull distributions) or more complicated probability models, no closed-form solutions exist for the MLEs, therefore numerical methods are required to maximize the likelihood function in terms of parameter $\boldsymbol{\theta}$.

1.4.3. Parametric likelihood ratio test

The likelihood ratio test (also known as the Wilks test), which dates back to Wilks (1938), is one of the classical approaches for hypothesis testing. The likelihood ratio test assesses the goodness of fit of two competing statistical models by the ratio of their likelihood functions. The two likelihood functions are found by maximizing over the entire parameter space and maximizing over a restricted parameter space under some constraints. If the observed data support the constraints under the null hypothesis, then the two likelihood functions should not differ much. Therefore, the likelihood ratio test evaluates if these two models are significantly different from each other in the test procedure.

Suppose we are interested in testing the hypotheses

$$H_0 : \boldsymbol{\theta} \in \Theta_0 \text{ versus } H_1 : \boldsymbol{\theta} \in \Theta \setminus \Theta_0,$$

where $\boldsymbol{\theta}$ is the parameter of interest, Θ_0 is the restricted parameter space under the null hypothesis, and Θ is the parameter space of the probability model.

Suppose the observed data is $\boldsymbol{T}$, the parametric likelihood ratio statistic is defined as

$$\lambda_{LR} = -2 \ln \left[\frac{\sup_{H_0:\boldsymbol{\theta} \in \Theta_0} L(\boldsymbol{\theta}|\boldsymbol{T})}{\sup_{H_1:\boldsymbol{\theta} \in \Theta} L(\boldsymbol{\theta}|\boldsymbol{T})} \right].$$

By the Wilk's theorem (Wilks, 1938), the asymptotic null distribution of the likelihood ratio statistic λ_{LR} follow chi-square distribution with degrees of freedom ν, where ν is the difference in dimensionality of $\boldsymbol{\theta} \in \Theta_0$ and $\boldsymbol{\theta} \in \Theta$. The p-value of the parametric likelihood ratio test based on test statistic λ_{LR} can be calculated as $\Pr(\chi^2_\nu > \lambda_{LR})$ where χ^2_ν is a random variable follows the chi-square distribution with degrees of freedom ν. When the sample size is small, the distribution of the likelihood ratio test statistic may not follow the asymptotic results, the Monte Carlo resampling simulation approach may be used to calculate the p-value.

1.5. Empirical Likelihood Method

The empirical likelihood method is a nonparametric method for statistical inference which shares many of the merits of the parametric likelihood method.

1.5.1. Empirical Likelihood

Thomas and Grunkemeier (1975) proposed the idea of empirical likelihood by inverting a nonparametric version of the likelihood ratio test to obtain confidence intervals. Efron (1981) showed that nonparametric statistical inference can be conducted by applying parametric techniques to suitable distributions supported on the data. Based on these ideas, Owen (1990, 1988) formally introduced the empirical likelihood to construct confidence regions in for nonparametric inference and mentioned that the empirical likelihood method has several advantages over the other nonparametric methods such as rank-based methods and bootstrap method.

After the empirical likelihood method was introduced, DiCiccio et al. (1991) obtained some significant results on high-order asymptotics including the Bartlett correctability. Then, Qin (1993) deals with a creative problem formulation in mixing empirical likelihood and parametric likelihood and combining multiple biased samples. Qin and Lawless (1994) established the link between estimating functions/equations and empirical likelihood and developed methods of combining information about parameters. Mykland (1995) applied the empirical likelihood method to improve the accuracy in inference with a martingale setting. For interval estimation, Chen and Hall (1993) proposed smoothed empirical likelihood confidence intervals for quantiles, and Chen and Qin (2000) developed the empirical likelihood confidence intervals for local linear smoothers. Chen (1996) discussed several applications of empirical likelihood in nonparametric density estimation and regression function estimation. Empirical likelihood approaches for censored and truncated data were developed by Li (1995a,b) and Murphy and van der Vaart (1997); Murphy and Van Der Vaart (1999); Murphy and Van der Vaart (2000). Kitamura (1997) studied the large deviations properties of empirical likelihood and investigated the connections between empirical likelihood and modern econometrics. Jing et al. (2009) developed the Jackknife empirical likelihood method. The idea of empirical likelihood has been applied to different areas of statistics such as time series analysis (Nordman and Lahiri, 2014), survival analysis (Zhou, 2019), longitudinal data analysis (Nadarajah et al., 2014), and regression analysis (Chen and Keilegom, 2009). For comprehensive reviews on the theory and applications of the empirical likelihood method, one may refer to the books written by Owen (2001) and Zhou (2019).

For the methods proposed in this thesis, we introduce the following results in the empirical likelihood approach. Suppose $X_1, X_2, \cdots, X_n$ is a random sample of size n from a population distribution with CDF F, which is assumed to be unknown. The empirical likelihood function based on the random sample $X_1, X_2, \cdots, X_n$ is defined as

$$EL(F) = \prod_{j=1}^{n} P_F(X = X_j) = \prod_{i=1}^{n} p_j \tag{1.6}$$

where $p_j = P_F(X = X_j) = F(X_j) - F(X_j -)$.

The following theorem shows that nonparametric likelihood is maximized by the empirical likelihood function. Thus, the value that maximizes the empirical likelihood function denoted as the empirical CDF (ECDF), is the nonparametric maximum likelihood estimate (NPMLE) of the CDF F.

Theorem 1 (Owen (2001)): Let $X_1, \cdots, X_n \in \mathbf{R}$ be a random sample with a common CDF F. Let F_n be their ECDF, and F_0 be any CDF. If $F_0 \neq F_n$, then the likelihood function $EL(F_0) < EL(F_n)$. $\square$

We now define empirical SF given data $X_1, \cdots, X_n$ as following

$$\hat{\bar{F}}_n(X_j) = \sum_{j=1}^{n} I_{(x,\infty)}(X_j), \tag{1.7}$$

where

$$I_A(b) = \begin{cases} 1, & \text{if } b \in A, \\ 0, & \text{otherwise.} \end{cases}$$

The empirical SF $\hat{\bar{F}}_n$ defined in Eq. (1.7) maximizes the empirical likelihood function. The empirical SF in Eq. (1.7) is equivalent to the Kaplan–Meier estimator (Kaplan and Meier, 1958) which is a non-parametric estimator for the survival function based on lifetime data. The Kaplan–Meier estimator of the SF $\bar{F}(t)$ is given by:

$$\bar{F}(t) = \prod_{j:\ x_j \leq t} \left(1 - \frac{d_j}{n_j}\right),$$

where x_j is a time when at least one event happened, d_j is the number of events (e.g., failure) that happened at time x_j, and n_j is the number of individuals known to have survived (have not yet had an event or been censored) up to time x_j (i.e., the number of observations at risk at time x_j). The Kaplan–Meier curve can provide a nonparametric estimate of the SF by taking right-censoring into account.

1.5.2. Empirical Likelihood Ratio

In Section 1.4.3, we discussed the parametric likelihood ratio test for hypothesis testing. If the likelihood under the null hypothesis is much smaller than the likelihood under the alternative hypothesis, then there is enough evidence to reject the null hypothesis. A similar idea can be applied in nonparametric inference in which the ratio of the nonparametric maximum likelihood values can be used as a basis for hypothesis testing. The empirical likelihood ratio is defined as (Owen, 1990)

$$R(F) = \frac{EL(F)}{EL(\hat{F}_n)},$$

where $\hat{F}_n$ is the ECDF that maximizes the empirical likelihood function $EL(F)$.

Suppose F belongs to a set of distributions $\mathcal{F}$ and we are interested in testing $T(F) = \delta$ for some function T of the distribution F, then using the empirical likelihood ratio $R(F)$, the hypothesis that $T(F) = \delta$ is rejected if $\delta \notin \{T(F) | R(F) \geq r_0\}$, where r_0 is a critical value. Here, $\delta \notin \{T(F)|R(F) \geq r_0\}$ can be interpreted as there is no distribution F under the hypothesis $T(F) = \delta$ has an empirical likelihood $EL(F)$ greater than or equal to r_0 times the maximum empirical likelhood (i.e., $r_0 EL(F_n)$).

It is known that the empirical likelihood ratio behaves like an ordinary parametric likelihood ratio (Wilks, 1938), such as the asymptotic chi-square distribution for the empirical likelihood ratio test statistic. Based on the idea of the empirical likelihood ratio, Zhang (2002) proposed a new approach of parameterization to construct a general goodness-of-fit test based on the likelihood ratio. He proposed several test statistics and showed that these tests are more powerful than the Kolmogorov-Smirnov, the Cramér-von Mises, and the Anderson-Darling tests. Later on, Zhang (2006) proposed a new non-parametric statistic for testing the homogeneity of distributions based on empirical likelihood ratio and showed that the proposed statistic provides better power performance compared to those traditional tests such as the two-sample Kolmogorov-Sminov test.

1.6. Scope of Thesis

Suppose there are two different coherent systems, System 1 and System 2, with n_1 and n_2 i.i.d. components and system signatures $\mathrm{s}_1 = (s_{11}, \cdots, s_{1n_1})$ and $\mathrm{s}_2 = (s_{21}, \cdots, s_{2n_2})$, respectively. The system lifetime data are obtained based on putting M_1 System 1 and M_2 System 2 on a life test. We denote the system lifetimes for System i as $\boldsymbol{T}_i = (T_{i1}, T_{i2}, \ldots, T_{iM_i})$, and the lifetimes of the components in the j-th System i (T_{ij}) as $\boldsymbol{X}_{ij} = (X_{ij1}, X_{ij2}, \ldots, X_{ijn_i})$, $i = 1, 2$, $j = 1, 2, \ldots, M_i$. The cumulative distribution function (CDF), survival function (SF), and probability density function (PDF) of the lifetimes of System i (i.e., T_{ij}) are denoted by F_{T_i}, $\overline{F}_{T_i}$, and f_{T_i}, and the CDF, SF, and PDF of the lifetimes of the components in System i (i.e., X_{ijk}) are denoted by F_{X_i}, $\overline{F}_{X_i}$, and f_{X_i}, for $i = 1, 2$ respectively.

In this thesis, based on the system lifetime data $\boldsymbol{T}_i$, $i = 1, 2$, we are interested in testing the homogeneity of the component lifetime distributions, which can be formulated as a hypothesis testing problem as testing

$$\begin{aligned} H_0 : \quad & F_{X_1}(t) = F_{X_2}(t), \text{for all } t \in (0, \infty) \\ \text{versus } H_1 : \quad & F_{X_1}(t) \neq F_{X_2}(t), \text{for some } t \in (0, \infty). \end{aligned} \tag{1.8}$$

In Chapter 2, we develop distribution-free testing procedures for the homogeneity of the component lifetime distributions in Eq. (1.8) based on completed system lifetime data when the system structures are known. Several non-parametric testing statistics based on the empirical likelihood method are proposed for testing the homogeneity of two or more component lifetime distributions. The computational approaches to obtain the critical values of the proposed test procedures are provided. A Monte Carlo simulation study is used to evaluate the performances of the proposed empirical likelihood ratio test procedures and to compare with the non-parametric Mann-Whitney U test, the parametric likelihood ratio test, and asymptotic tests based on exponentially distributed component lifetimes. The simulation results show that the proposed test procedures provide comparable power performance under different sample sizes and underlying component lifetime distributions.

In Chapter 3, we generalize the empirical likelihood methods for testing the homogeneity of the component lifetime distributions when the system lifetime data are censored. We consider the case that the system lifetime data are Type-II censored. We proposed testing procedures to handle the Type-II censored data using the empirical likelihood methods developed based on complete system lifetime data. The critical values for the test procedures based on the censoring data are obtained by the Monte Carlo method. The power performances of the proposed empirical likelihood ratio tests are studied and compared to different non-parametric and parametric tests by means of Monte Carlo simulation. We consider different sample sizes, censoring proportions, and underlying component lifetime distributions in the Monte Carlo simulation study.

Finally, in Chapter 4, we summarize the contributions of the thesis and present some concluding remarks. Furthermore, we discuss several possible future research directions.

CHAPTER 2

Empirical Likelihood Ratio Tests for Homogeneity of Component Lifetime Distributions Based on System Lifetime Data

2.1. Introduction

In this chapter, based on the empirical likelihood ratio, we propose two nonparametric test procedures for testing the homogeneity of component lifetime distributions given the system lifetimes and the system signatures.

The rest of this chapter is organized as follows. In Section 2.2, we introduce the mathematical notations and formulate the homogeneity test as a statistical hypothesis testing problem. We also review the existing parametric and nonparametric testing procedures for the homogeneity of component lifetime distributions based on system-level data. Section 2.3 introduces the two proposed test procedures based on the empirical likelihood ratio. Then, the computational approach to obtain the null distributions of the nonparametric test statistics by means of the Monte Carlo method is described in Section 2.4. In Section 2.5, A numerical example is used to illustrate the methodologies proposed in this chapter. Monte Carlo simulation studies are used to evaluate the performances of those parametric and nonparametric test procedures in Section 2.6. Finally, concluding remarks are provided in Section 2.7.

2.2. Tests for Homogeneity Based on System Lifetime Data

In this section, we introduce the mathematical notations and formulate the homogeneity test as a statistical hypothesis testing problem. Then, we review some existing parametric and nonparametric test procedures.

Suppose there are two different coherent systems, System 1 and System 2, with n_1 and n_2 i.i.d. components following the same lifetime distribution, respectively. The system lifetime data are obtained based on putting M_1 System 1 and M_2 System 2 on a life test. We denote the system lifetimes for System i as $\boldsymbol{T}_i = (T_{i1}, T_{i2}, \ldots, T_{iM_i})$, and the lifetimes of the components in the j-th System i (T_{ij}) as $\boldsymbol{X}_{ij} = (X_{ij1}, X_{ij2}, \ldots, X_{ijn_i})$, $i = 1, 2$, $j = 1, 2, \ldots, M_i$. Suppose system signatures are $\mathrm{s}_i = (s_{i1}, \cdots, s_{in_i})$ for System i, $i = 1, 2$.

In addition to the system signatures, Navarro et al. (2007) noted that the SF of the system lifetime T_i can be expressed in terms of the SF of k-component series system lifetimes, $X_{i,1:k} = \min(X_{i1}, X_{i2}, \ldots, X_{ik})$, $k = 1, 2, \ldots, n_i$:

$$\bar{F}_{T_i}(t) = \sum_{k=1}^{n_i} a_{ik} \bar{F}_{X_{i,1:k}}(t),$$

where $\bar{F}_{X_{i,1:k}}(t)$ is the SF of a k-component series system lifetime, for some non-negative and negative integers $a_{i1}, a_{i2}, \ldots, a_{in_i}$, that do not depend on the component lifetime distribution with $\sum_{k=1}^{n_i} a_{ik} = 1$ (see also, Ng et al., 2012). The vector $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in_i})$ is called the minimal signature of System i.

Based on the system lifetime data $\boldsymbol{T}_i$, $i = 1, 2$, given system signatures, we are interested in testing the homogeneity of the component lifetime distributions, which can be formulated as a hypothesis testing problem as testing

$$
\begin{aligned}
H_0: \quad & F_{X_1}(t) = F_{X_2}(t), \text{ for all } t \in (0, \infty) \\
\text{versus } H_1: \quad & F_{X_1}(t) \neq F_{X_2}(t), \text{ for some } t \in (0, \infty).
\end{aligned} \tag{2.1}
$$

2.2.1. Parametric Test Procedures

For comparative purposes, we consider here two parametric tests – an asymptotic test under the assumption of exponentially distributed component lifetimes and a parametric likelihood ratio test – for testing the hypotheses in Eq. (2.1) based on system lifetime data $\boldsymbol{T}_i$, $i = 1, 2$.

2.2.1.1. Asymptotic Tests Under Exponential Distributed Component Lifetimes Assumption

We consider the asymptotic parametric test developed in Zhang et al. (2015) based on the assumption that the underlying lifetimes of components follow an exponential distribution. Specifically, we assume that the lifetimes of components from System i $(i = 1, 2)$ follow an exponential distribution with scale parameter $\theta_i > 0$ (denoted as $Exp(\theta_i)$) with PDF

$$
f_{X_i}(t) = \frac{1}{\theta_i} \exp\left(-\frac{t}{\theta_i}\right), \text{ for } t \geq 0. \tag{2.2}
$$

To estimate the scale parameter θ_i using the system lifetime data $\boldsymbol{T}_i$, $i = 1, 2$, we use the method of moments estimator (MME) (Ng et al., 2012) defined as

$$\tilde{\theta}_i = \frac{\sum_{j=1}^{M_i} T_{ij}}{M_i \Delta_i},$$

where T_{ij} is the j-th system lifetime of System i and $\Delta_i = \sum_{k=1}^{n_i}(a_{ik}/k)$. The MME estimator $\tilde{\theta}_i$ is an unbiased estimator and the variance of $\tilde{\theta}_i$ is

$$Var(\tilde{\theta}_i) = \frac{\theta_i^2}{M_i}\left(\frac{2\Delta_i^{(2)}}{\Delta_i} - \Delta_i\right),$$

where $\Delta_i^{(2)} = \sum_{k=1}^{n_i}(a_{ik}/k^2)$.

Under the assumption of exponentially distributed component lifetime, testing the hypotheses of homogeneity in Eq. (2.1) is equivalent to testing

$$H_0^* : \theta_1/\theta_2 = 1 \quad \text{versus} \quad H_1^* : \theta_1/\theta_2 \neq 1. \tag{2.3}$$

We consider the test statistic based on the logarithm transformation of $R = \tilde{\theta}_1/\tilde{\theta}_2$ as follows:

$$Z_L = \frac{\ln R}{\sqrt{Var(\ln R)}} = \frac{\ln \tilde{\theta}_1 - \ln \tilde{\theta}_2}{\sqrt{Var(\ln \tilde{\theta}_1) + Var(\ln \tilde{\theta}_2)}},$$

where $Var(\ln \tilde{\theta}_i)$ can be approximated by the delta method as $Var(\ln \tilde{\theta}_i) \approx Var(\tilde{\theta}_i)/\tilde{\theta}_i^2$ for $i = 1, 2$. Under the null hypothesis H_0^* in Eq. (2.3), the test statistic Z_L is asymptotically standard normally distributed. Therefore, the p-value of the asymptotic test based on test statistic Z_L can be calculated as $2(1 - \Phi(|Z_L|)$ where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

2.2.1.2. Parametric Likelihood Ratio Test

Suppose that the component lifetime distributions for the components in System 1 and System 2 follow the same parametric family of distributions with PDF $f_{X_i}(t; \boldsymbol{\theta}_i)$ $i = 1, 2$ with parameter vector $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, then testing the hypotheses of homogeneity in Eq. (2.1) is equivalent to testing

$$H_0^{**} : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 \quad \text{versus} \quad H_1^{**} : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2. \tag{2.4}$$

From Eq. (1.3), the likelihood function based on System i lifetime data $\boldsymbol{T}_i = (T_{i1}, T_{i2}, \cdots, T_{iM_i})$, $i = 1, 2$, is

$$\begin{aligned} L_i(\boldsymbol{\theta}_i | \boldsymbol{T}_i) &= \prod_{j=1}^{M_i} f_{T_i}(T_{ij}; \boldsymbol{\theta}_i) \\ &= \prod_{j=1}^{M_i} \sum_{k=1}^{n_i} s_{ik} \binom{n_i}{k} k f_{X_i}(T_{ij}; \boldsymbol{\theta}_i) [F_{X_i}(T_{ij}; \boldsymbol{\theta}_i)]^{k-1} [\bar{F}_{X_i}(T_{ij}; \boldsymbol{\theta}_i)]^{n_i - k}. \end{aligned}$$

The MLE of $\boldsymbol{\theta}_i$ based on $\boldsymbol{T}_i = (T_{i1}, T_{i2}, \cdots, T_{iM_i})$ alone, denoted as $\hat{\boldsymbol{\theta}}_i$, can be obtained by maximizing $L_i(\boldsymbol{\theta}_i | \boldsymbol{T}_i)$ with respect to the $\boldsymbol{\theta}_i$.

Under the null hypothesis $H_0^{**} : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \boldsymbol{\theta}$ in Eq. (2.4), the likelihood function based on $(\boldsymbol{T}_1, \boldsymbol{T}_2)$ can be expressed as

$$L_{12}(\boldsymbol{\theta} | \boldsymbol{T}_1, \boldsymbol{T}_2) = L_1(\boldsymbol{\theta} | \boldsymbol{T}_1) L_2(\boldsymbol{\theta} | \boldsymbol{T}_2)$$

and the MLE of $\boldsymbol{\theta}$ based on the pooled data $(\boldsymbol{T}_1, \boldsymbol{T}_2)$, denoted as $\hat{\boldsymbol{\theta}}$, can be obtained by maximizing $L_{12}(\boldsymbol{\theta}|\boldsymbol{T}_1, \boldsymbol{T}_2)$ with respect to $\boldsymbol{\theta}$. The parametric likelihood ratio statistic is defined as

$$\begin{aligned}\lambda_{LR} &= -2\ln\left[\frac{\sup_{H_0^{**}} L_{12}(\boldsymbol{\theta}|\boldsymbol{T}_1, \boldsymbol{T}_2)}{\sup_{H_1^{**}} L_1(\boldsymbol{\theta}_1|\boldsymbol{T}_1)L_2(\boldsymbol{\theta}_2|\boldsymbol{T}_2)}\right] \\ &= -2\ln\left[\frac{L_{12}(\hat{\boldsymbol{\theta}}|\boldsymbol{T}_1, \boldsymbol{T}_2)}{L_1(\hat{\boldsymbol{\theta}}_1|\boldsymbol{T}_1)L_2(\hat{\boldsymbol{\theta}}_2|\boldsymbol{T}_2)}\right].\end{aligned}$$

By the Wilk's theorem (Wilks, 1938), the asymptotic distribution of the likelihood ratio statistic λ_{LR} under null hypothesis is chi-square with degrees of freedom ν, where ν is the difference in dimensionality of $\boldsymbol{\theta}$ and $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. The p-value of the parametric likelihood ratio test based on test statistic λ_{LR} can be approximated as $\Pr(\chi^2_\nu > \lambda_{LR})$ where χ^2_ν is a random variable following the chi-square distribution with degrees of freedom ν.

2.2.2. Nonparametric Mann-Whitney U Statistic

To test the hypotheses in Eq. (2.1), Zhang et al. (2015) proposed a nonparametric test procedure based on the Mann-Whitney U test, also called the Mann–Whitney–Wilcoxon test (Mann and Whitney, 1947; Wilcoxon, 1992), which is a nonparametric test for comparing two component lifetime distributions. Based on the system lifetime data from System 1 and System 2, we define the indicator function between the two systems as

$$D_{j_1,j_2} = \begin{cases} 1, & \text{if } T_{1j_1} < T_{2j_2}, \\ 0, & \text{otherwise,} \end{cases}$$

for $j_1 = 1, 2, \ldots, M_1$ and $j_2 = 1, 2, \ldots, M_2$. The U statistic for testing the homogeneity of component lifetime distributions is defined as

$$U = \sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_2} D_{j_1,j_2} = \sum_{j=1}^{M_2} (M_2 - j + 1) R_j, \tag{2.5}$$

where R_j is the number of ordered observations from sample $\boldsymbol{T}_1$ which are in between the $(j-1)$-th and j-th order statistics from sample $\boldsymbol{T}_2$. The support of the statistic U is $\{0, 1, \ldots, M_1 M_2\}$ and the null hypothesis in Eq. (2.1) is rejected if U is too large or too small. Here, we reject the null hypothesis in Eq. (2.1) at α level if $U \leq c_{U1}$ or $U \geq c_{U2}$, where c_{U1} and c_{U2} are critical values and can be determined by $\Pr(U \leq c_{U1} | H_0) \leq \alpha/2$ and $\Pr(U \geq c_{U2} | H_0) \leq \alpha/2$, respectively. The exact p-value can be calculated as presented in Zhang et al. (2015). Since the computation of the exact p-value requires the numerical evaluation of integration which can be computationally intensive, we consider using the Monte Carlo method to obtain the null distribution of U under different scenarios in this chapter. The procedure to get the simulated null distribution of U is described in Section 2.4 below.

2.3. Proposed Test Procedures Based on Empirical Likelihood Ratio

In this section, we develop test procedures based on empirical likelihood ratio to test the hypotheses in Eq. (2.1) nonparametrically. From Hall et al. (2015), the empirical likelihood function for System i based on $\boldsymbol{T}_i$ only ($i = 1, 2$) can be expressed as

$$L_{T_i}(t) = \binom{M_i}{Y_i(t)} h_i(p_i(t))^{Y_i(t)} [1 - h_i(p_i(t))]^{M_i - Y_i(t)}, \tag{2.6}$$

where $Y_i(t) = \sum_{j=1}^{M_i} I_{(t,\infty)}(T_{ij}),\ i = 1, 2$, and $I_{(t,\infty)}(T_{ij})$ is the indicator function and can be defined as

$$I_A(b) = \begin{cases} 1, & \text{if } b \in A, \\ 0, & \text{otherwise.} \end{cases}$$

The empirical likelihood function $L_{T_i}(t)$ in Eq. (2.6) is maximized at

$$\hat{p}_i(t) = h_i^{-1}(1 - \hat{F}_{T_i}(t)), \tag{2.7}$$

where $\hat{F}_{T_i}(t)$ is the empirical CDF of $F_{T_i}(t)$ defined as

$$\hat{F}_{T_i}(t) = 1 - \frac{1}{M_i} \sum_{j=1}^{M_i} I_{(t,\infty)}(T_{ij}). \tag{2.8}$$

That is, based on $\boldsymbol{T}_i$,

$$\sup\{L_{T_i}(t)\} = \binom{M_i}{Y_i(t)} h_i(\hat{p}_i(t))^{Y_i(t)} [1 - h_i(\hat{p}_i(t))]^{M_i - Y_i(t)}.$$

Under the null hypothesis that $F_{X_1}(t) = F_{X_2}(t)$ (or equivalently $p_1(t) = p_2(t) = p(t)$), we can pool the samples from System 1 and System 2 into an ordered sample of size $M = M_1 + M_2$, denoted as $\boldsymbol{T}^* = (< T^*_{(1)} < T^*_{(2)} < \cdots < T^*_{(M)})$. Following Hall et al. (2015)

and the idea of Kaplan-Meier estimator (Kaplan and Meier, 1958), the empirical likelihood function based on the pooled data is

$$L_T^*(t) = \prod_{i=1,2} \binom{M_i}{Y_i(t)} h_i(p(t))^{Y_i(t)}[1 - h_i(p(t))]^{M_i - Y_i(t)}. \tag{2.9}$$

The nonparametric MLE of $p(t)$, denoted as $\hat{p}(t)$, can be obtained by maximizing $L_{T_i}^*(t)$ with respect to $p(t)$, i.e.,

$$\sup\{L_T^*(t)\} = \prod_{i=1,2} \binom{M_i}{Y_i(t)} h_i(\hat{p}(t))^{Y_i(t)}[1 - h_i(\hat{p}(t))]^{M_i - Y_i(t)}.$$

Since a closed-form solution $\hat{p}(t)$ cannot be obtained, the maximization can be approximated by numerical methods such as the Newton-Raphson method. The nonparametric MLE $\hat{p}(t)$ can be written as

$$\hat{p}(t) = \begin{cases} 1, & t < T_{(1)}^*; \\ q_j, & t \in [T_{(j)}^*, T_{(j+1)}^*); \\ 0, & t \geq T_{(M)}^*. \end{cases}$$

where q_j maximizes the likelihood function in Eq. (2.9) for t in $[T_{(j)}^*, T_{(j+1)}^*)$, $j = 1, 2, \ldots, M-1$. For the initial values of q_j (denoted as $q_j^{(0)}$) of the iterative maximization algorithm, for $t \in [T_{(j)}^*, T_{(j+1)}^*)$, we consider a weighted function of $\hat{p}_i(t)$, $i = 1, 2$, in Eq. (2.7)

$$q_j^{(0)} = \frac{M_1 \hat{p}_1(t) + M_2 \hat{p}_2(t)}{M}, \quad j = 1, 2, \ldots, M-1.$$

The empirical likelihood ratio at time t is

$$R_T(t) = \frac{\sup\{L_T^*(t) : p_1(t) = p_2(t)\}}{\sup\{L_{T_1}(t)\}\sup\{L_{T_2}(t)\}}.$$

Then, we obtain the log empirical likelihood ratio as

$$\begin{aligned}
\ln R_T(t) &= \ln\left\{\frac{\sup\{L_T^*(t) : p_1(t) = p_2(t)\}}{\sup\{L_{T_1}(t)\}\sup\{L_{T_2}(t)\}}\right\} \\
&= \ln\left\{\frac{\prod_{i=1,2}\binom{M_i}{Y_i(t)} h_i(\hat{p}(t))^{Y_i(t)}[1-h_i(\hat{p}(t))]^{M_i - Y_i(t)}}{\prod_{i=1,2}\binom{M_i}{Y_i(t)} h_i(\hat{p}_i(t))^{Y_i(t)}[1-h_i(\hat{p}_i(t))]^{M_i - Y_i(t)}}\right\} \\
&= \sum_{i=1}^{2}\left\{Y_i(t)\ln\left[\frac{h_i(\hat{p}(t))}{h_i(\hat{p}_i(t))}\right] + (M_i - Y_i(t))\ln\left[\frac{1-h_i(\hat{p}(t))}{1-h_i(\hat{p}_i(t))}\right]\right\}.
\end{aligned}$$

Based on the log empirical likelihood ratio, we define

$$\begin{aligned}
G_t^2 &= 2\ln R_T(t) \\
&= 2\sum_{i=1}^{2}\left\{Y_i(t)\ln\left[\frac{h_i(\hat{p}(t))}{h_i(\hat{p}_i(t))}\right] + (M_i - Y_i(t))\ln\left[\frac{1-h_i(\hat{p}(t))}{1-h_i(\hat{p}_i(t))}\right]\right\},
\end{aligned}$$

which is a function of time t. We consider two functions based on G_t^2:

$$Z^* = \sup_{t\in(0,\infty)}[G_t^2 \times w(t)], \tag{2.10}$$

$$Z = \int_0^\infty G_t^2 dw(t), \tag{2.11}$$

where $w(t)$ is a weight function. Following Zhang (2006), we propose two test statistics for testing the hypotheses in Eq. (2.1) with different weight functions:

1. Take $w(t) = 1$ in Eq. (2.10), we obtain the test statistic

$$\begin{aligned} Z_K &\triangleq \sup_{t\in(0,\infty)} [G_t^2] \qquad (2.12) \\ &= \max_{1\le k\le M} \left\{ 2\sum_{i=1}^{2} Y_i(T^*_{(k)}) \ln\left[\frac{h_i(\hat{p}_i(T^*_{(k)}))}{h_i(\hat{p}(T^*_{(k)}))}\right] + (M_i - Y_i(T^*_{(k)})) \ln\left[\frac{1-h_i(\hat{p}_i(T^*_{(k)}))}{1-h_i(\hat{p}(T^*_{(k)}))}\right] \right\}. \end{aligned}$$

2. We denote the maximum likelihood estimator of component CDF for the pooled data as $\hat{F}_X(t)$ and define $\hat{F}_X(T^*_{(0)}) = 0$. Then, $\hat{F}_X(t) = 1 - \hat{p}(t)$. In Eq.(2.11), take the weight function

$$dw(t) = \frac{1}{\hat{F}_X(t)(1-\hat{F}_X(t))} d\hat{F}_X(t).$$

Then, the test statistic can be written as

$$\begin{aligned} Z_A &\triangleq \int_0^\infty G_t^2 \frac{1}{\hat{F}_X(t)(1-\hat{F}_X(t))} d\hat{F}_X(t) \\ &= 2\sum_{k=1}^{M} \frac{\hat{p}(T^*_{(k)}) - \hat{p}(T^*_{(k-1)})}{\hat{p}(T^*_{(k)})(1-\hat{p}(T^*_{(k)}))} \times \qquad (2.13) \\ &\left\{ \sum_{i=1}^{2} \left[Y_i(T^*_{(k)}) \ln \frac{h_i(\hat{p}_i(T^*_{(k)}))}{h_i(\hat{p}(T^*_{(k)}))} + (M_i - Y_i(T^*_{(k)})) \ln \frac{1-h_i(\hat{p}_i(T^*_{(k)}))}{1-h_i(\hat{p}(T^*_{(k)}))} \right] \right\}. \end{aligned}$$

Large values of Z_K and Z_A support the alternative hypothesis in Eq. (2.1), which leads to the rejection of the null hypothesis in Eq. (2.1).

2.4. Null Distributions of Z_K, Z_A, and U Based on Monte Carlo Method

Since the distributions of Z_K and Z_A are intractable theoretically in general, even under the null hypothesis, we rely on the Monte Carlo method to obtain the null distributions of Z_K and Z_A.

We simulate the lifetimes of M_1 systems for System 1 and the lifetimes of the M_2 systems for System 2 from any component lifetime distribution F_X with $F_{X_1} = F_{X_2} = F_X$ for given n_1, n_2, M_1, M_2, and system signatures s_1 and s_2. The statistics Z_K and Z_A are computed from the simulated lifetimes.

In practice, we can simulate the null distributions of Z_K, Z_A, and U by choosing F_X as a distribution that is easy to simulate (e.g., standard exponential distribution, $Exp(1)$) with a large number of Monte Carlo simulations (say, $1,000,000$ times). For example, the simulated 90-th, 95-th, and 99-th percentage points for Z_K and Z_A with $\mathrm{s}_1 = (0, 0, 0, 1)$ (i.e., $n_1 = 4$) and $\mathrm{s}_2 = (1, 0, 0)$ (i.e., $n_2 = 3$) based on $1,000,000$ simulations with $F_X \sim Exp(1)$ are presented in Table 2.1.

Table 2.1: Simulated 90-th, 95-th, and 99-th percentage points of the null distributions of Z_K and Z_A with $\mathrm{s}_1 = (0, 0, 0, 1)$ (i.e., $n_1 = 4$) and $\mathrm{s}_2 = (1, 0, 0)$ (i.e., $n_2 = 3$) for different sample sizes of $M_1 = M_2$.

	Z_K			Z_A		
Sample sizes	90%	95%	99%	90%	95%	99%
$M_1 = M_2 = 10$	4.2826	6.2433	10.0858	4.2609	6.6789	13.4692
$M_1 = M_2 = 15$	5.7377	6.6794	10.2433	4.7174	7.0945	13.4382
$M_1 = M_2 = 20$	7.6502	7.6502	10.7033	5.1977	7.1561	13.2830
$M_1 = M_2 = 30$	6.9730	7.7890	11.4753	5.9456	7.6422	13.4232
$M_1 = M_2 = 50$	7.1904	8.7383	11.8581	6.5791	8.3773	13.6362

For the null distribution of the Mann-Whitney U statistic, since U is discrete and the number of possible values of U can be small when the sample sizes M_1 and M_2 are small, the actual percentage when U equals to or less than the critical value might be much larger than the nominal significance level. Therefore, instead of using the exact distribution presented in Zhang et al. (2015), we propose to use a Monte Carlo simulation method to obtain the approximated percentage points of the null distribution and use a randomization procedure to obtain a test procedure based on U with the required significance level. To illustrate the procedure for obtaining the critical values, we consider the following example with parallel and series systems with signatures $\mathbf{s}_1 = (0,0,0,1)$ and $\mathbf{s}_2 = (1,0,0)$, respectively. In Table 2.2, we present the critical values of the Mann-Whitney statistic U and the corresponding simulated probability that U is more extreme than or equal to the critical value for different significance levels and sample sizes. The values in Table 2.2 are generated based on $1,000,000$ simulations. For illustrative purposes, we consider the case of $M_1 = M_2 = 10$ to demonstrate how to conduct the hypothesis test at 5% significance level.

Table 2.2: Simulated 0.5-th, 2.5-th, 5-th, 95-th, and 99-th percentage points of the null distributions of U with $\mathbf{s}_1 = (0,0,0,1)$ (i.e., $n_1 = 4$) and $\mathbf{s}_2 = (1,0,0)$ (i.e., $n_2 = 3$) for different sample sizes $M_1 = M_2$

Percentage / Sample Size	0.5%			2.5%			5%		
$M_1 = M_2$	c_{U1}	$\Pr(U = c_{U1})$	$\Pr(U \le c_{U1})$	c_{U1}	$\Pr(U = c_{U1})$	$\Pr(U \le c_{U1})$	c_{U1}	$\Pr(U = c_{U1})$	$\Pr(U \le c_{U1})$
10	0	0.3254	0.3254	0	0.3254	0.3254	0	0.3254	0.3254
15	0	0.1483	0.1483	0	0.1483	0.1483	0	0.1483	0.1483
20	0	0.0648	0.0648	0	0.0648	0.0648	0	0.0648	0.0648
30	0	0.0114	0.0114	2	0.0110	0.0301	4	0.0149	0.0571
50	8	0.0013	0.0062	17	0.0034	0.0280	23	0.0051	0.0548
Percentage / Sample Size	95%			97.5%			99.5%		
$M_1 = M_2$	c_{U2}	$\Pr(U = c_{U2})$	$\Pr(U \ge c_{U2})$	c_{U2}	$\Pr(U = c_{U2})$	$\Pr(U \ge c_{U2})$	c_{U2}	$\Pr(U = c_{U2})$	$\Pr(U \ge c_{U2})$
10	0	0.3254	0.3254	0	0.3254	0.3254	0	0.3254	0.3254
15	0	0.1483	0.1483	0	0.1483	0.1483	0	0.1483	0.1483
20	0	0.0648	0.0648	0	0.0648	0.0648	0	0.0648	0.0648
30	0	0.0114	0.0114	2	0.0110	0.0301	4	0.0149	0.0571
50	8	0.0013	0.0062	17	0.0034	0.0280	23	0.0051	0.0548

From Table 2.2, for $\alpha = 0.05$, the critical values are $c_{U1} = 0$ and $c_{U2} = 12$ and we have the following probability

$$\Pr(U \leq 0) = 0.3254,\ \Pr(U \geq 12) = 0.02520.$$

If we use the critical values $c_{U1} = 0$ and $c_{U2} = 12$ directly, the significance level will be $0.3254 + 0.0252 = 0.3506$, which is much higher than the 5% level. Therefore, we use the simulated null distribution with a randomization procedure to control the significance level of the test based on U. For example, to control $\Pr($reject H_0 when U is too small$|H_0)$ ≤ 0.025, we do not reject H_0 if the observed value of U is greater than 0 and less than 12. If the observed value of U is 0, we reject H_0 with probability $0.025/0.3254$, i.e.,

$$\Pr(\text{reject } H_0 \text{ when } U \text{ is too small}|H_0) = \Pr(U \leq 0|H_0)(0.025/0.3254) = 0.025.$$

Similarly, to control $\Pr($reject H_0 when U is too large$|H_0) \leq 0.025$, we do not reject H_0 if the observed value of U is less than 12 and reject H_0 if the observed value of U is larger than 12. If the observed value of U is 12, we reject H_0 with probability $(0.025 - \Pr(U > 12)/\Pr(U = 12) = (0.025 - (\Pr(U \geq 12) - \Pr(U = 12))/\Pr(U = 12) = 0.9737$, i.e.,

$$\begin{aligned}
& \Pr(\text{reject } H_0 \text{ when } U \text{ is too large}|H_0) \\
= \ & \Pr(U > 12|H_0) + \Pr(U = 12|H_0)\left[\frac{0.025 - \Pr(U > 12|H_0)}{\Pr(U = 12|H_0)}\right] \\
= \ & (\Pr(U \geq 12|H_0) - \Pr(U = 12|H_0)) + \Pr(U = 12|H_0) \times 0.9737 \\
= \ & 0.025.
\end{aligned}$$

Obviously, when $\Pr(U \leq c_{U1}) = 0.025$ and $\Pr(U \geq c_{U2}) = 0.025$, no randomization procedure is needed. Following this procedure, we can obtain approximate p-values of the two-sided U test under H_0 for $\alpha = 0.01,\ 0.05,\ 0.1$ using the critical values c_{U1} and c_{U2} and the corresponding values of $\Pr(U \leq c_{U1})$ and $\Pr(U \geq c_{U2})$ when these probabilities are not equal to $\alpha/2$.

2.5. Illustrative Example

To illustrate the test procedures developed in this chapter, we analyze a data set based on the example presented in Yang et al. (2016) and Frenkel and Khvatskin (2006). The example given in Yang et al. (2016) and Frenkel and Khvatskin (2006) described the phosphor acid filter system as a real-life prototype of a consecutive 2-out-of-n system. For a consecutive 2-out-of-n system, the system fails when any 2 adjacent components fail. For illustrative purposes, we consider that System 1 is a consecutive 2-out-of-8 system with system signature $\mathbf{s}_1 = (0, 1/4, 11/28, 2/7, 1/14, 0, 0, 0)$ and component lifetimes follow a Birnbaum-Saunders distribution (Birnbaum and Saunders, 1969) with CDF

$$F_X(t; a, b) = \Phi\left[\frac{1}{a}\left(\left(\frac{t}{b}\right)^{1/2} - \left(\frac{b}{t}\right)^{1/2}\right)\right], \quad t > 0,$$

where the shape parameter is $a = 1$ and scale parameter $b = 1$ and System 2 is a 4-component with system signature $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ and component lifetimes follow a $Weibull(3, 2)$ distribution. The system lifetime data for System 1 and System 2 with sample sizes $M_1 = M_2 = 20$ are presented in Table 2.3.

The lifetime data is listed in the following table. A hypothesis test is conducted to determine if the components from two different systems follow the same lifetime distribution.

Table 2.3: Simulated lifetime data from System 1 and System 2 with $M_1 = M_2 = 20$

j	1	2	3	4	5	6	7	8	9	10
T_{1j}	0.2598	0.2803	0.3329	0.4172	0.4532	0.459	0.5541	0.5769	0.5842	0.7784
T_{2j}	0.5890	0.6423	0.7774	0.9879	1.0754	1.1200	1.1685	1.2410	1.2412	1.2642
j	11	12	13	14	15	16	17	18	19	20
T_{1j}	0.7917	0.8565	0.8895	0.9186	0.9348	1.1130	1.2049	1.3938	1.4406	1.6351
T_{2j}	1.4164	1.4401	1.4924	1.5415	1.6912	2.0695	2.4374	2.5627	2.6197	2.7791

The nonparametric MLE of the SF of the component lifetime distribution F_{X_1} based on $\boldsymbol{T}_1$ from Eq. (2.8) (denoted as $\hat{\bar{F}}_{X_1}$), the nonparametric MLE of the SF of the component lifetime distribution F_{X_2} based on $\boldsymbol{T}_2$ from Eq. (2.8) (denoted as $\hat{\bar{F}}_{X_2}$), the average of $\hat{\bar{F}}_{X_1}$ and $\hat{\bar{F}}_{X_2}$, and the nonparametric MLE of the component lifetime distribution based on the pooled data ($\boldsymbol{T}_1$, $\boldsymbol{T}_2$) under $H_0 : F_{X_1} = F_{X_2}$ by maximizing Eq. (2.9), are plotted in Figure 2.1.

To test the hypotheses in Eq. (2.1), using the nonparametric Mann-Whitney U test and the two proposed empirical likelihood ratio tests at 1% level of significance for the data set in Table 2.3, we obtain the critical values based on the procedures described in Section 2.4 with $1,000,000$ simulations. The critical values for the empirical likelihood ratio tests based on Z_K and Z_A are 12.6394 and 18.4978, respectively, and the 0.5% and 99.5% percentile of the Mann-Whitney U statistic are 172 and 314, respectively.

For the data presented in Table 2.3, we can compute the test statistics $Z_K = 13.5608$, $Z_A = 22.0917$, and $U = 334$. We observe that all these test statistics are larger than their corresponding critical value at a 1% level of significance. Therefore, we reject the null hypothesis in Eq. (2.1) at 1% level with p-value of 2×10^{-6} based on test statistics Z_A and Z_K and p-value of 0.01 based on the test statistic U. These results agree with our expectation since the component lifetimes in System 1 are simulated from a Birnbaum-Saunders distribution, and the component lifetimes in System 2 are simulated from a Weibull distribution.

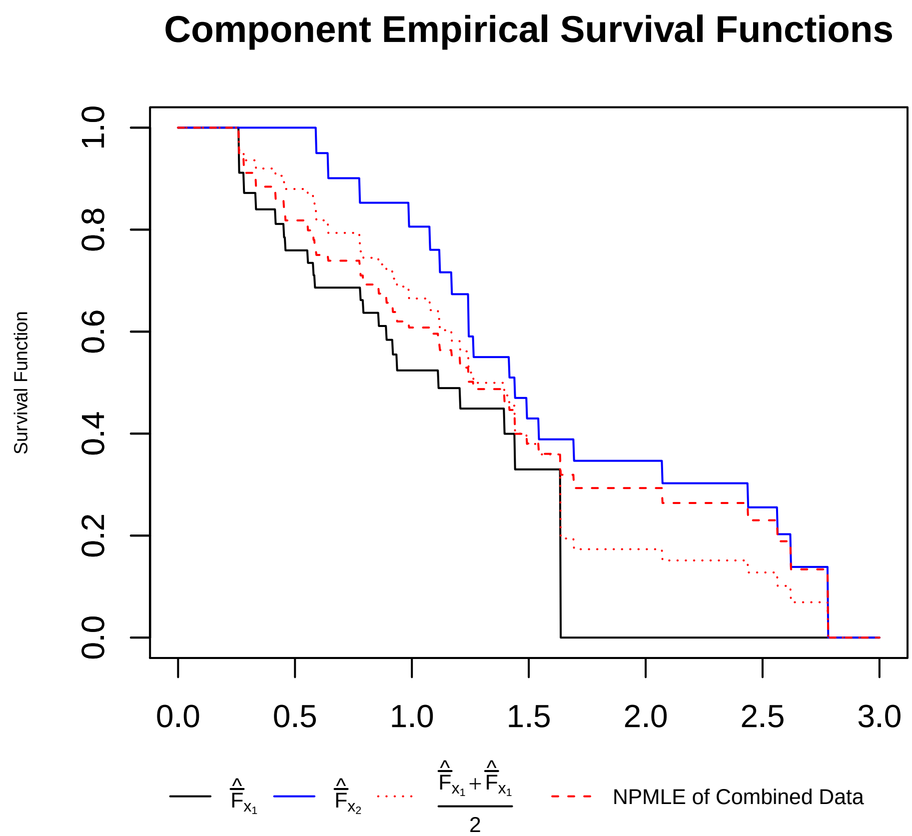

Figure 2.1: The nonparametric MLE of the SF of the component lifetime distribution F_{X_1} based on $\boldsymbol{T}_1$ from Eq. (2.8) (denoted as $\hat{\hat{F}}_{X_1}$), the nonparametric MLE of the SF of the component lifetime distribution F_{X_2} based on $\boldsymbol{T}_2$ from Eq. (2.8) (denoted as $\hat{\hat{F}}_{X_2}$), the average of $\hat{\hat{F}}_{X_1}$ and $\hat{\hat{F}}_{X_2}$, and the nonparametric MLE of the component lifetime distribution based on the pooled data ($\boldsymbol{T}_1$, $\boldsymbol{T}_2$) under $H_0 : F_{X_1} = F_{X_2}$ by maximizing Eq. (2.9)

2.6. Monte Carlo Simulation Studies

In this section, Monte Carlo simulation studies are used to evaluate the performances of those parametric and nonparametric test procedures described in Sections 2.2 and 2.3 for testing the hypotheses in Eq. (2.1).

In the first simulation study, we conduct parametric tests to examine how the Type-I error rates vary when the underlying distributions are misspecified. We consider applying the asymptotic parametric test under exponentially distributed component lifetimes assumption presented in Section 2.2.1.1 to cases when the component lifetimes follow different statistical distributions described above. The PDFs of the distributions considered in the first simulation study are plotted in Figure 2.2. We consider that System 1 is a 4-component parallel system and System 2 is a 3-component series system with system signatures $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$, respectively, with different sample sizes $M_1 = M_2 = 10,\ 15,\ 20,\ 30$, and 50. The simulated rejection rates of the asymptotic parametric test based on $10,000$ simulations under the null hypothesis that the components in System 1 and System 2 have the same distribution with 5% level of significance are presented in Table 2.4.

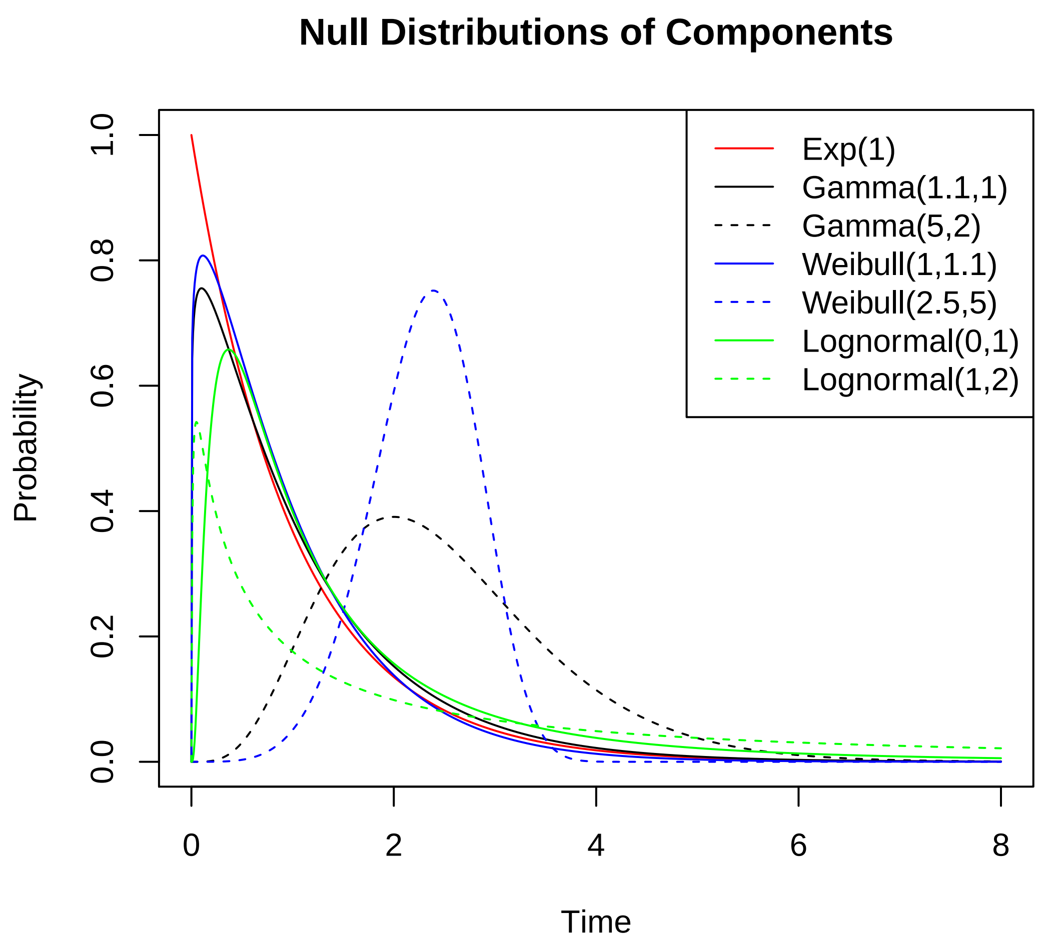

Figure 2.2: PDFs of the distributions considered in the Monte Carlo simulation studies.

Table 2.4: Simulated rejection rates of the asymptotic parametric test under the exponentially distributed component lifetimes assumption when the null hypothesis is true at 5% significant level for different underlying component lifetime distributions and different sample sizes.

True Dist. \ Sample Size	$M_1 = M_2 = 10$	$M_1 = M_2 = 15$	$M_1 = M_2 = 20$	$M_1 = M_2 = 30$	$M_1 = M_2 = 50$
$Exp(1)$	0.0531	0.0527	0.0491	0.0524	0.0482
$Exp(0.1)$	0.0531	0.0523	0.0499	0.0525	0.0492
$Gamma(5, 2)$	0.9707	0.9998	1.0000	1.0000	1.0000
$Gamma(1.1, 1)$	0.0465	0.0487	0.0546	0.0547	0.0617
$Gamma(1.5, 1)$	0.0949	0.1374	0.1918	0.2949	0.5014
$Gamma(1.1, 2)$	0.0465	0.0487	0.0546	0.0547	0.0617
$Weibull(1, 1.1)$	0.0474	0.0535	0.0639	0.0730	0.1058
$Weibull(2.5, 1.5)$	0.2466	0.4061	0.5563	0.7691	0.9621
$Weibull(2.5, 5)$	1.0000	1.0000	1.0000	1.0000	1.0000
$Lognormal(0, 1)$	0.0406	0.0437	0.0457	0.0461	0.0490
$Lognormal(1, 2)$	0.8821	0.9448	0.9740	0.9930	0.9993

Table 2.4 shows that the asymptotic parametric test developed under the exponential assumption inflates the simulated type-I error rates when the underlying component lifetime distributions deviate from the exponential, such as $Gamma(5, 2)$, $Weibull(2.5, 5)$, and $Lognormal(1, 2)$ (see Figure 2.2). On the other hand, the simulated Type-I error rates are close to the nominal level of 5% when the underlying component lifetime distributions are exponential or similar to the exponential distribution, such as $Gamma(1.1, 1)$, $Weibull(1, 1.1)$ and $Lognormal(0, 1)$ (see Figure 2.2). These results indicate that those parametric tests for homogeneity of component lifetime distributions may not be appropriate, especially when the underlying distribution is unknown.

In the second simulation study, we evaluate the power performances of the proposed empirical likelihood ratio tests Z_K and Z_A, and the Mann-Whitney U test and compare them with the parametric tests by assuming that the underlying distributions agree with the distributions the data are generated from.

For the parametric likelihood ratio test with a two-parameter distribution, we assume that both F_{X_1} and F_{X_2} are from the same class of distributions and the two parameters are unknown, but one of the parameters in F_{X_1} and F_{X_2} are the same under the alternative hypothesis. For example, in the simulation where the component lifetimes follow the Weibull distribution, F_{X_1} and F_{X_2} are $Weibull(\lambda_1, \gamma_1)$ and $Weibull(\lambda_2, \gamma_2)$, respectively, we use the hypotheses

$$H_0^{**} : \lambda_1 = \lambda_2 \text{ and } \gamma_1 = \gamma_2$$
$$\text{versus } H_1^{**} : \lambda_1 \neq \lambda_2 \text{ and } \gamma_1 = \gamma_2.$$

As shown in the first simulation study, the asymptotic parametric test developed under the exponentially distributed components assumption may not be appropriate for distributions other than exponential distribution; we only consider the asymptotic parametric test when the data are generated from an exponential distribution. In this simulation study, we consider the following settings for the system structures of System 1 and System 2 (Navarro et al., 2007):

[S1] System 1: 4-component parallel system with system signature $\mathbf{s}_1 = (0, 0, 0, 1)$;
System 2: 3-component series system with system signature $\mathbf{s}_2 = (1, 0, 0)$.

[S2] System 1: 3-component system with system signature $\mathbf{s}_1 = (0, 2/3, 1/3)$;
System 2: 4-component system with system signature $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$.

[S3] System 1: 3-component system with system signature $\mathbf{s}_1 = (0, 2/3, 1/3)$;
System 2: 4-component system with system signature $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$.

For comparative purposes, we also consider the nonparametric test procedures based on complete component-level data, i.e., all $n_k M_k$ component lifetimes are observed for

System k. This is equivalent to considering System 1 and System 2 to be 1-component systems such that the lifetime of the component is equal to the lifetime of the system. Since this is an ideal scenario with complete information on the component lifetimes, we denote this case as "Full data". Note that the power performance of the test procedures based on complete component-level data can be served as a benchmark for the power comparisons since this is the case that all the $n_1M_1 + n_2M_2$ component lifetimes are observed. The critical values of nonparametric tests are obtained from the simulation mentioned in Section 2.4 based on $1,000,000$ simulation times.

The component lifetimes are generated from the following distribution settings:

[D1] Exponential distributions with changes in the scale parameter:

System 1: $Exp(\theta_1)$ with $\theta_1 = 1$ (i.e., $\ln\theta_1 = 0$);
System 2: $Exp(\theta_2)$ with $\ln\theta_2$ varies from -1.6 to 1.6 with increment 0.1 (denoted by $-1.6\ (0.1)\ 1.6$);

[D2] Weibull distributions with changes in the shape parameter:

System 1: $Weibull(\lambda_1, \gamma_1)$ with $\lambda_1 = 1$ and $\gamma_1 = 1$;
System 2: $Weibull(\lambda_2, \gamma_2)$ with $\lambda_2 = 1$ and γ_2 varies from 0.5 to 1.5 with increment 0.1 (denoted by $0.5(0.1)1.5$);

[D3] Lognormal distributions with changes in the standard deviation on the log-scale (i.e., change in the shape parameter):

System 1: $Lognormal(\mu_1, \sigma_1)$ with $\mu_1 = 0$ and $\sigma_1 = 2$;
System 2: $Lognormal(\mu_2, \sigma_2)$ with $\mu_2 = 0$ and σ_2 varies from 1 to 3 with increment 0.1 (denoted by $1(0.1)3$).

[D4] Weibull distributions with changes in the scale parameter:

System 1: $Weibull(\lambda_1, \gamma_1)$ with $\lambda_1 = 2.5$ and $\gamma_1 = 5$;
System 2: $Weibull(\lambda_2, \gamma_2)$ with λ_2 varies from 1.5 to 3.5 with increment 0.1 (denoted by $1.5(0.1)3.5$) and $\gamma_2 = 5$;

[D5] Gamma distribution with changes in the shape parameter:

System 1: $Gamma(\alpha_1, \beta_1)$ with $\alpha_1 = 5$ and $\beta_1 = 2$;

System 2: $Gamma(\alpha_2, \beta_2)$ with α_2 varies from 3 to 7 with increment 0.2 (denoted by $3(0.2)7$) and $\beta_2 = 2$;

[D6] Gamma distribution with changes in the rate parameter:

System 1: $Gamma(\alpha_1, \beta_1)$ with $\alpha_1 = 5$ and $\beta_1 = 2$;

System 2: $Gamma(\alpha_2, \beta_2)$ with $\alpha_2 = 5$ and β_2 varies from 1 to 3 with increment 0.1 (denoted by $1(0.1)3$);

[D7] Lognormal distributions with changes in the mean on the log-scale (i.e., change in the scale parameter):

System 1: $Lognormal(\mu_1, \sigma_1)$ with $\mu_1 = 0$ and $\sigma_1 = 1$;

System 2: $Lognormal(\mu_2, \sigma_2)$ with μ_2 varies from -1.6 to 1.6 with increment 0.1 (denoted by $-1.6(0.1)1.6$) and $\sigma_2 = 1$.

The rejection rates with significance level 5% under different settings are estimated based on $10,000$ simulations. For the sake of saving space, we present here the simulated power curves for setting **[D1]** with **[S1]** in Figure 2.3, the simulated power curves for setting **[D2]** with **[S1]** in Figure 2.4, the simulated power curves for setting **[D3]** with **[S1]**, and the simulated power curves for settings **[D4]** with **[S1]**, **[S2]**, and **[S3]** in Figures 2.6–2.8, respectively. We present the simulated power curves for each figure under the sample sizes $M_1 = M_2 = 10$, 15, 20, 30, and 50. The simulated power curves for other settings, including [**D5**]–[**D7**] with system structures [**S1**]–[**S3**] are presented in Figures A.1–A.15 in Appendix A.

The simulated power curves centered at the simulated rejection rates under the null hypothesis (i.e., $F_{X_1} = F_{X_2}$), which are expected to be close to the nominal significance level of 5%. When the differences between the parameters increase (i.e., moving away from the center), we expect the simulated power values to increase. The closer the power values are to one, the better the performance of the test procedure. As mentioned above, the simulated power curves with “Full data" can serve as benchmarks for comparisons as the power values based on complete component-level data are larger than those based on system-level data. Moreover, we expect that the power values for the parametric likelihood ratio tests under the correct model specification of underlying distributions are better than those of nonparametric tests.

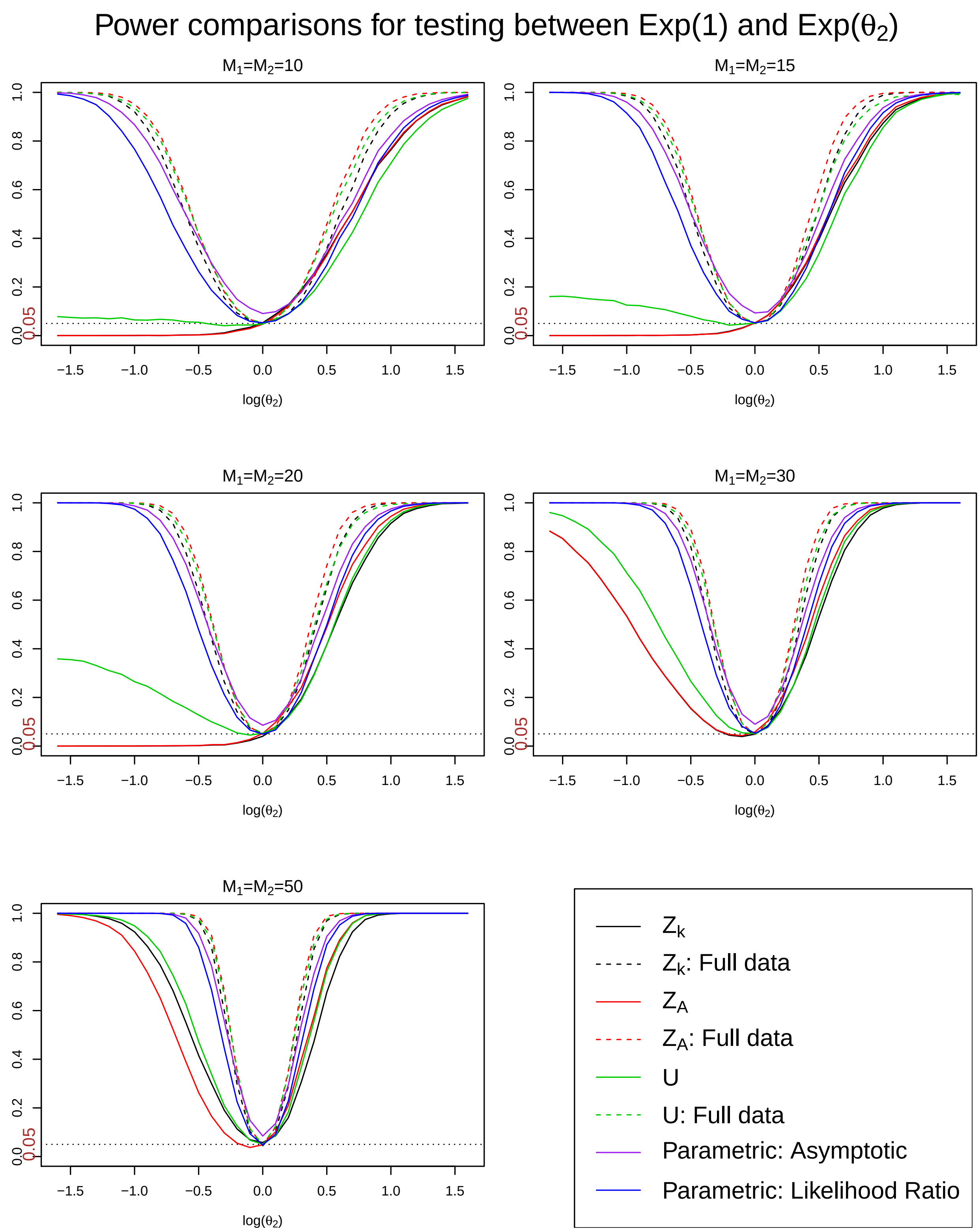

Figure 2.3: Simulated power curves of the parametric and nonparametric test for $\ln\theta_1 = 0$ and $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ with $M_1 = M_2$ for exponential components from systems $\mathbf{s}_1 = (0,0,0,1)$ and $\mathbf{s}_2 = (1,0,0)$ (i.e., [**D1**] with system structures [**S1**]).

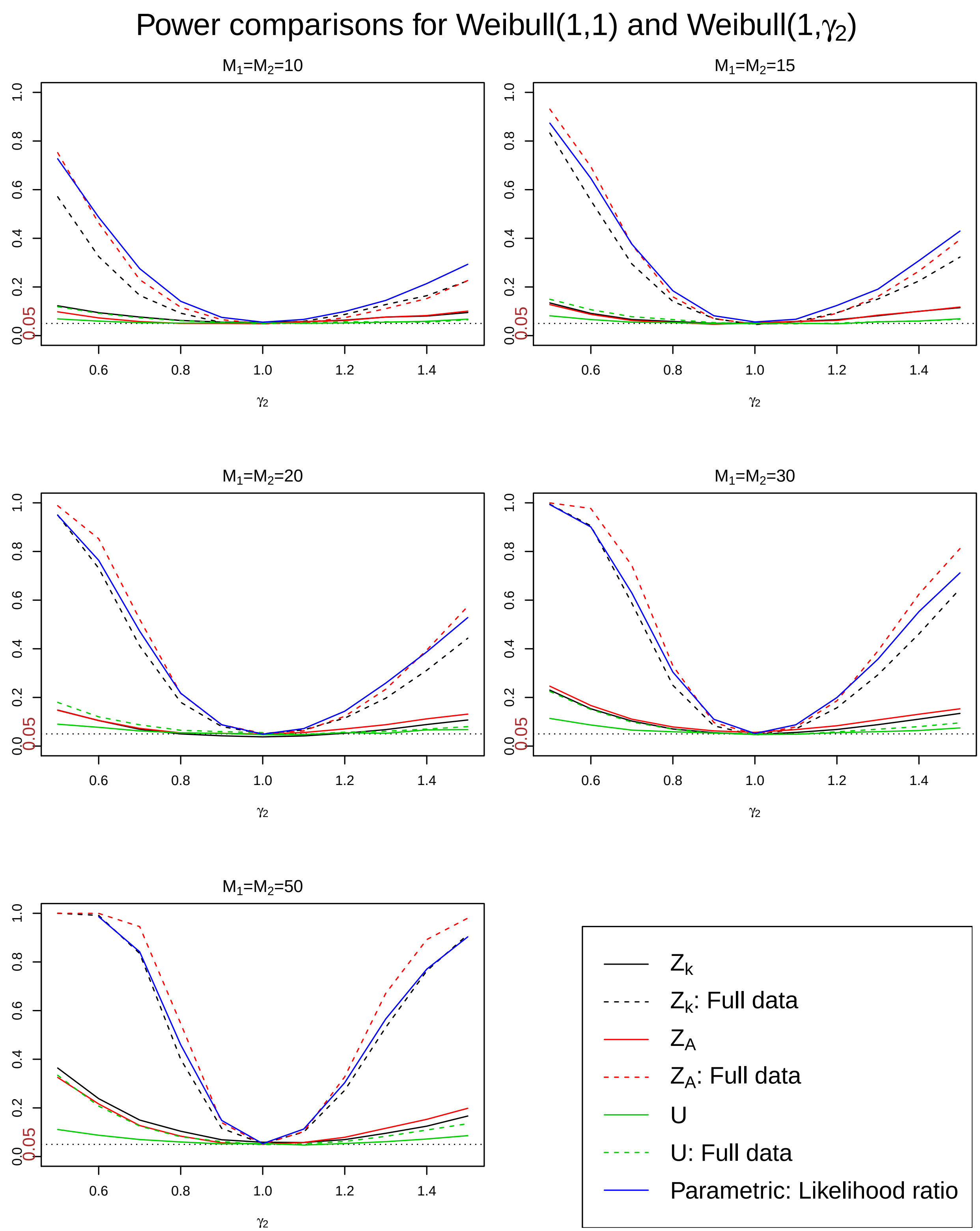

Figure 2.4: Simulated power curves of the parametric and nonparametric test for $\gamma_1 = 1$ and $\gamma_2 = 0.5\ (0.1)\ 1.5$, $\lambda_1 = \lambda_2 = 1$ with $M_1 = M_2$ for Weibull components from systems $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$ (i.e., [**D2**] with system structures [**S1**]).

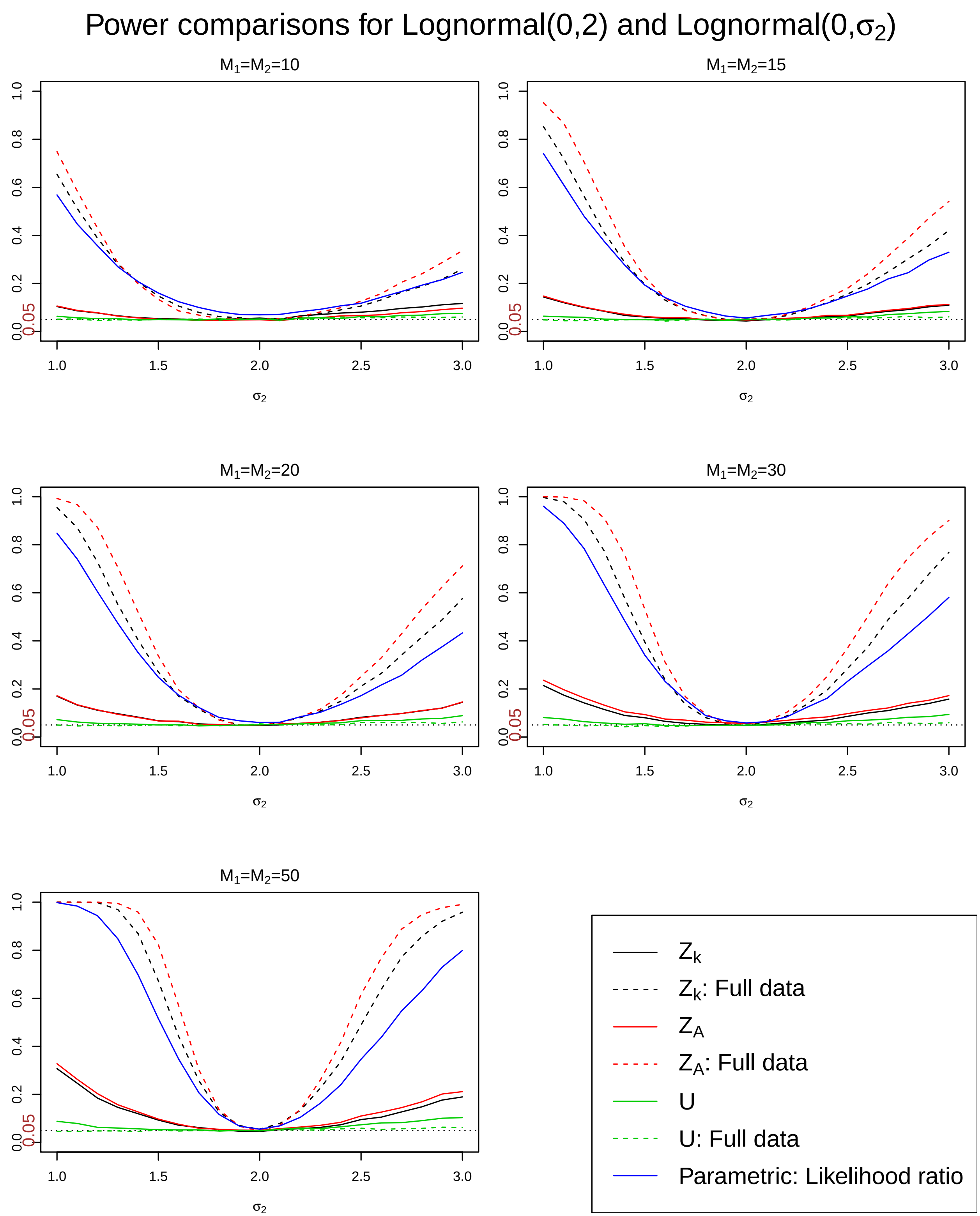

Figure 2.5: Simulated power curves of the parametric and nonparametric test for $\mu_1 = \mu_2 = 0$, $\sigma_1 = 2$ and $\sigma_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for lognormal components from systems $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$ (i.e., [**D3**] with system structures [**S1**]).

From Figures 2.3 and 2.4, we observe that for the extreme setting **[S1]** with System 1 being a parallel system and System 2 being a series system, when the sample sizes are small (say, $M_1 = M_2 \leq 20$), the power values are low when the mean lifetime of components in System 1 is smaller than the mean lifetime of components in System 2 (i.e., the left-hand side of the power curve) for the nonparametric tests. As pointed out by Zhang et al. (2015), it is due to the nature of the problem since we are comparing the worst component in one system to the best component in another system to determine if the lifetime characteristics of the components are the same.

From Figures 2.3–2.8, the power values of the nonparametric tests with complete component-level data and the parametric tests under the correct specification of the underlying component lifetime distribution are larger than the power values of the nonparametric tests based on system-level data. We observe that the proposed empirical likelihood ratio tests provide comparative power performance in most cases compared to the Mann-Whitney U test. After considering the upper bound of the Monte Carlo error $\sqrt{(0.5)(1-0.5)/10000} = 0.005$, the proposed empirical likelihood ratio tests are more powerful for small sample sizes on the right-hand side. Between the two empirical likelihood ratio tests, the test based on Z_A tends to have better power performance than the test based on Z_K in most cases.

For comparative purposes, the differences between the simulated power values of the test procedures based on U statistic and Z_K, and the test procedures based on U statistic and Z_A for Weibull distribution with $M_1 = M_2 = 30$ and for lognormal distribution with different sample sizes are plotted in Figures 2.9 and 2.10, respectively. Negative values of the differences indicate that the proposed tests based on Z_A and Z_K provide better power performance than the Mann-Whitney U test. We also included the lines for plus and minus three Monte Carlo errors ($\pm 3MCE$) to indicate if the differences are significant.

When the mean lifetime of the component lifetime distribution of System 2 is larger than the mean lifetime of the component lifetime distribution of System 1, the proposed empirical likelihood ratio test based on Z_A provides better power values among the three nonparametric tests considered here. The advantage of the test based on Z_A is more significant when the two systems are extremely different (i.e., setting **[S1]**). When the mean lifetime of the component lifetime distribution of System 2 is smaller than the mean lifetime of the component lifetime distribution of System 1, the Mann-Whitney U test provides better power values among the three nonparametric tests considered here.

The proposed nonparametric tests outperform when the shape of the component lifetime distributions are different (i.e., settings **[D2]** and **[D3]**). For example, from the top left panel of Figure 2.9 and Figure 2.10 show that the proposed empirical likelihood ratio tests provide better power performance than the Mann-Whitney U test in detecting changes in the shape parameters. Despite the alteration of the shape parameter in setting **[D5]** (see Figures A.1, A.7, and A.13 in Appendix A), the proposed tests do not outperform the U test, which may be due to the deviations in the shape of the gamma distributions being minor.

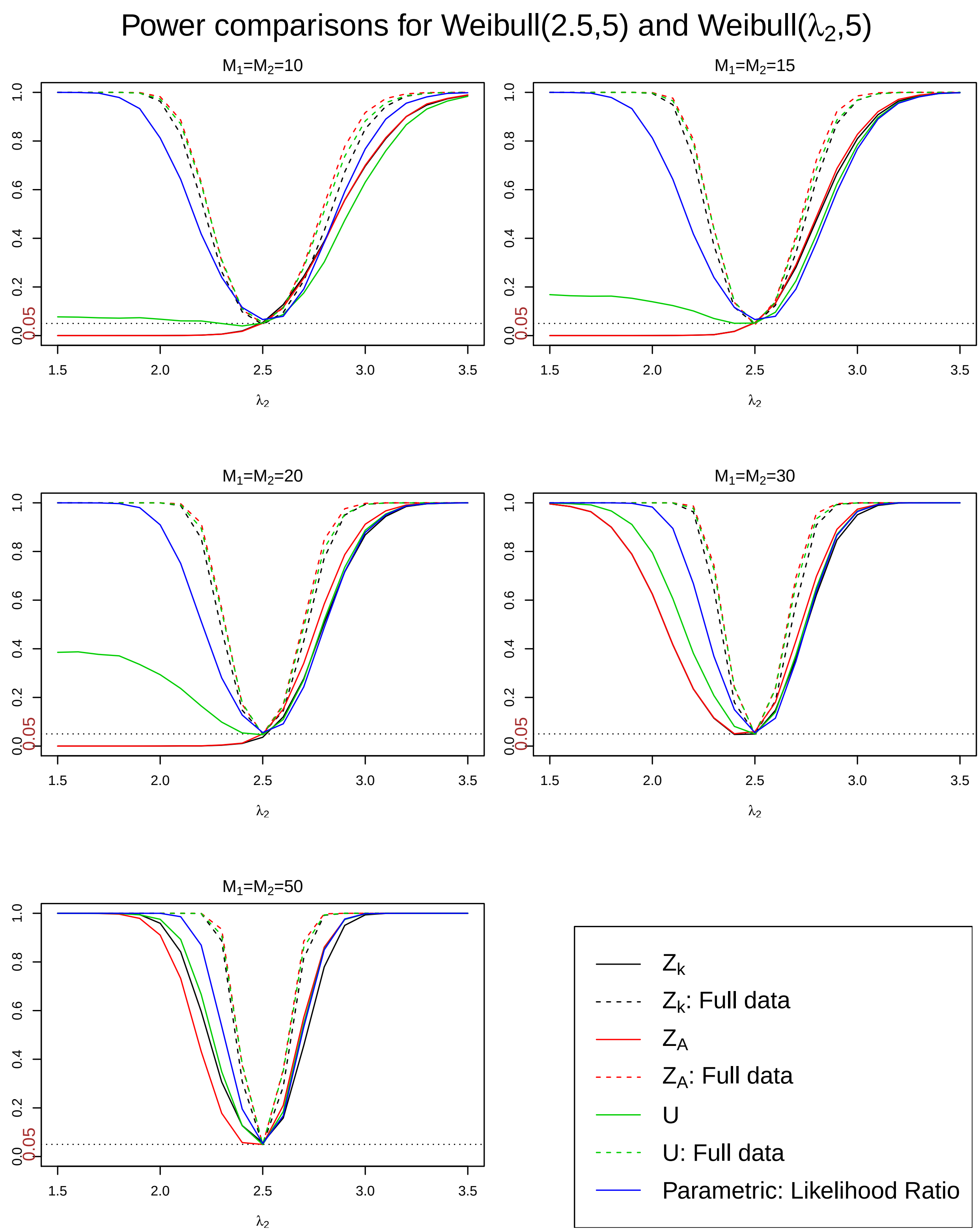

Figure 2.6: Simulated power curves of the parametric and nonparametric test for $\lambda_1 = 2.5$ and $\lambda_2 = 1.5\ (0.1)\ 3.5$, $\gamma_1 = \gamma_2 = 5$ with $M_1 = M_2$ for Weibull components from systems $\mathrm{s}_1 = (0,0,0,1)$ and $\mathrm{s}_2 = (1,0,0)$ (i.e., **[D4]** with system structures [**S1**]).

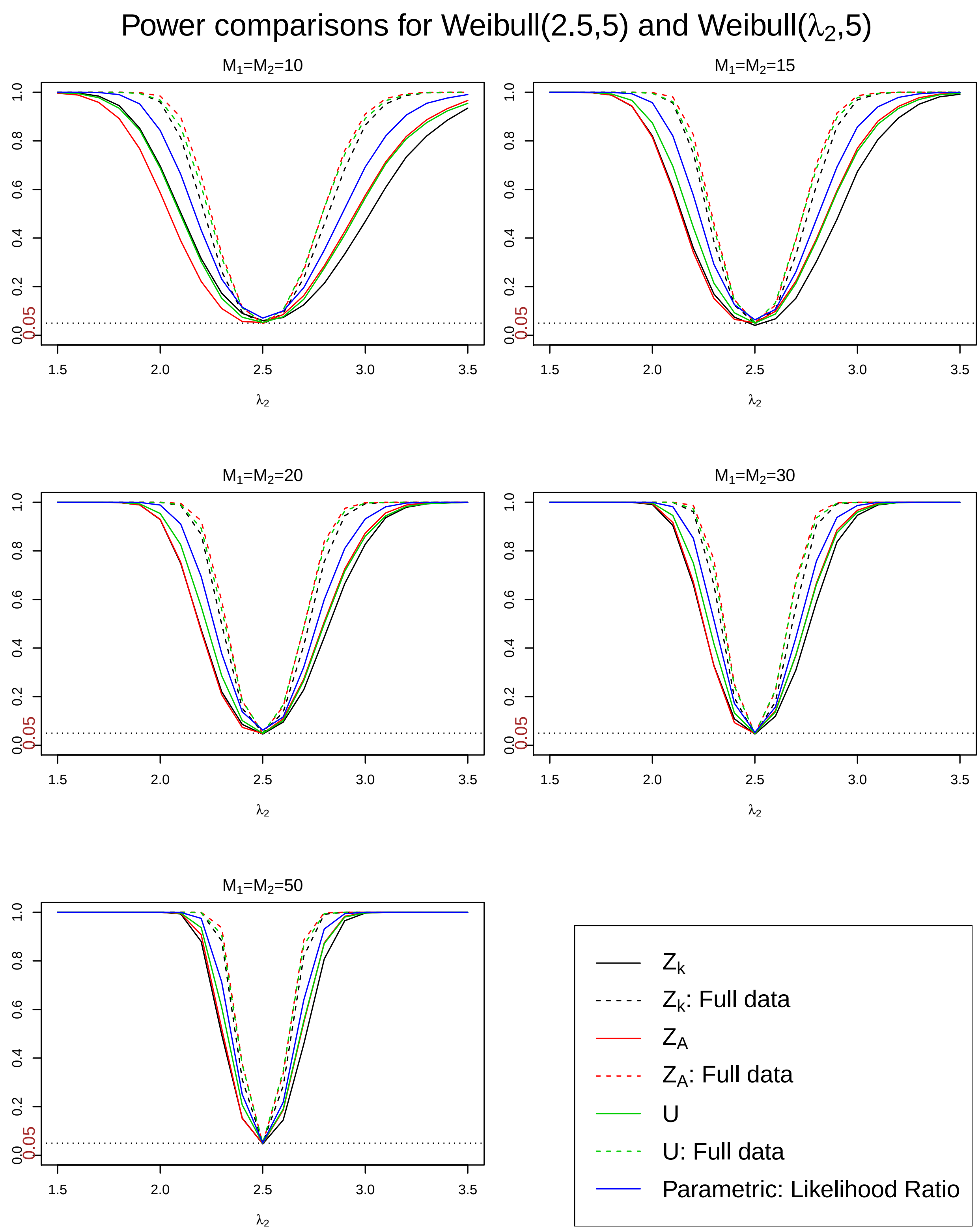

Figure 2.7: Simulated power curves of the parametric and nonparametric test for $\lambda_1 = 2.5$ and $\lambda_2 = 1.5\ (0.1)\ 3.5$, $\gamma_1 = \gamma_2 = 5$ with $M_1 = M_2$ for Weibull components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D4**] with system structures [**S2**]).

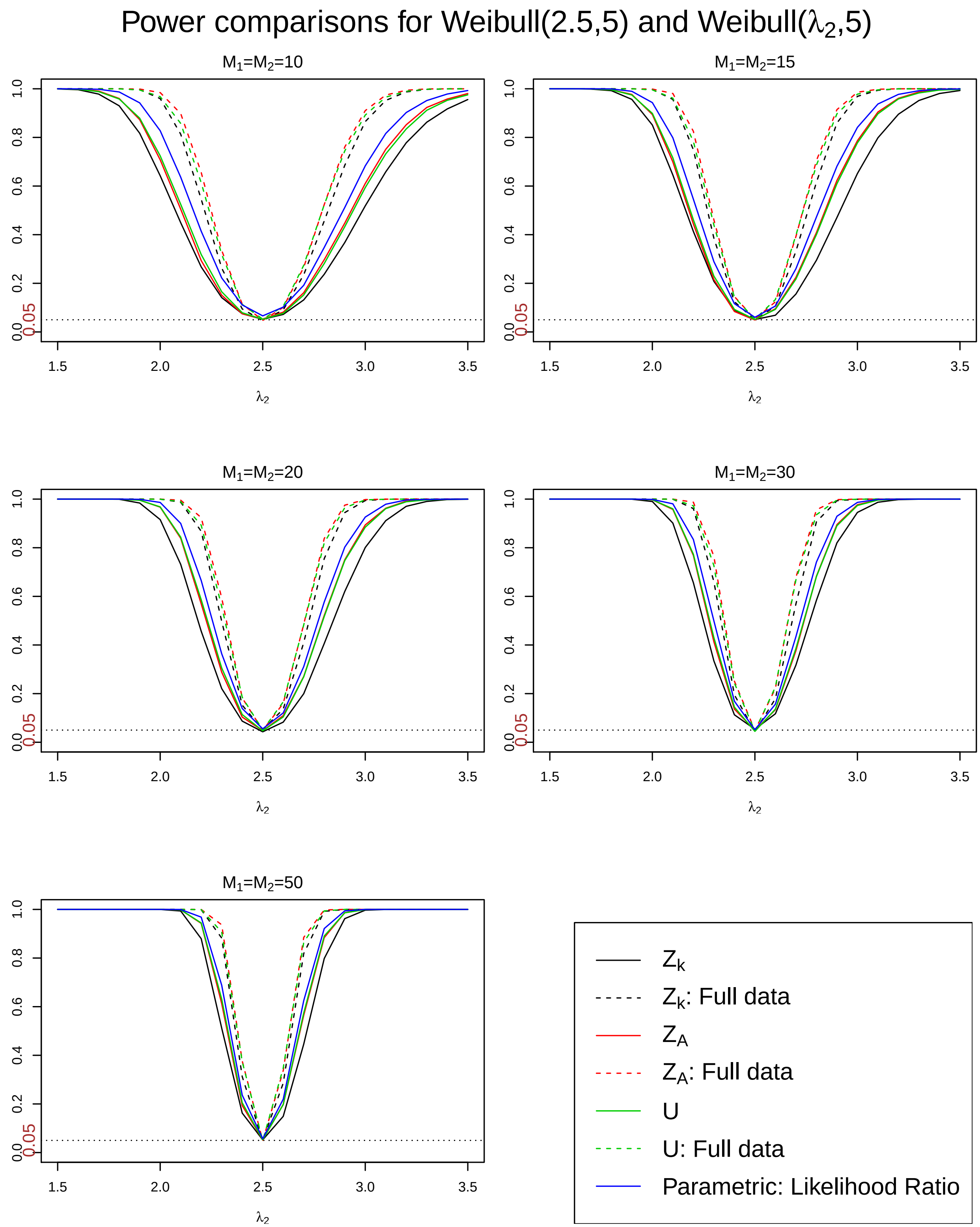

Figure 2.8: Simulated power curves of the parametric and nonparametric test for $\lambda_1 = 2.5$ and $\lambda_2 = 1.5\ (0.1)\ 3.5$, $\gamma_1 = \gamma_2 = 5$ with $M_1 = M_2$ for Weibull components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D4**] with system structures [**S3**]).

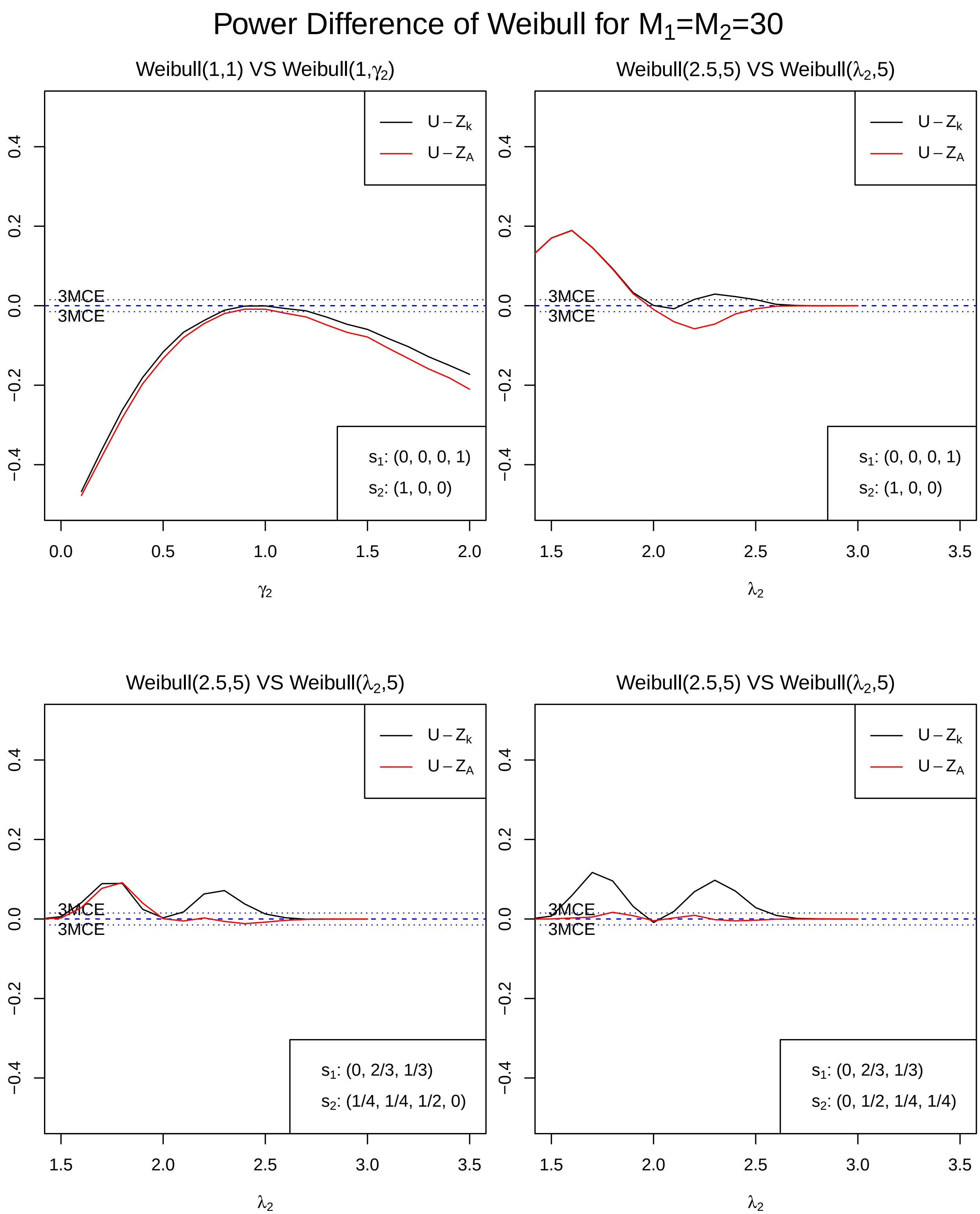

Figure 2.9: Differences between the simulated power values of the test procedures based on U statistic and Z_K, and the test procedures based on U statistic and Z_A for Weibull distribution with $M_1 = M_2 = 30$.

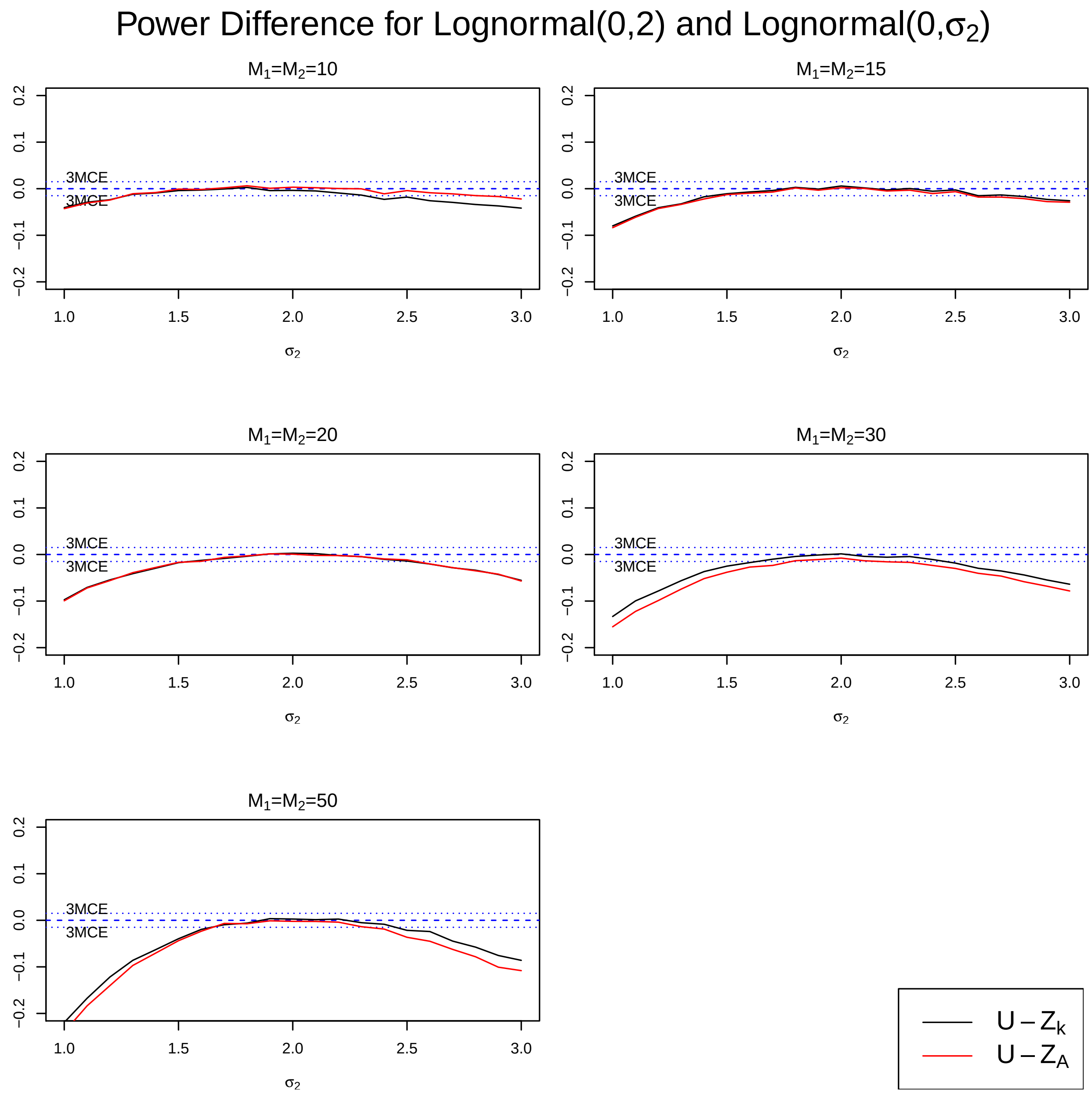

Figure 2.10: Differences between the simulated power values of the test procedures based on U statistic and Z_K, and the test procedures based on U statistic and Z_A for $\mu_1 = \mu_2 = 0$, $\sigma_1 = 2$ and $\sigma_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for lognormal distributed components from systems $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$.

2.7. Concluding Remarks

In this chapter, we studied the problem of testing the homogeneity of component lifetime distributions based on system-level data, which can be applied to many practical situations in life testing procedures involving systems with known structures. We showed that those existing parametric test procedures might suffer from the inflation of Type-I error rates when the underlying probability distributions of the component lifetimes are misspecified. To address this issue, we focus on developing nonparametric statistical test procedures for the homogeneity of component lifetime distributions based on system-level data. We proposed two empirical likelihood ratio tests based on the empirical likelihood ratio and the nonparametric estimation of component lifetime distributions. We provided the computational algorithms for obtaining the null distributions of the test statistics using the Monte Carlo method. Our simulation results show that the proposed nonparametric procedures provide comparative power values with those existing tests. These proposed test procedures have advantages in power performance when the two systems are very different.

CHAPTER 3

Empirical Likelihood Ratio Tests for Homogeneity of Component Lifetime Distributions Based on Type-II Censored System Lifetime Data

3.1. Introduction

In this chapter, we focus on Type-II censoring, in which the life-testing experiment will be terminated as soon as a prefixed number of failures are observed. In this situation, the number of observed failures is prefixed while the duration of the experiment is a random variable. For right-censored survival data, the log-rank test (Mantel, 1966; Peto and Peto, 1972) is one of the most commonly used test procedures for testing the equality of two lifetime distributions. The log-rank test has been proven to be asymptotically most efficient under the proportional hazards alternatives (Aalen, 1978; Fleming et al., 1987). However, the log-rank test cannot be applied to test the equality of two component lifetime distributions based on system lifetime data. Although nonparametric tests for the homogeneity of component lifetime distributions based on complete system lifetime data have been discussed, test procedures for the same purpose based on Type-II censored data, which is a more practical situation, have not been developed. Therefore, we develop several nonparametric procedures for testing the homogeneity of component lifetimes based on Type-II censored system lifetime data with known system structures.

In this chapter, we develop several nonparametric procedures for testing the homogeneity of component lifetimes based on Type-II censored system lifetime data with known system structures. The rest of the chapter is organized as follows. In Section 3.2, we extend the nonparametric test procedures based on complete system lifetime data to Type-II censored system lifetime data by considering the permutations of the censored system lifetimes. Due to the numerous number of permutations that can be involved in the computation when the number of censored systems is large, in order to reduce the computation burden, we propose the test statistics under a restrictive assumption on the equality of the censored system lifetimes. In Section 3.3, the Monte Carlo method is used to approximate the null distributions of the proposed test statistics. The simulated critical values and the randomization procedure to control the type-I error rate at the desired level are also presented in Section 3.3. In Section 3.4, a practical example is presented to illustrate the proposed nonparametric test procedures. Then, in Section 3.5, a Monte Carlo simulation study is used to evaluate and compare the performance of proposed testing procedures under different scenarios. Discussions of the simulation results are also provided in Section 3.5. Finally, in Section 3.6, concluding remarks are provided.

3.2. Extensions to Type-II Censored System Lifetime Data

In this section, we extend upon the nonparametric test procedures discussed in Chapter 2 for testing the hypotheses in Eq. (1.8) based on Type-II censored system lifetime data. The life-testing experiment is performed on M_i experimental systems for System i and is terminated as soon as the r_i-th failure is observed, where r_i is a predetermined number such that $r_i \leq M_i$. Any remaining systems beyond the r_i-th failure are considered right-censored. We denote the ordered observed system lifetimes as $\boldsymbol{T}_i^O = (T_{i,(1)} < T_{i,(2)} < \ldots < T_{i,(r_i)})$, the unobserved ordered Type-II censored system lifetimes as $\boldsymbol{T}_i^C = (T_{i,(r_i+1)} < \ldots < T_{i,(M_i)})$, and the censoring proportion for System i as $\rho_i = r_i/M_i$, $i = 1, 2$.

To conduct the Mann-Whitney U test in Section 2.2.2 and the empirical likelihood ratio tests in Section 2.3, it is essential that the ordering of all the system lifetimes from System

1 and System 2 are known. Due to the nonparametric nature of the test procedures considered in Chapter 2.3, the actual values of the failure times are not required but the ranks of the failure times. For Type-II censored data, the complete ranking information of the censored data is missing. To handle this situation, one possible approach is to consider all possible ranks of the censored data by arranging the unobserved censored samples $\boldsymbol{T}_1^C$ and $\boldsymbol{T}_2^C$. For each possible arrangement of the unobserved censored samples $\boldsymbol{T}_1^C$ and $\boldsymbol{T}_2^C$, we can determine the rankings of the system lifetime data for System 1 and System 2, and the corresponding test statistics can be calculated. Then, test procedures based on going through all the possible permutations of the unobserved censored samples $\boldsymbol{T}_1^C$ and $\boldsymbol{T}_2^C$ can be developed. The following proposition gives the formula for calculating the number of unique permutations for different M_1, M_2, r_1, r_2 and observed number of failure occurs in between the observed system failures for System 1 ($T_{1,(r_1)}$) and for System 2 ($T_{2,(r_2)}$).

Proposition 3.2.1. Suppose $\boldsymbol{T}_i = (T_{1,(1)}, \cdots, T_{i,(r_i)})$ denotes the observed Type-II censored system lifetimes for System i and the remaining $(M_i - r_i)$ failures are censored $(i = 1, 2)$, and let L be the number of observed system failures that occur between the last observed failure for System 1 $(T_{1,(r_1)})$ and the last observed failure for System 2 $(T_{2,(r_2)})$, the number of all permutations, denoted N_c, is given by

$$N_c = \begin{cases} \sum\limits_{k=0}^{M_1-r_1} \binom{L+k}{k}\binom{M_1-r_1+M_2-r_2-k}{M_2-r_2}, & T_{1,(r_1)} < T_{2,(r_2)}, \\ \sum\limits_{k=0}^{M_2-r_2} \binom{L+k}{k}\binom{M_1-r_1+M_2-r_2-k}{M_1-r_1}, & T_{1,(r_1)} > T_{2,(r_2)}, \\ \binom{M_1-r_1+M_2-r_2}{M_2-r_2}, & T_{1,(r_1)} = T_{2,(r_2)}. \end{cases} \tag{3.1}$$

For the special case $M_1 = M_2 = M$ and $r_1 = r_2 = r$, then Eq. (3.1) simplifies to:

$$N_c = \begin{cases} \sum_{k=0}^{M-r} \binom{L+k}{k}\binom{2M-2r-k}{M-r}, & T_{1,(r_1)} \neq T_{2,(r_2)}, \\ \binom{M_1-r_1+M_2-r_2}{M_2-r_2}, & T_{1,(r_1)} = T_{2,(r_2)}. \end{cases} \tag{3.2}$$

Proof. For the case when $T_{1,(r_1)} < T_{2,(r_2)}$, the number of permutations for the $(M_1 - r_1)$ unobserved censored failures from System 1, and the $(M_2 - r_2)$ censored failures from System 2, is given by

$$\binom{M_1 - r_1 + M_2 - r_2}{M_2 - r_2}.$$

For the case when $T_{1,(r_1)} < T_{2,(r_2)}$, this implies that the L observed failures occur in between $T_{1,(r_1)}$ and $T_{2,(r_2)}$ are from System 2, i.e., $T_{1,(r_1)} < T_{2,(r_2-L)} < T_{2,(r_2-L+1)} < \ldots < T_{2,(r_2-1)} < T_{2,(r_2)}$.

To enumerate the number of permutations of the unobserved censored failures, among the $M_1 - r_1$ unobserved failures from System 1, we consider k of them occur before $T_{2,(r_2)}$ and the remaining $M_1 - r_1 - k$ of them occur after $T_{2,(r_2)}$, $k = 0, 1, \ldots, (M_1 - r_1)$. Since the $M_2 - r_2$ unobserved censored failures from System 2 must be greater than $T_{2,(r_2)}$, there are $(M_2 - r_2) + (M_1 - r_1 - k)$ unobserved failures after $T_{2,(r_2)}$ and the number of permutations for these $(M_2 - r_2) + (M_1 - r_1 - k)$ unobserved failures is

$$\binom{M_1 - r_1 + M_2 - r_2 - k}{M_2 - r_2}. \tag{3.3}$$

The number of permutations for allocating k failures before $T_{2,(r_2)}$ can be calculated by treating the problem as placing k balls into $L+1$ boxes.

$$(T_{1,(r_1)}, T_{2,(r_2-L)}], (T_{2,(r_2-L)}, T_{2,(r_2-L+1)}], \ldots, (T_{2,(r_2-1)} < T_{2,(r_2)}].$$

Hence, the number of permutations for those k failures allocate before $T_{2,(r_2)}$ is given by

$$\binom{k+(L+1)-1}{(L+1)-1} = \binom{L+k}{L}. \tag{3.4}$$

From Eqs. (3.3) and (3.4), the number of permutations for the unobserved censored failures when $T_{1,(r_1)} < T_{2,(r_2)}$ is

$$\sum_{k=0}^{M_1-r_1} \binom{L+k}{L} \binom{M_1-r_1+M_2-r_2-k}{M_2-r_2}.$$

Following the same argument, the number of permutations for the unobserved censored failures when $T_{1,(r_1)} > T_{2,(r_2)}$ can be obtained as

$$\sum_{k=0}^{M_2-r_2} \binom{L+k}{L} \binom{M_1-r_1+M_2-r_2-k}{M_1-r_1}.$$

$\square$

When the stopping times of the two samples are different, which is a more practical situation, the situation becomes more complex and the number of permutations can increase significantly. Hence, going through all the possible permutations can be computationally expensive.

For the purpose of illustrating the idea of going through all the permutations of the censored samples, we consider a simulated data set for System 1 with system signature $\boldsymbol{s}_1 = (0, 2/3, 1/3)$ and System 2 with system signature $\boldsymbol{s}_2 = (1/4, 1/4, 1/2, 0)$, with sample sizes $M_1 = M_2 = 10$ and the number of observed failures $r_1 = r_2 = 8$. In other words, the Type-II censored lifetimes $T_{i,(9)}$ and $T_{i,(10)}$, $i = 1, 2$, are unobserved. The simulated data set is presented in Table 3.1. The stopping times for System 1 and 2 are $T_1 = 0.88$ and $T_2 = 1.08$, respectively. Based on the data set in Table 3.1, from Proposition 3.1, since $T_{1,(8)} < T_{2,(8)}$ and $L = 0$, the number of permutations is

$$\sum_{k=0}^{2} \binom{k}{0} \binom{2+2-k}{2} = \sum_{k=0}^{2} \binom{4-k}{2} = \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = 6 + 3 + 1 = 10.$$

The 10 distinct scenarios are illustrated in Figure 3.1. For each arrangement, we have the information equivalent to the complete system lifetime data and hence, the test statistics presented in Section 2.3 can be computed.

Table 3.1: Simulated lifetime data for systems with signatures $\mathrm{s}_1 = (0, 2/3, 1/3)$ and $\mathrm{s}_2 = (1/4, 1/4, 1/2, 0)$ with sample sizes $M_1 = M_2 = 10$ and number of observed failures $r_1 = r_2 = 8$ (+ indicates right-censoring).

j	1	2	3	4	5	6	7	8	9	10
$T_{1(j)}$	0.25	0.33	0.35	0.38	0.43	0.81	0.82	0.88	0.88+	0.88+
$T_{2(j)}$	0.03	0.06	0.08	0.14	0.18	0.44	0.48	1.08	1.08+	1.08+

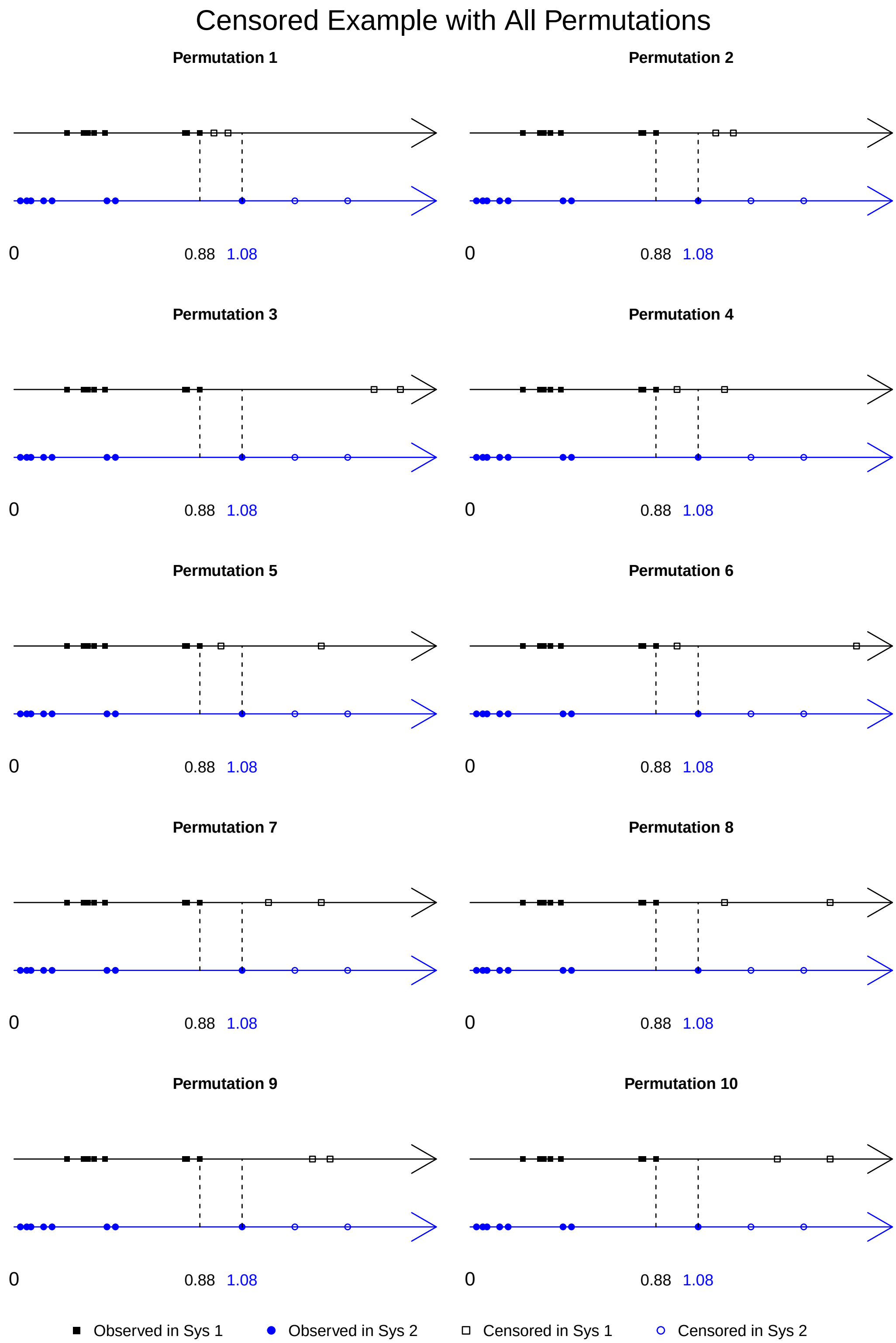

Figure 3.1: The 10 distinct possible permutations of the censored samples based on the data presented in Table 3.1

It is worth mentioning that if the last observed failures of the two samples are not following each other, the number of possible permutations can increase significantly. In this example, we observe that even for a simple case with only two censored system lifetimes in each sample and the last observed failures of the two samples following each other, going through all the possible permutations can be complex. To reduce the computational burden, we propose to simplify the procedure by grouping all the unobserved censored lifetimes in each system together, i.e., making an assumption that all the unobserved failure times from System i are close to each other such that none of the unobserved censored lifetimes from the other sample can occur in between them. As an example, for the data set in Table 3.1, there are three possible permutations under this assumption (see, Figure 3.2). This assumption reduces the number of possible permutations one needs to go through, and the number of possible permutations does not depend on the numbers of censored systems $(M_1 - r_1)$ and $(M_2 - r_2)$. Therefore, this assumption simplifies the calculation of corresponding test statistics for testing the hypotheses in Eq. (2.1).

The following proposition provides the number of unique possible permutations based on the assumption that all the unobserved failure times from System i are the same, for $i = 1, 2$.

Proposition 3.2.2. Suppose L is the number of observed system failures (from System 1 and System 2) between the last observed failure for System 1 ($T_{1,(r_1)}$) and the last observed failure for System 2 ($T_{2,(r_2)}$), i.e., $T_{2,(r_2-\ell)} \in (T_{1,(r_1)}, T_{2,(r_2)})$, $\ell = 1, 2, \ldots, L$, and $T_{2,(r_2-L-1)} < T_{1,(r_1)}$. Assume that all the unobserved failure times from System i are close to each other and none of the censored lifetimes from another system occurs between them. Under this assumption, for $i = 1, 2$, the number of possible unique permutations, denoted as N_r, is given by

$$N_r = \begin{cases} L+3, & T_{1,(r_1)} \neq T_{2,(r_2)}, \\ 2, & T_{1,(r_1)} = T_{2,(r_2)}. \end{cases} \tag{3.5}$$

Proof. For $T_{1,(r_1)} = T_{2,(r_2)}$, given the assumption stated in the condition, there are two permutations $C_1 > C_2$ and $C_1 < C_2$.

Without loss of generality, suppose $T_{1,(r_1)} < T_{2,(r_2)}$, it follows that the L failures that occur in the interval $(T_{1,(r_1)}, T_{2,(r_2)})$ must be from System 2. Under the assumption that all the unobserved failure times from System i are close each other that none of the failures from the other system can occur between them, we have $C_1 > T_{1,(r_1)}$ and $C_2 > T_{2,(r_2)} > T_{1,(r_1)}$. Then, we consider the following two cases:

Case 1: $C_1 > T_{2,(r_2)}$: There are two possible permutations in this case:

(1) $C_1 > C_2$;

(2) $C_2 > C_1$.

Case 2: $C_1 < T_{2,(r_2)}$: There are $L + 1$ possible permutations in this case:

(1) $C_1 \in (T_{1,(r_1)}, T_{2,(r_2-L)})$ and $C_2 > T_{2,(r_2)}$;

(2) $C_1 \in (T_{2,(r_2-L)}, T_{2,(r_2-L+1)})$ and $C_2 > T_{2,(r_2)})$;

$\vdots$

$(L+1)$ $C_1 \in (T_{2,(r_2-1)}, T_{2,(r_2)})$ and $C_2 > T_{2,(r_2)}$.

Therefore, the number of possible unique permutations $N_r = L+3$. The result for $T_{1,(r_1)} > T_{2,(r_2)}$ can be obtained in a similar manner and the proof follows. $\square$

From Proposition 3.2, since the maximum value of L is $\max\{r_1-1, r_2-1\}$, the maximum number of permutations under the assumption that all the unobserved failure times from System i ($i = 1, 2$) are the same is $\max\{r_1 - 1, r_2 - 1\} + 3$.

To develop test procedures based on the statistics presented in Section 2.3, for the h-th permutation ($h \in 1, \ldots, N_r$), after we set the values of C_1 and C_2 based on the permutation, we can treat the data as complete system lifetime data and compute the test statistics U in Eq. (2.5), Z_K in Eq. (2.12), and Z_A in Eq. (2.14), denote as $U^{(h)}$, $Z_K^{(h)}$, and $Z_A^{(h)}$, respectively. This process generates N_r test statistics based on the Type-II censored system lifetime data, namely $U^{(h)}$, $Z_K^{(h)}$, and $Z_A^{(h)}$ for $h \in \{1, \ldots, N_r\}$. Then, we propose to consider the following functions of the statistics as test statistics for testing the hypotheses in Eq. (2.1):

$$U_{\min} = \min\{U^{(1)}, \ldots, U^{(N_r)}\}, U_{\max} = \max\{U^{(1)}, \ldots, U^{(N_r)}\}, U_{\text{mean}} = \sum_{h=1}^{N_r} U^{(h)}/N_r, \tag{3.6}$$

$$Z_{K,\min} = \min\{Z_k^{(1)}, \ldots, Z_k^{(N_r)}\}, Z_{K,\max} = \max\{Z_k^{(1)}, \ldots, Z_k^{(N_r)}\}, Z_{K,\text{mean}} = \sum_{h=1}^{N_r} Z_k^{(h)}/N_r, \tag{3.7}$$

$$Z_{A,\min} = \min\{Z_A^{(1)}, \ldots, Z_A^{(N_r)}\}, Z_{K,\max} = \max\{Z_A^{(1)}, \ldots, Z_A^{(N_r)}\}, Z_{A,\text{mean}} = \sum_{h=1}^{N_r} Z_A^{(h)}/N_r. \tag{3.8}$$

Based on the property of U, Z_K, and Z_A, larger values of $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$, and $Z_{A,\text{mean}}$ lead to the rejection of the null hypothesis in Eq. (2.1), while large or small values of $U_{\min}$, $U_{\max}$ and U_{mean} lead to the rejection of the null hypothesis in Eq. (2.1).

For the data set presented in Table 3.1, $L = 0$ and there are $(L + 3) = 3$ possible permutations of the unobserved censored system lifetimes under the assumption that that all the unobserved failure times from System i ($i = 1, 2$) are the same. The three possible permutations under the restriction are presented in Figure 3.2.

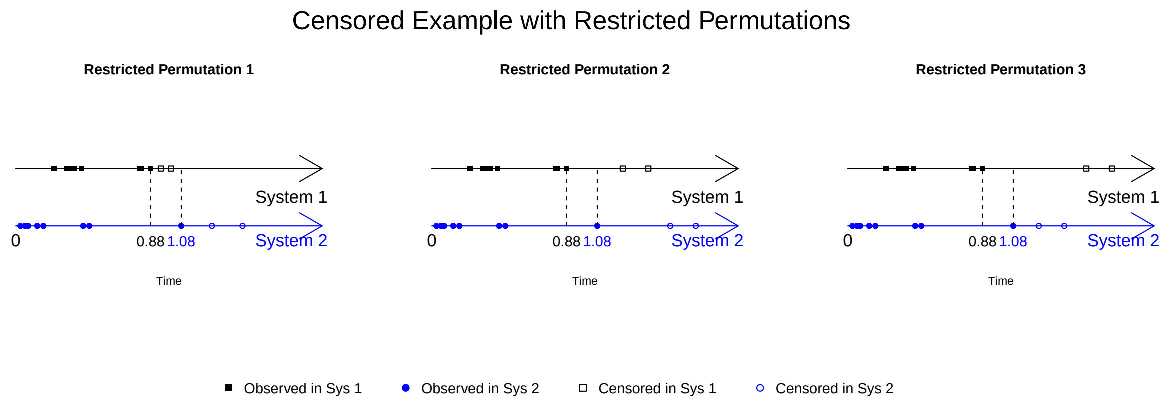

Figure 3.2: The Type-II censored system lifetime data presented in Table 3.1 and the three distinct possible permutations of the censored samples under the assumption that all the unobserved failure times from System i ($i = 1, 2$) are close to each other.

To justify the propriety to consider the restricted permutations, we conducted a simulation study comparing the maximum and minimum values of the test statistics U, Z_A, and Z_K among all the possible permutations and all the restricted permutations. We consider different systems, and the underlying component lifetime distributions are exponential distributions with sample sizes $M_1 = M_2 = 10$ and $r_1 = r_2 = 8$. In Table 3.2, we report the percentages of cases where the minimum and maximum test statistics were the same for all possible permutations and restricted permutations in $10,000$ simulated data sets.

From Table 2, except for the minimum of test statistic Z_A, the majority of test statistics obtained based on the restricted permutations are equivalent to the test statistics obtained based on going through all the possible permutations.

Table 3.2: Simulated percentage of equality of the maximum and minimum of the test statistics based on two permutation methods for different systems with sample sizes $M_1 = M_2 = 10$ and $r_1 = r_2 = 8$ based on $10,000$ simulations.

	System		Z_k		Z_A		U	
System (i)	Signatures	F_{X_i}	Min.	Max.	Min.	Max.	Min.	Max.
1 2	$\mathbf{s}_1 = (0, 2/3, 1/3)$ $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$	$Exp(1)$ $Exp(1)$	88.15%	100%	0%	100%	100%	100%
1 2	$\mathbf{s}_1 = (0, 2/3, 1/3)$ $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$	$Exp(1)$ $Exp(0.2)$	99.88%	100%	0%	100%	100%	100%
1 2	$\mathbf{s}_1 = (0, 2/3, 1/3)$ $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$	$Exp(1)$ $Exp(1)$	95.47%	100%	0%	100%	100%	100%
1 2	$\mathbf{s}_1 = (0, 2/3, 1/3)$ $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$	$Exp(1)$ $Exp(0.2)$	100%	100%	0%	100%	100%	100%
1 2	$\mathbf{s}_1 = (0, 0, 0, 1)$ $\mathbf{s}_2 = (1, 0, 0)$	$Exp(1)$ $Exp(1)$	22.90%	100%	5.43%	100%	100%	100%
1 2	$\mathbf{s}_1 = (0, 0, 0, 1)$ $\mathbf{s}_2 = (1, 0, 0)$	$Exp(1)$ $Exp(0.2)$	0.08%	100%	0%	100%	100%	100%

On the other hand, conducting all permutations is often unrealistic due to the large number of unique permutations required. To illustrate the amount of computational effort that can be saved by considering the restricted permutations, we present the number of permutations required for going through all permutations (N_c) and restricted permutations (N_r) with different censored sample sizes $M_i - r_i$ ($i = 1, 2$) and the number of observed system failures L between the last observed failures for two systems in Table 3.3. From Table 3.3, we observe that the number of unique permutations required for conducting all permutations (N_c) increases dramatically as the size of censored data increases. However, considering the restriction on the censoring observations requires a much smaller number of permutations (N_r), which reduces the computational effort for computing the test statistics in Eqs. (3.6), (3.7), and (3.8). Tables 3.2 and 3.3 show that considering the restricted permutations results in similar minimum and maximum of the test statistics while requiring significantly less computation effort. Therefore, we consider the restricted permutations in this chapter and investigate the performance of the proposed test procedures in the subsequent sections.

Table 3.3: Number of permutations required for conducting all permutations (N_c) and restricted permutations (N_r) with censored sample sizes c_i ($i = 1, 2$) and number of observed system failures L between the last observed failures for two systems.

Parameters		Number of Permutations	
L	$M_1 - r_1 = M_2 - r_2$	N_c	N_r
$L = 0$	2	10	3
	4	126	3
	6	1716	3
	8	24310	3
	10	352716	3
$L = 1$	2	15	4
	4	210	4
	6	3003	4
	8	43758	4
	10	646646	4
$L = 2$	2	21	5
	4	330	5
	6	5005	5
	8	75582	5
	10	1144066	5
$L = 3$	2	28	6
	4	495	6
	6	8008	6
	8	125970	6
	10	1961256	6
$L = 4$	2	36	7
	4	715	7
	6	12376	7
	8	203490	7
	10	3268760	7

3.3. Null Distributions of the Test Statistics

In Section 3.2, we introduced different test statistics based on the statistics U, Z_K, and Z_A with the permutations of the Type-II censored observations. Since the theoretical distributions of these statistics under the null hypothesis that $F_{X_1} = F_{X_2} = F_X$ cannot be determined in general, and the null distributions do not depend on the underlying distribution F_X, Monte Carlo simulation method can be used to approximate the null distributions. Specifically, for specified values of $M_1,\ M_2,\ r_1(=\ [M_1(1-\rho_1)]),\ r_2(=\ [M_2(1-\rho_2)]),\ n_1$ and n_2, and system signatures s_1 and s_2, where ρ_1 and ρ_2 are the censoring proportions for System 1 and System 2, respectively, and $[a]$ is the integer part of a, we simulate the ordered Type-II censored system lifetimes from System 1 and System 2 based on component lifetime distribution F_X.

To approximate the null distribution based on the Monte Carlo simulation method, we use the standard exponential distribution (denoted as $Exp(1)$) with CDF $F_X(x) = 1-\exp(-x),\ x>0$ as it is easy to simulate. For each simulated Type-II censored system lifetimes data for System 1 and System 2, we compute the test statistics in Eqs. (3.6), (3.7), and (3.8) and approximate the null distribution using $200,000$ simulations.

For a specific level of significance α, based on the simulated null distributions, the critical values and the related probabilities for performing the test procedures based on test statistics $U_{\min}$, $U_{\max}$, U_{mean}, $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{K,\max}$, and $Z_{A,\text{mean}}$. Since the test statistics considered here depend on ordering the failures from System 1 and System 2 and the number of possible orderings is finite, the test statistics can take on a limited number of values. Therefore, to control the significance level of α, we consider using a randomization procedure. In Table 3.4, we present the critical values and the related randomization probabilities for the test procedures based on test statistics $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$, and $Z_{K,\text{mean}}$ with $\alpha = 0.05$, system signatures $\mathrm{s}_1 = (0, 2/3, 1/3)$ (i.e., $n_1 = 3$) and $\mathrm{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., $n_2 = 4$), sample sizes $M_1 = M_2 = 10$, and censoring proportions $\rho_1 = \rho_2 = 0.1,\ 0.2,\ 0.3$, and 0.4.

To illustrate the proposed test procedure with randomization, we use the test statistic $Z_{K,\min}$ with censoring proportions $\rho_1 = \rho_2 = 0.1$ as an example. For $\alpha = 0.05$, the critical value, denoted as $C_{Z_K}^{\min}$, is 8.0230 from Table 3.4, which means that we reject the null hypothesis in Eq. (2.1) when the test statistic $Z_{K,\min}$ is greater than or equal to 8.0230. However, the type-I error rate (significance level) will be

$$\begin{aligned}\Pr(Z_{K,\min} \geq 8.0230|H_0) &= \Pr(Z_{K,\min} > 8.0230|H_0) + \Pr(Z_{K,\min} = 8.0230|H_0) \\ &= 0.047420 + 0.005650 = 0.05307,\end{aligned}$$

which is greater than the nominal level $\alpha = 0.05$. To control the type-I error rate at $\alpha = 0.05$, we use a randomization procedure in which we generate a Bernoulli random variate with the probability of success equal to q to determine if the null hypothesis is rejected when $Z_{K,\min} = C_{Z_K}^{\min}$, where q can be obtained as

$$\begin{aligned}q &= \frac{0.05 - \Pr(Z_{K,\min} > C_{Z_K}^{\min}|H_0)}{\Pr(Z_{K,\min} = C_{Z_K}^{\min}|H_0)} = \frac{0.05 - \Pr(Z_{K,\min} > 8.0230)}{\Pr(Z_{K,\min} = 8.0230|H_0)} \\ &= \frac{0.05 - 0.047420}{0.005650} = 0.4566372.\end{aligned}$$

Since the null hypothesis in Eq. (2.1) is rejected if the Man-Whitney U test is too large or too small, we have two critical values for a specific significance level. Similar to the scenario in Z_K and Z_A, the null distributions of the minimum, maximum, and mean of U take a limited number of possible values, especially when the sample sizes M_1 and M_2

Table 3.4: Simulated critical values and the related randomization probabilities for the test procedures based on test statistics $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$, and $Z_{A,\text{mean}}$ with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ (i.e., $n_1 = 3$) and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., $n_2 = 4$), sample sizes $M_1 = M_2 = 10$, and censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4.

Proportions \ Percentile	$Z_{K,\min}$			$Z_{K,\max}$			$Z_{K,\text{mean}}$		
	$C_{Z_K}^{\min}$	$\Pr(Z_{K,\min} = C_{Z_K}^{\min})$	$\Pr(Z_{K,\min} > C_{Z_K}^{\min})$	$C_{Z_K}^{\max}$	$\Pr(Z_{K,\max} = C_{Z_K}^{\max})$	$\Pr(Z_{K,\max} > C_{Z_K}^{\max})$	$C_{Z_K}^{\text{mean}}$	$\Pr(Z_{K,\text{mean}} = C_{Z_K}^{\text{mean}})$	$\Pr(Z_{K,\text{mean}} > C_{Z_K}^{\text{mean}})$
$\rho_1 = \rho_2 = 0.1$	8.0230	0.005650	0.047420	10.9108	0.019190	0.042410	8.2597	0.000495	0.049970
$\rho_1 = \rho_2 = 0.2$	7.2234	0.006595	0.046325	13.5924	0.023015	0.039850	8.7972	0.016575	0.038005
$\rho_1 = \rho_2 = 0.3$	7.0348	0.006415	0.044780	16.5391	0.024670	0.030685	10.8558	0.013370	0.045180
$\rho_1 = \rho_2 = 0.4$	6.5068	0.046660	0.027495	20.8607	0.046660	0.012035	11.2753	0.003505	0.049380
Proportions \ Percentile	$Z_{A,\min}$			$Z_{A,\max}$			$Z_{A,\text{mean}}$		
	$C_{Z_A}^{\min}$	$\Pr(Z_{A,\min} = C_{Z_A}^{\min})$	$\Pr(Z_{A,\min} > C_{Z_A}^{\min})$	$C_{Z_A}^{\max}$	$\Pr(Z_{A,\max} = C_{Z_A}^{\max})$	$\Pr(Z_{A,\max} > C_{Z_A}^{\max})$	$C_{Z_A}^{\text{mean}}$	$\Pr(Z_{A,\text{mean}} = C_{Z_A}^{\text{mean}})$	$\Pr(Z_{A,\text{mean}} > C_{Z_A}^{\text{mean}})$
$\rho_1 = \rho_2 = 0.1$	11.7583	0.000010	0.049995	16.1745	0.000020	0.049985	13.5612	0.000010	0.049995
$\rho_1 = \rho_2 = 0.2$	11.0522	0.000270	0.049950	19.2596	0.000010	0.049995	14.7029	0.000100	0.049970
$\rho_1 = \rho_2 = 0.3$	10.9364	0.000225	0.049815	22.7476	0.000110	0.049955	16.0579	0.017285	0.047135
$\rho_1 = \rho_2 = 0.4$	11.5465	0.000180	0.049865	26.9272	0.046660	0.029530	17.7945	0.000235	0.049820

are small. As a result, the lower and upper critical values may not accurately provide a test procedure with significance level α. Therefore, a randomization procedure can be used to adjust the actual significance level. In Table 3.5, we present the upper and lower critical values and the randomization probabilities for the test procedures based on test statistics $U_{\min}$, $U_{\max}$, and U_{mean}, with $\alpha = 0.05$, system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ (i.e., $n_1 = 3$) and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., $n_2 = 4$), sample sizes $M_1 = M_2 = 10$, and censoring proportions $\rho_1 = \rho_2 = 0.1,\ 0.2,\ 0.3$, and 0.4.

From Table 3.5, for $\alpha = 0.05$ and $\rho_1 = \rho_2 = 0.1$, the upper and lower critical values of $U_{\min}$ are $C_{U_1}^{\min} = 10$ and $C_{U_2}^{\min} = 60$, respectively. Since the null hypothesis should be rejected when $U_{\min}$ is too large or too small, we consider the probabilities

$$\begin{aligned}
\Pr(U_{\min} \leq 10|H_0) &= \Pr(U_{\min} = 10|H_0) + \Pr(U_{\min} < 10|H_0) \\
&= 0.006055 + 0.020545 = 0.0266, \\
\Pr(U_{\min} \geq 60|H_0) &= \Pr(U_{\min} = 60|H_0) + \Pr(U_{\min} > 60|H_0) \\
&= 0.00440 + 0.023935 = 0.028335.
\end{aligned}$$

Table 3.5: Simulated upper and lower critical values and the related randomization probabilities for the test procedures based on test statistics $U_{\max}$, $U_{\min}$, and U_{mean} with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ (i.e., $n_1 = 3$) and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., $n_2 = 4$), sample sizes $M_1 = M_2 = 10$ and censoring proportions $\rho_1 = \rho_2 = 0.1,\ 0.2,\ 0.3$, and 0.4.

2.5% Percentile / Proportions	$U_{\min}$			$U_{\max}$			U_{mean}		
	$C_{U_1}^{\min}$	$\Pr(U_{\min} = C_{U_1}^{\min})$	$\Pr(U_{\min} < C_{U_1}^{\min})$	$C_{U_1}^{\max}$	$\Pr(U_{\max} = C_{U_1}^{\max})$	$\Pr(U_{\max} < C_{U_1}^{\max})$	$C_{U_1}^{\text{mean}}$	$\Pr(U_{\text{mean}} = C_{U_1}^{\text{mean}})$	$\Pr(U_{\text{mean}} < C_{U_1}^{\text{mean}})$
$\rho_1 = \rho_2 = 0.1$	10	0.006055	0.020545	17	0.007130	0.019765	14	0.003800	0.024535
$\rho_1 = \rho_2 = 0.2$	7	0.005710	0.021660	24	0.011265	0.023715	15.2	0.002915	0.024455
$\rho_1 = \rho_2 = 0.3$	4	0.011615	0.017285	31	0.011615	0.017285	15.3	0.011615	0.017285
$\rho_1 = \rho_2 = 0.4$	0	0.046660	0	40	0.046660	0	15.5	0.046660	0

97.5% Percentile / Proportions	$U_{\min}$			$U_{\max}$			U_{mean}		
	$C_{U_2}^{\min}$	$\Pr(U_{\min} = C_{U_2}^{\min})$	$\Pr(U_{\min} > C_{U_2}^{\min})$	$C_{U_2}^{\max}$	$\Pr(U_{\max} = C_{U_2}^{\max})$	$\Pr(U_{\max} > C_{U_2}^{\max})$	$C_{U_2}^{\text{mean}}$	$\Pr(U_{\text{mean}} = C_{U_2}^{\text{mean}})$	$\Pr(U_{\text{mean}} > C_{U_2}^{\text{mean}})$
$\rho_1 = \rho_2 = 0.1$	60	0.004440	0.023935	63	0.004125	0.023965	62	0.002615	0.022670
$\rho_1 = \rho_2 = 0.2$	58	0.004495	0.022070	66	0.004020	0.024755	62.6	0.001180	0.023865
$\rho_1 = \rho_2 = 0.3$	54	0.006055	0.023615	71	0.003150	0.023325	64	0.001675	0.024225
$\rho_1 = \rho_2 = 0.4$	49	0.007350	0.024395	76	0.003705	0.023110	66	0.001315	0.023995

These probabilities show that rejecting the null hypothesis when $U_{\min} \leq C_{U_1}^{\min} = 10$ and $U_{\min} \geq C_{U_2}^{\min} = 60$ will result in a significance level larger than the 5% level. To maintain a Type-I error rate of $\alpha = 0.025$ for each side of the test, a randomization procedure is employed. This involves generating a Bernoulli random variate with probability of success equal to q_1 and q_2 to determine whether the null hypothesis is rejected when the observed test statistics $U_{\min} = C_{U_1}^{\min}$ and $U_{\min} = C_{U_2}^{\min}$, respectively. Based on the probabilities provided in Table 3.5, the values of q_1 and q_2 for this example can be obtained as

$$
\begin{aligned}
q_1 &= \frac{0.025 - \Pr(U_{\min} < C_{U_1}^{\min} | H_0)}{\Pr(U_{\min} = C_{U_1}^{\min} | H_0)} = \frac{0.025 - \Pr(U_{\min} < 10 | H_0)}{\Pr(U_{\min} = 10 | H_0)} \\
&= \frac{0.025 - 0.020545}{0.006055} = 0.7357556. \\
q_2 &= \frac{0.025 - \Pr(U_{\min} > C_{U_2}^{\min} | H_0)}{\Pr(U_{\min} = C_{U_2}^{\min} | H_0)} = \frac{0.025 - \Pr(U_{\min} > 60 | H_0)}{\Pr(U_{\min} = 60 | H_0)} \\
&= \frac{0.025 - 0.023935}{0.004440} = 0.2398649.
\end{aligned}
$$

Since the simulated critical values and the related probabilities for the randomization procedure depends on the values of n_1, n_2, $\mathbf{s}_1$, $\mathbf{s}_2$, M_1, M_2, r_1, r_2, and α, instead of providing the extensive tables for some particular settings, we provide the computer program written in R (R Core Team, 2022) to simulate the critical values and the related probabilities under the null hypothesis for the nine test statistics.

3.4. Practical Example

To illustrate the testing procedures proposed in Section 3.2, we apply these procedures to test the homogeneity of component lifetime distributions based on a system lifetime data set from Yang et al. (2016) and Frenkel and Khvatskin (2006). In this data set, the phosphor acid filter system is a consecutive 2-out-of-n system that fails when two adjacent components fail. We consider that System 1 is a consecutive 2-out-of-8 ($n_1 = 8$) system with signature $\mathbf{s}_1 = (0, 1/4, 11/28, 2/7, 1/14, 0, 0, 0)$ and System 2 is a 4-component system (i.e., $n_2 = 4$) with signature $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, and the component lifetimes for System 1 follow a Birnbaum-Saunders distribution (Birnbaum and Saunders, 1969) with CDF

$$F_X(t; a, b) = \Phi \left\{ \frac{1}{a} \left[\left(\frac{t}{b} \right)^{1/2} - \left(\frac{b}{t} \right)^{1/2} \right] \right\}, \quad t > 0,$$

where the shape parameter is $a = 1$ and scale parameter $b = 1$ and the component lifetimes for System 2 follow a Weibull distribution with scale parameter $\lambda = 3$ and shape parameter $\gamma = 2$ (with PDF presented in Eq. (1.4)). The complete system lifetime data with sample sizes $M_1 = M_2 = 20$ for both systems are presented in Table 3.6.

Table 3.6: Simulated lifetime data from System 1 and System 2 with $M_1 = M_2 = 20$

j	1	2	3	4	5	6	7	8	9	10
$T_{1,(j)}$	0.2598	0.2803	0.3329	0.4172	0.4532	0.459	0.5541	0.5769	0.5842	0.7784
$T_{2,(j)}$	0.5890	0.6423	0.7774	0.9879	1.0754	1.1200	1.1685	1.2410	1.2412	1.2642

j	11	12	13	14	15	16	17	18	19	20
$T_{1,(j)}$	0.7917	0.8565	0.8895	0.9186	0.9348	1.1130	1.2049	1.3938	1.4406	1.6351
$T_{2,(j)}$	1.4164	1.4401	1.4924	1.5415	1.6912	2.0695	2.4374	2.5627	2.6197	2.7791

For illustrative purposes, we consider several Type-II censored data with different censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.3, and 0.5. For instance, when $\rho_1 = \rho_2 = 0.5$, the observed data from System 1 is $\boldsymbol{T}_1 = (T_{1,(1)}, \cdots, T_{2,(10)})$ and the observed data from System 2 is $\boldsymbol{T}_2 = (T_{2,(1)}, \cdots, T_{2,(10)})$ the remaining ten lifetimes from System 1 and System 2 are right-censored. To visualize the lifetime data for this example, we plot the Kaplan-Meier survival functions for System 1 and System 2 when censoring proportions are $\rho_1 = \rho_2 = 0.5$ in Figure 3.3.

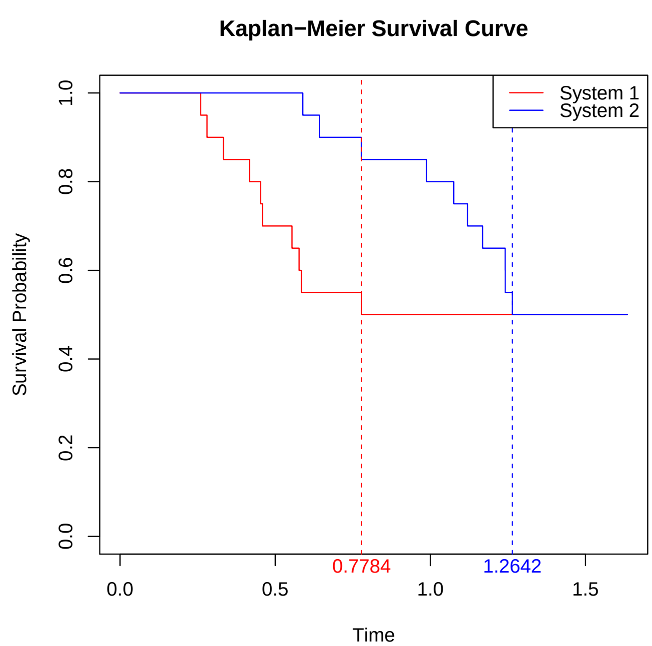

Figure 3.3: Kaplan-Meier survival functions for System 1 and System 2 with the data presented in Table 3.6 when censoring proportions are $\rho_1 = \rho_2 = 0.5$.

For the data set presented in Table 3.6, when $\rho_1 = \rho_2 = 0.5$, the experiment for System 1 ended at $T_{1,(10)} = 0.7784$ and the experiment for System 2 ended at $T_{2,(10)} = 1.2642$, and there are $L = 6$ failures from System 2 in between $T_{1,(10)} = 0.7784$ and $T_{2,(10)} = 1.2642$. Therefore, according to Proposition 3.2, there are $N_r = L + 3 = 9$ permutations under the restriction on the censoring observations. In contrast to the number of permutations $N_c = 8436285$, we only need to compute the test statistics under 9 permutations. Based on the the Type-II censored data with censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.3, and 0.5, we compute the test statistics $U_{\min}$, $U_{\max}$, U_{mean}, $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{K,\max}$, and $Z_{A,\text{mean}}$ under the restriction on the censoring observations discussed in Section 3.2 and the values of the test statistics are presented in Table 3.7.

Table 3.7: The proposed test statistics for Type-II censored data obtained from the data set in Table 3.6

Censoring Proportions	$Z_{K,\min}$	$Z_{K,\max}$	$Z_{K,\text{mean}}$	$Z_{A,\min}$	$Z_{A,\max}$	$Z_{A,\text{mean}}$	$U_{\min}$	$U_{\max}$	U_{mean}
$\rho_1 = \rho_2 = 0.1$	13.5608	13.5608	13.5608	21.3711	26.2878	22.6879	320	340	330.8
$\rho_1 = \rho_2 = 0.3$	13.5608	24.2200	15.9444	20.0818	35.2745	25.6503	265	367	328.7
$\rho_1 = \rho_2 = 0.5$	13.5608	30.8581	18.0847	19.9003	44.7490	27.5182	197	367	317

To test the hypotheses in Eq. (2.1) based on the observed test statistics in Table 3.6, we simulate the null distributions for $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$, $Z_{A,\text{mean}}$, $U_{\min}$, $U_{\max}$, and U_{mean} with $200,000$ Monte Carlo simulations. Since the randomization procedure is not needed for this example, we only present the simulated critical values at 5% level of significance for the test procedures based on $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$, $Z_{A,\text{mean}}$, $U_{\min}$, $U_{\max}$, and U_{mean} in Table 3.8.

By comparing the observed values of the test statistics in Table 3.7 to the critical values presented in Table 3.8, we determine that the null hypothesis is rejected at 0.05 significance level for all the proposed test procedures under the censoring proportions 0.1, 0.3, and 0.5. These results agree with our expectation since the component lifetimes in System 1 follow a Birnbaum-Saunders distribution, and the component lifetimes in System

Table 3.8: Simulated critical values for the test statistics with sample size $M_1 = M_2 = 20$, $r_1 = M_1(1-\rho_1)$, $r_2 = M_2(1-\rho_2)$, $n_1 = 8$, $\mathbf{s}_1 = (0, 1/4, 11/28, 2/7, 1/14, 0, 0, 0)$, $n_2 = 4$, $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, and significance level $\alpha = 5\%$.

Censoring proportion	Test Statistics					
	$Z_{K,\min}$	$Z_{K,\max}$	$Z_{K,\text{mean}}$	$Z_{A,\min}$	$Z_{A,\max}$	$Z_{A,\text{mean}}$
$\rho_1 = \rho_2 = 0.1$	8.1751	13.0109	9.3828	11.8865	17.2713	13.7522
$\rho_1 = \rho_2 = 0.3$	7.9638	23.5806	16.4214	11.6480	28.4890	19.3265
$\rho_1 = \rho_2 = 0.5$	7.2502	38.5601	29.1758	12.7950	44.6873	32.9071

Censoring proportion	Lower critical values			Upper critical values		
	Test Statistics					
	$U_{\min}$	$U_{\max}$	U_{mean}	$U_{\min}$	$U_{\max}$	U_{mean}
$\rho_1 = \rho_2 = 0.1$	166	174	170.5	298	321	310.8
$\rho_1 = \rho_2 = 0.3$	139	187	159	249	339	305.5
$\rho_1 = \rho_2 = 0.5$	79	221	119.9	187	360	308

2 follow a Weibull distribution.

3.5. Monte-Carlo Simulation Study

In this section, we use a Monte Carlo simulation study to assess the effectiveness and power performance of the nonparametric test procedures described in Section 3.2 for testing the hypotheses in Eq. (2.1). The simulation settings, including the system structures and the underlying distributions of the component lifetimes, are described in Section 3.5.1. The simulation results and discussions are provided in Section 3.5.2.

3.5.1. Simulation settings

In the simulation study, we consider the following system signatures described in Navarro et al. (2007) for System 1 and System 2:

[S1] System 1: 3 components system with system signature $\mathbf{s}_1 = (0, 2/3, 1/3)$;
System 2: 4 components system with system signature $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$.

[S2] System 1: 3 components system with system signature $\mathbf{s}_1 = (0, 2/3, 1/3)$;
System 2: 4 components system with system signature $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$.

To compare the power of hypothesis testing, we fix the parameter settings of the underlying component distribution for the components in System 1 and vary the parameter settings of the underlying component distribution for the components in System 2. The following distribution settings are used to generate the system lifetime data for System 1 and System 2:

[D1] Exponential distributions with variations in the scale parameter:

System 1: $Exp(\theta_1)$ with $\theta_1 = 1$ (i.e., $\ln \theta_1 = 0$);
System 2: $Exp(\theta_2)$ with $\ln \theta_2$ varies from -1.6 to 1.6 in increments of 0.1. (denoted by $-1.6\ (0.1)\ 1.6$);

[D2] Gamma distributions with variations in the shape parameter:

System 1: $Gamma(\alpha_1, \beta_1)$ with $\alpha_1 = 5$ and $\beta_1 = 2$;
System 2: $Gamma(\alpha_2, \beta_2)$ with α_2 varies from 3 to 7 in increments of 0.2 (denoted by $3\ (0.2)\ 7$) and $\beta_2 = 2$;

[D3] Gamma distributions with variations in the rate parameter:

System 1: $Gamma(\alpha_1, \beta_1)$ with $\alpha_1 = 5$ and $\beta_1 = 2$;
System 2: $Gamma(\alpha_2, \beta_2)$ with $\alpha_2 = 5$ and β_2 varies from 1 to 3 in increments of 0.1 (denoted by $1\ (0.1)\ 3$);

[D4] Weibull distributions with variations in the scale parameter:

System 1: $Weibull(\lambda_1, \gamma_1)$ with $\lambda_1 = 2.5$ and $\gamma_1 = 5$;

System 2: $Weibull(\lambda_2, \gamma_2)$ with λ_2 varies from 1.5 to 3.5 in increments of 0.1 (denoted by $1.5\ (0.1)\ 3.5$) and $\gamma_2 = 5$;

[D5] Lognormal distributions with variations in the standard deviation on the log-scale, which corresponds to changes in the shape parameter.:

System 1: $Lognormal(\mu_1, \sigma_1)$ with $\mu_1 = 0$ and $\sigma_1 = 1$;

System 2: $Lognormal(\mu_2, \sigma_2)$ with $\sigma_2 = 1$ and μ_2 varies from -1.6 to 1.6 in increments of 0.1 (denoted by $-1.6\ (0.1)\ 1.6$).

For the simulation study, we fix the significance level $\alpha = 0.05$. Different sample sizes $M_1 = M_2 = 10$, 20, and 30, and censoring proportions $\rho_1 = \rho_2 = 0.1,\ 0.2,\ 0.3$, and 0.4 are considered. The estimated the rejection rates under various settings (i.e., different combinations of **[S1]** and **[S2]** with **[D1]**–**[D5]**) are obtained based on $5,000$ simulations. We also consider the test procedures based on complete system lifetime data (i.e., $r_i = M_i,\ i = 1, 2$) as a benchmark for the power comparisons. For the sake of simplicity, we only present the simulated power curves for test statistic Z_A based on complete system lifetime data as Z_A is more powerful compared to Z_K and U in most cases.

3.5.2. Results and Discussions

After simulating the rejection rate of the test procedures based on the nine statistics, $U_{\min}$, $U_{\max}$, U_{mean}, $Z_{K,\min}$, $Z_{K,\max}$, $Z_{K,\text{mean}}$, $Z_{A,\min}$, $Z_{A,\max}$ and $Z_{A,\text{mean}}$ under the settings described in Section 3.5.1, we summarize the results by plotting the simulated power curves of these nice proposed test procedures. These simulated power curves are aim to provide a clear and concise way of visualizing the performance of each test procedure in detecting the hypothesized effect sizes under varying conditions. These simulated power curves are centered at the rejection rates obtained under the null hypothesis that $F_{X_1} = F_{X_2}$, which are expected to be close to the nominal significance level $\alpha = 5\%$. As the differences between the parameters in the underlying component lifetime distributions for components in System 1 and System 2 increase (i.e., the simulated power curves moving away from the center), we anticipate that the simulated power values will increase.

Based on our preliminary observations, the performance of the proposed test procedures are similar for different sample sizes M_1 and M_2. For the sake of saving spaces, we present the simulated power curves for five different underlying distribution settings **[D1]**–**[D5]**, with system structures **[S1]** and **[S2]**, for sample sizes of $M_1 = M_2 = 20$ in Figures 3.4–3.13. For the simulated power curves for sample sizes $M_1 = M_2 = 10$ and 30 are presented in Figure B.1–B.19 in Appendix.

In addition to the power values, we also consider the computation times required to compute the test statistic. In Table 3.9, we present the total computation time for the test statistics based on Z_K, Z_A, and U for $5,000$ simulations when the underlying distributions $Exp(1)$ and $Exp(\theta_2)$, where $\ln \theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., **[D1]** with system structures **[S1]**.) From Table 3.9, we observe that the computation time for the test procedures based on the U statistic is at least 90% lower than the computation time for the test procedures based on Z_A and Z_K, while the computation time for the test procedures based on Z_A is similar to that for those based on Z_K.

Table 3.9: Total computation time (in minutes) for the test statistics based on Z_K, Z_A, and U for 5000 simulations for different settings

Settings	$Z_{K,\min}, Z_{K,\max}, Z_{K,\text{mean}}$	$Z_{A,\min}, Z_{A,\max}, Z_{A,\text{mean}}$	$U_{\min}, U_{\max}, U_{\text{mean}}$
$\rho_1 = \rho_2 = 0.1$, $M_1 = M_2 = 10$	75.68	75.73	6.08
$\rho_1 = \rho_2 = 0.2$, $M_1 = M_2 = 20$	195.6	196.8	11.02
$\rho_1 = \rho_2 = 0.4$, $M_1 = M_2 = 20$	174.30	174.61	11.42

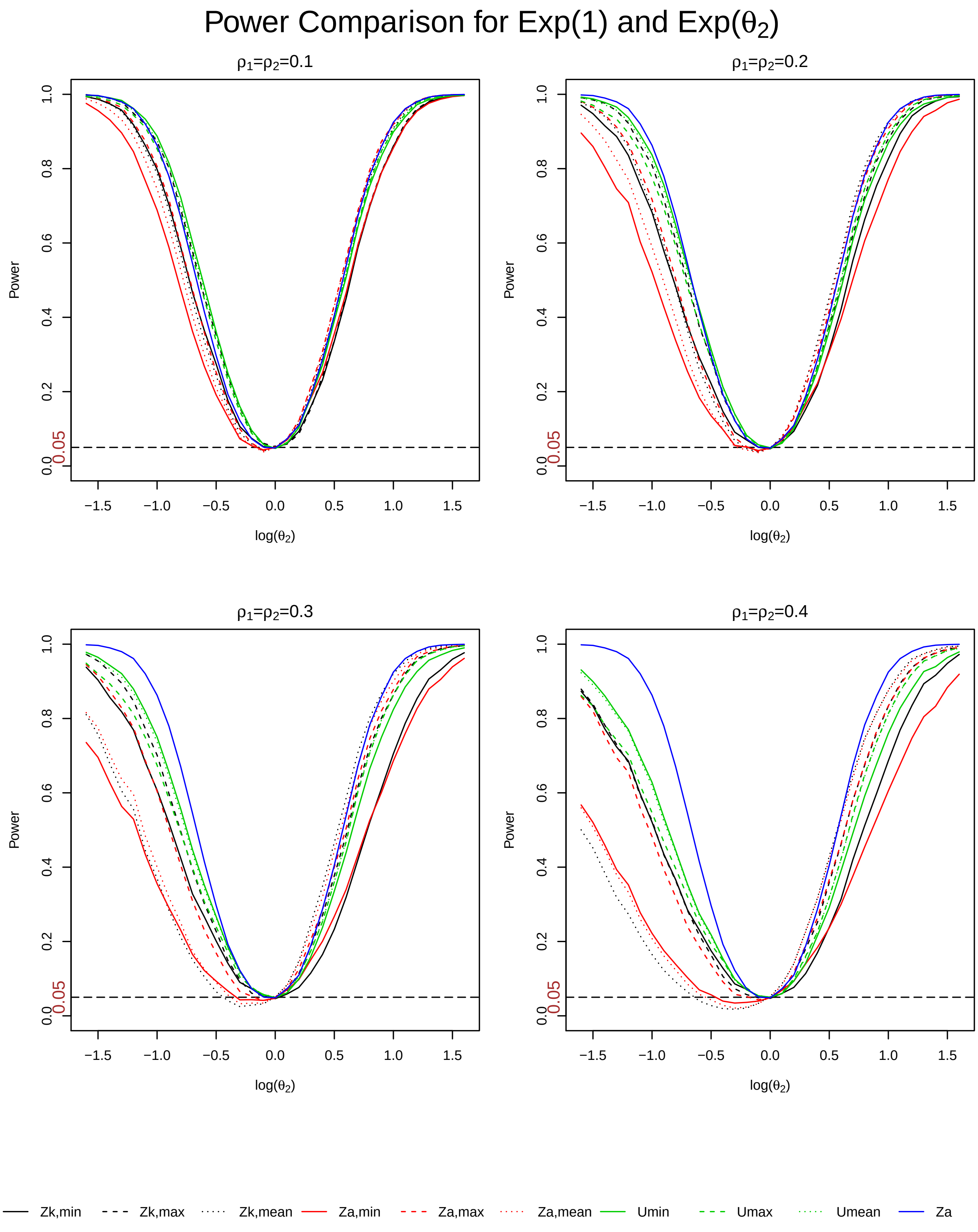

Figure 3.4: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$), sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Exp(1)$ and $Exp(\theta_2)$, where $\ln \theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S1**]).

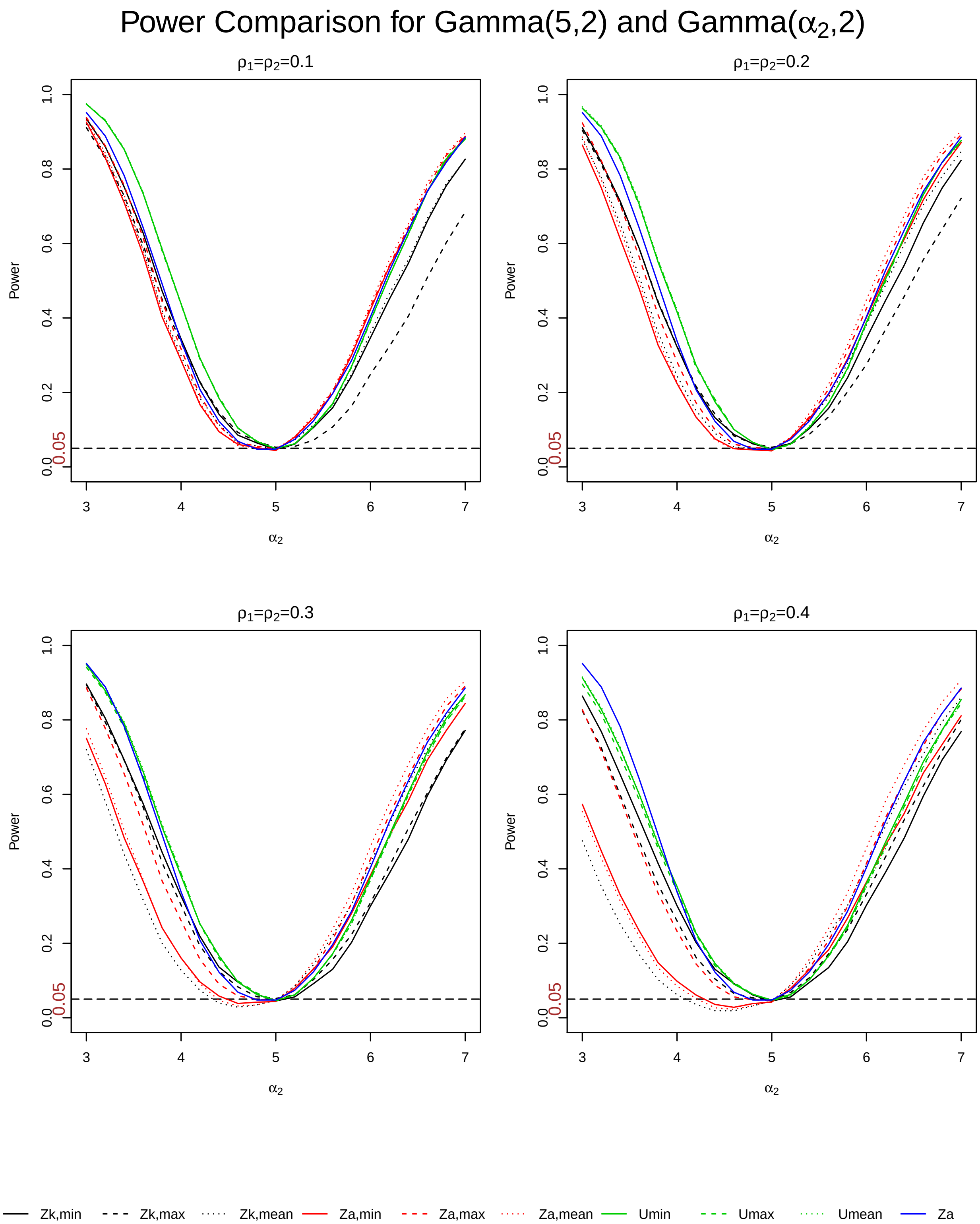

Figure 3.5: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where α_2=3 (0.2) 7 (i.e., [**D2**] with system structures [**S1**]).

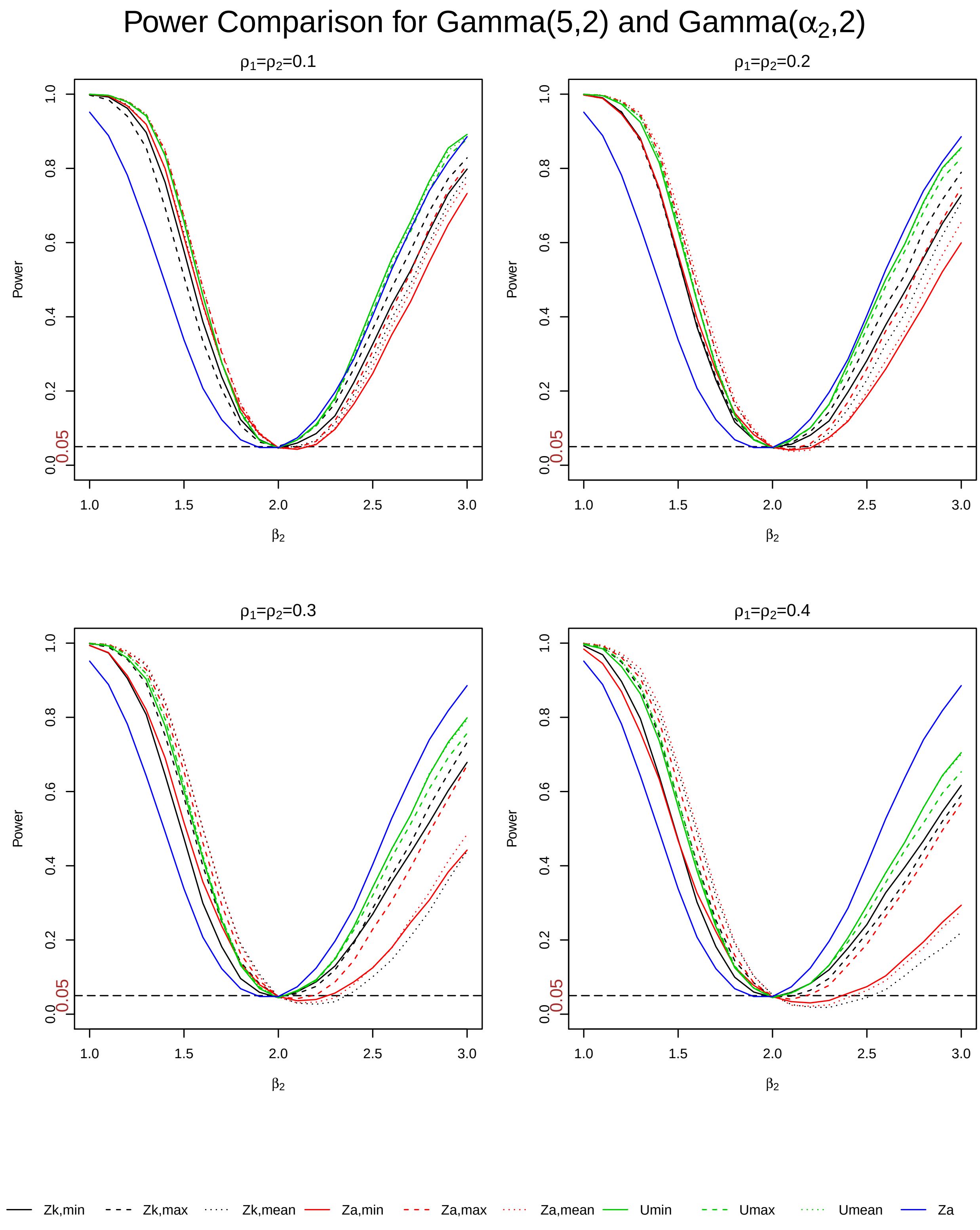

Figure 3.6: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2, where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S1**]).

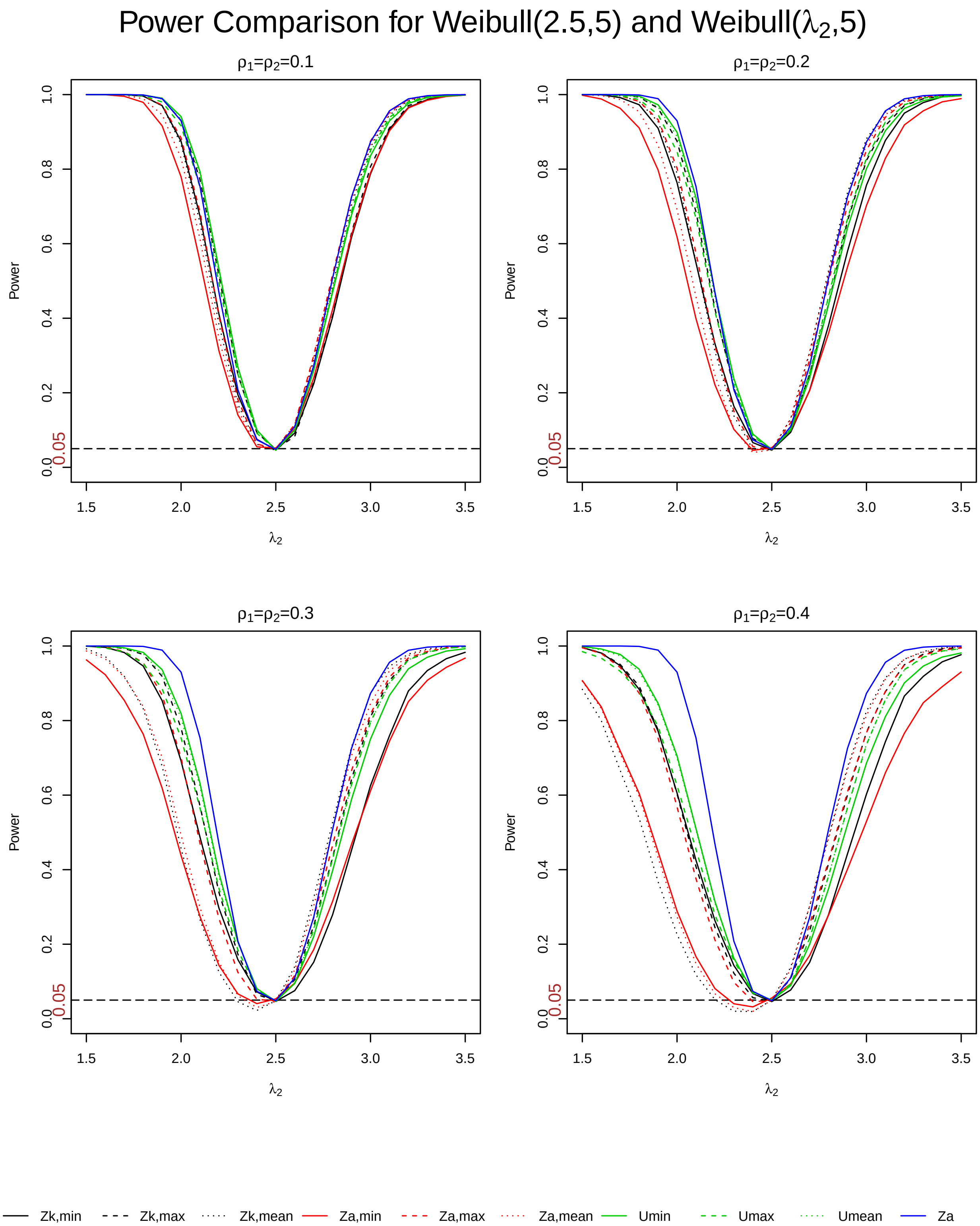

Figure 3.7: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2, where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D4**] with system structures [**S1**]).

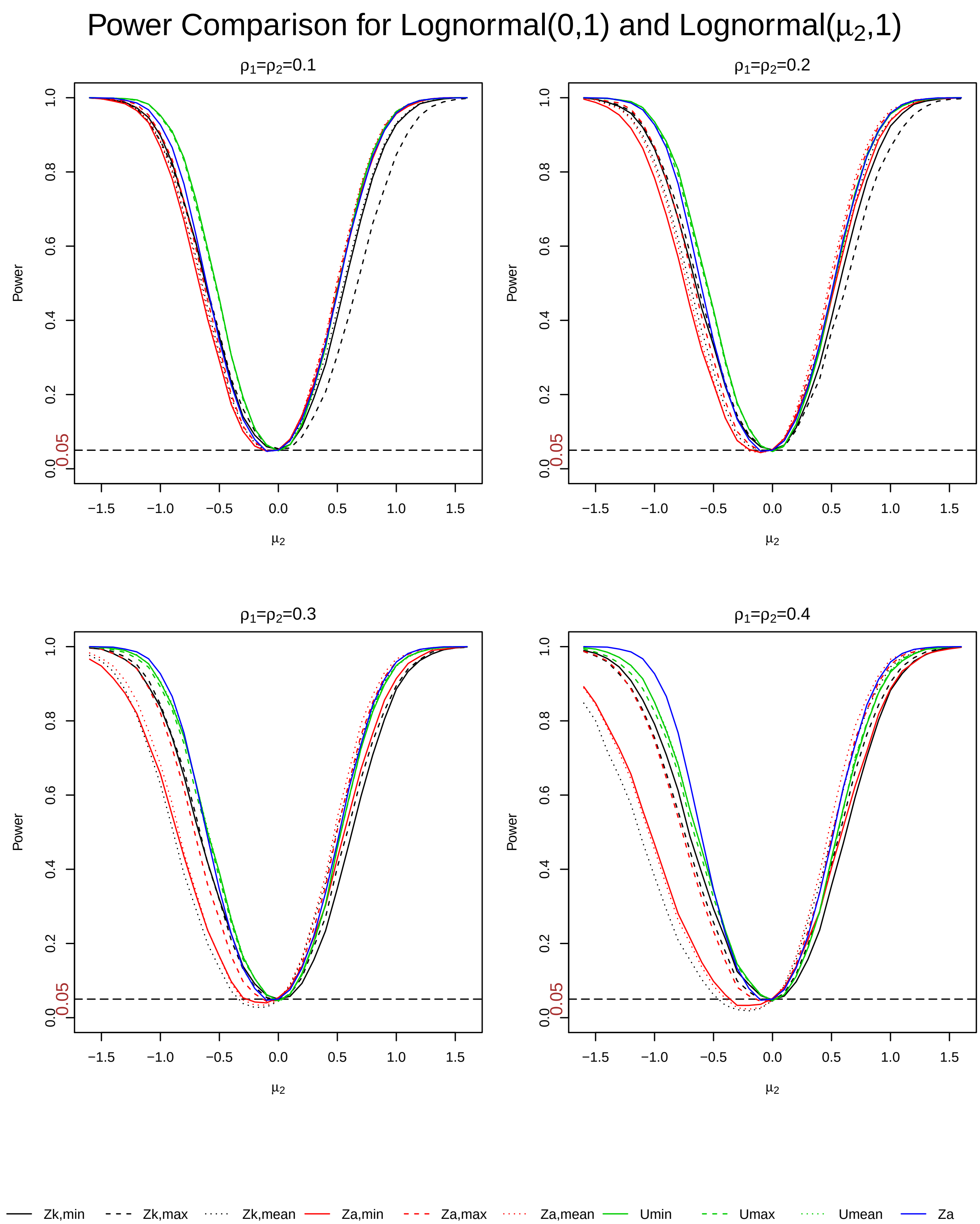

Figure 3.8: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Lognormal(0, 1)$ for System 1 and $Lognormal(\mu_2, 1)$ for System 2, where $\mu_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D5**] with system structures [**S1**]).

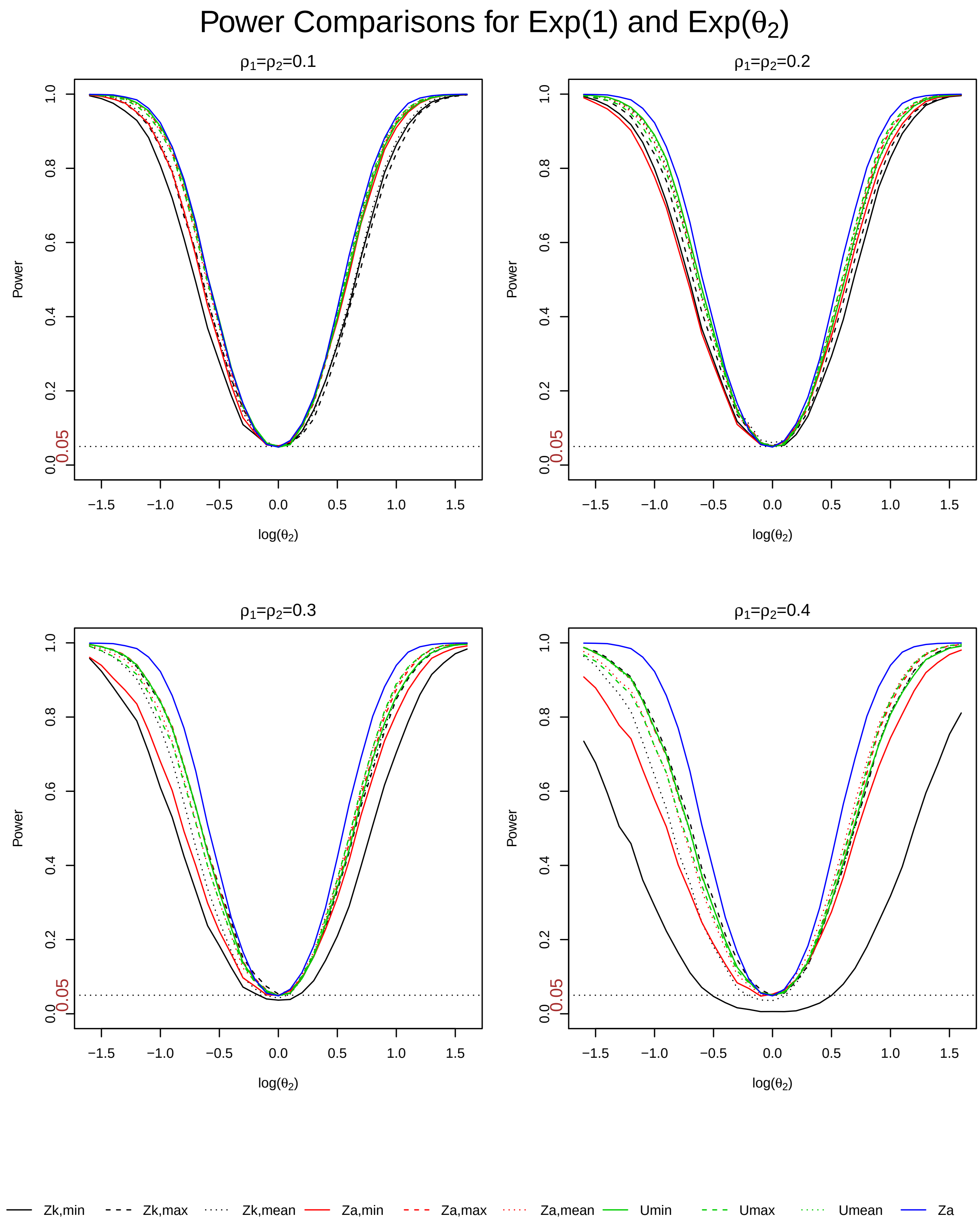

Figure 3.9: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Exp(1)$ for System 1 and $Exp(\theta_2)$ for System 2, where $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S2**]).

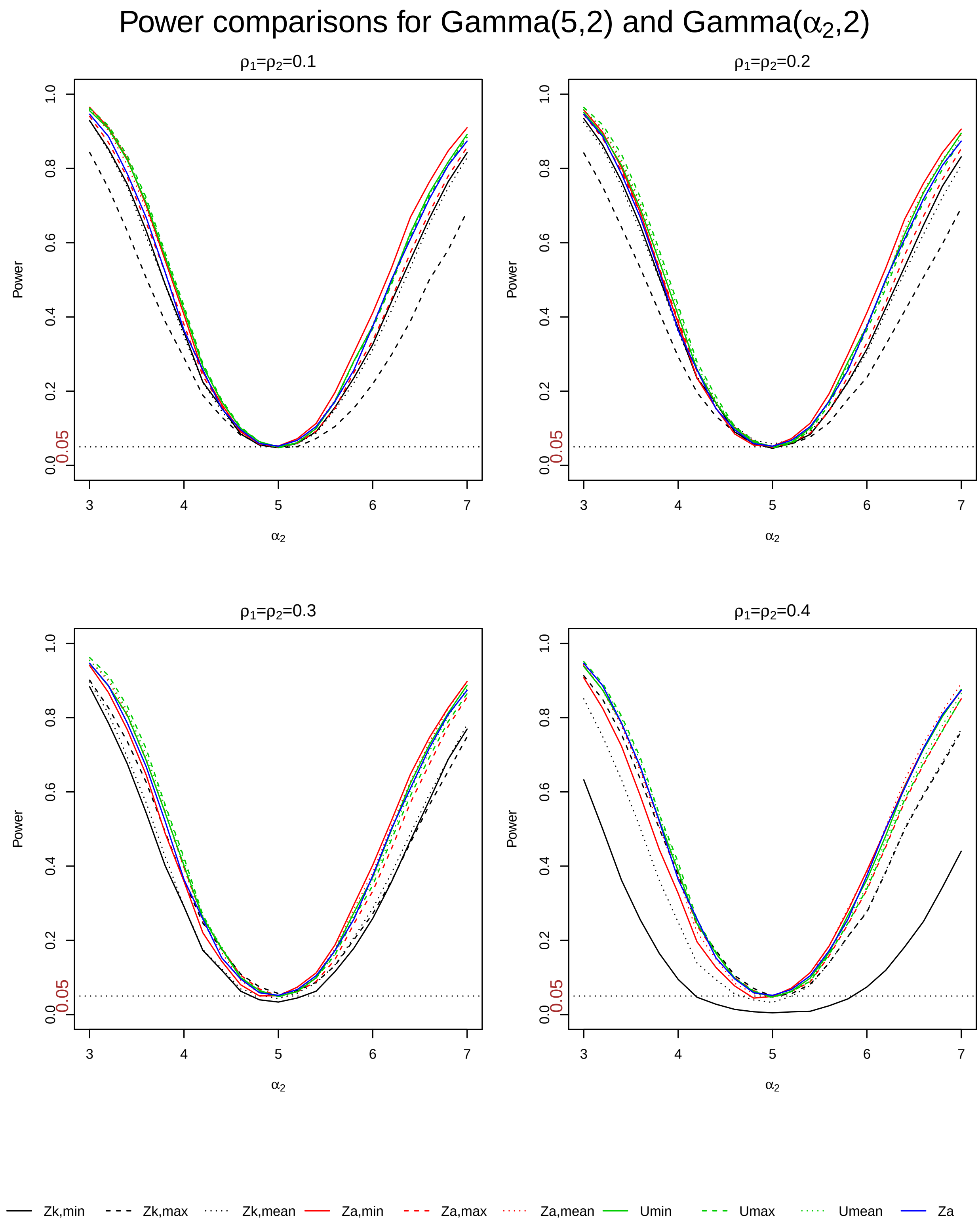

Figure 3.10: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where $\alpha_2 = 3\ (0.2)\ 7$ (i.e., [**D2**] with system structures [**S2**]).

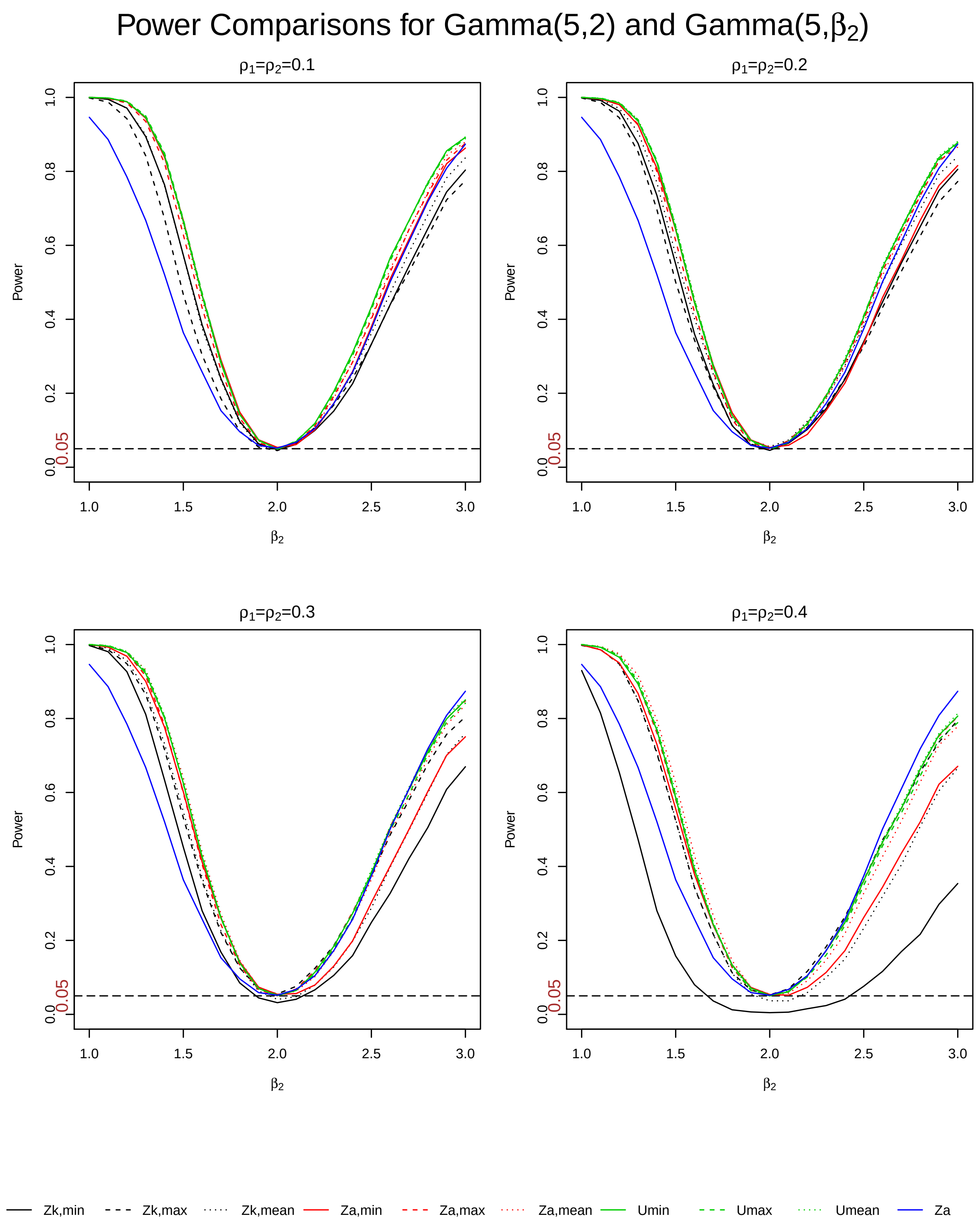

Figure 3.11: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2 where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S2**]).

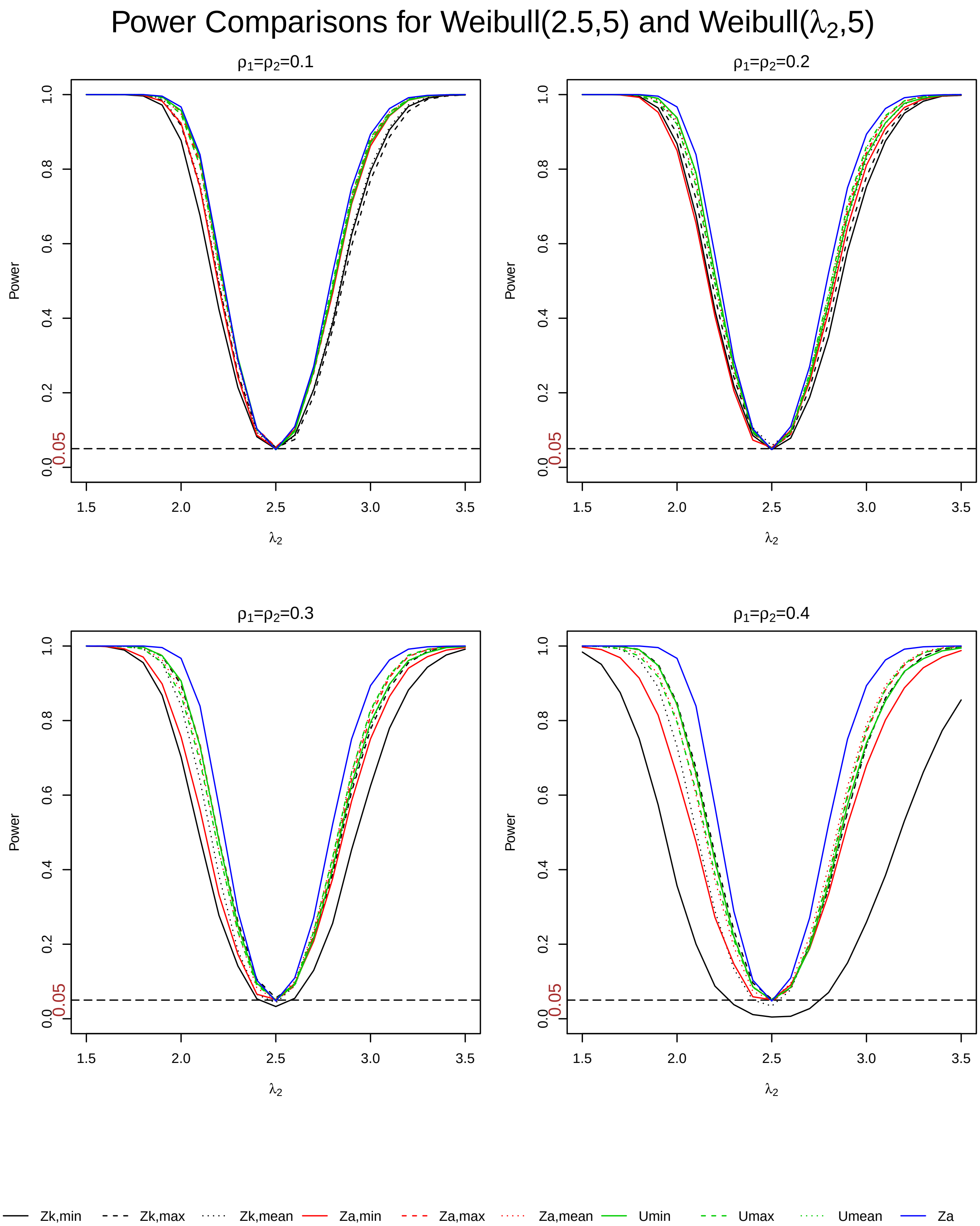

Figure 3.12: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2, where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D4**] with system structures [**S2**]).

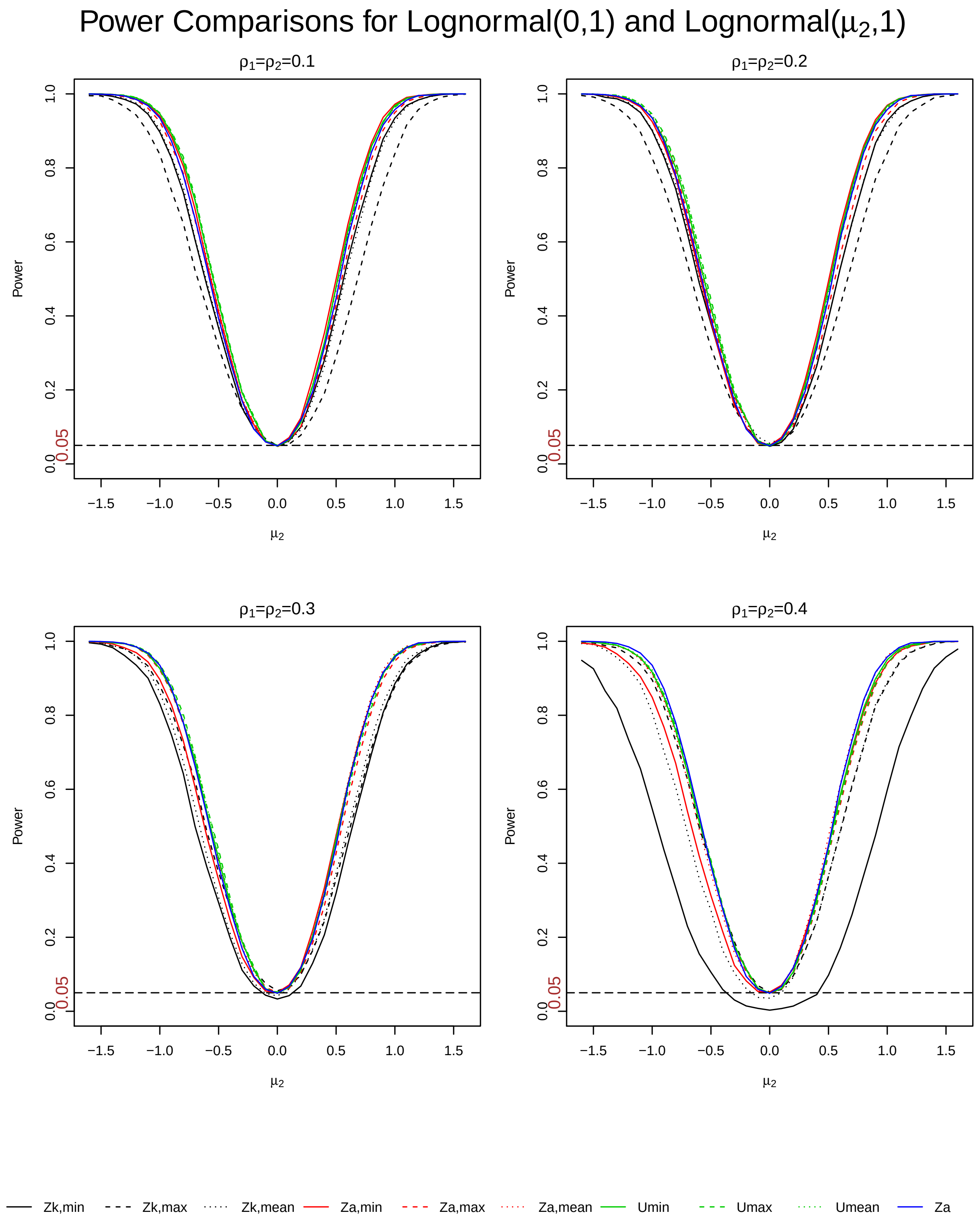

Figure 3.13: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 20$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Lognormal(0, 1)$ for System 1 and $Lognormal(\mu_2, 1)$ for System 2, where μ_2=-1.6 (0.1) 1.6 (i.e., [**D5**] with system structures [**S2**]).

From the simulation results, we observe that the proposed test procedures for testing the homogeneity of component lifetime distributions based on Type-II censored data provide reasonable power values across all the settings considered here. As was foreseeable, we observe an increase in the power values for all the proposed test procedures as the sample sizes M_1 and M_2 increase. We also observe a decrease in the power values for all the proposed test procedures when the censoring proportions increase. Interestingly, no single test procedure consistently outperforms all the other test procedures under all the settings considered in this simulation study.

The simulation results demonstrate that our proposed procedures for Type-II censored data yield comparable power values to the test procedure Z_A based on complete system lifetime data. When the censoring proportion is small (say, $\rho_1 = \rho_2 = 0.1$ or 0.2), the power values based on Type-II censored data are similar to those based on complete system lifetime data. As censoring proportions increase (say, $\rho_1 = \rho_2 = 0.3$ or 0.4), as we expected, the power values based on complete system lifetime data are larger than those based on the censored system lifetime data.

To study the performance of the proposed test procedures for Type-II censored system lifetime data, first, we compare the power performance of the statistics by taking the minimum, maximum, and mean of the statistics U, Z_K, and Z_A for all the restricted permutations. When we compare the simulated power curves by taking the upper bound of the Monte Carlo error, $\sqrt{0.5(1-0.5)/5000} = 0.007$, into account, we have the following observations:

- In comparing $Z_{K,\min}$, $Z_{K,\max}$, and $Z_{K,\text{mean}}$ (indicated by black lines in the figures), we observe that $Z_{K,\max}$ outperforms $Z_{K,\min}$ and $Z_{K,\text{mean}}$ in most cases. It is noteworthy that the test procedure based on $Z_{K,\min}$ may not be appropriate in some cases when the censoring proportions are $\rho_1 = \rho_2 = 0.3$ and 0.4 (see, for example, Figure 3.9 and Figure B.18 in Appendix) since there are limited values of Z_k for the restricted permutations and the distribution of $Z_{k,\min}$ is highly discrete.

- In comparing $Z_{A,\min}$, $Z_{A,\max}$, and $Z_{A,\text{mean}}$ (indicated by red lines in the figures), we observe that the test statistic $Z_{A,\min}$ has worse power performance than $Z_{A,\max}$ and $Z_{A,\text{mean}}$ in most cases.
- In comparing $U_{\min}$, $U_{\max}$, and U_{mean} (indicated by green lines in the figures), we observe that the test statistic $U_{\min}$ has better power performance than $U_{\max}$ and U_{mean} in most cases.

In comparing all the nine proposed test statistics, we compare the power performance under (i) the mean lifetime of components in System 1 is smaller than the mean lifetime of components in System 2; and (ii) the mean lifetime of components in System 2 is smaller than the mean lifetime of components in System 1. Based on the simulation results, we have the following observations:

- For changes in the shape parameter (i.e., distribution settings **[D2]** and **[D5]**): (i) $U_{\min}$ has the best power performance on the left-hand side of the power curve in general; and (ii) $Z_{A,\text{mean}}$ has the best power performance on the right-hand side of the power curve (see, for example, Figures 3.5 and 3.8);
- For changes in the scale parameter (i.e., distribution settings **[D1]**, **[D3]** and **[D4]**): (i) $U_{\min}$ has the best power performance on the left-hand side of the power curve in general; and (ii) $Z_{A,\text{mean}}$ has the best power performance on the right-hand side of the power curve (see, for example, Figures 3.4 and 3.7).

When the sample sizes are large ($M_1 = M_2 = 30$), the tests based on U generally exhibit better performance, or are among the tests with better performance, across all censoring proportions considered as shown in Figures B.6-B.10.

Overall, among the nine test statistics proposed in this chapter, we observe that the power performance of $Z_{A,\max}$ and $U_{\min}$ are satisfactory compared to all the other test statistics in most of the scenarios considered in the Monte Carlo simulation study. If we take into account the computation time requires to compute the test statistics, it is more advisable to use $U_{\min}$ as the computation time for $U_{\min}$ is significantly shorter than $Z_{A,\max}$.

3.6. Concluding Remarks

In this manuscript, we proposed different nonparametric statistical testing procedures for testing the homogeneity of component lifetime distributions based on Type-II censored system-level data with known structures, which is a practical scenario in life testing procedures involving systems. The empirical likelihood ratio statistics and Mann-Whitney U statistic developed for testing the homogeneity of component lifetime distributions based on complete system-level data are considered. The proposed test procedures are based on the permutations of the unobserved censored system lifetimes, which can be considered a generalization of the test procedures based on complete system-level data. In order to reduce the number of permutations required in the computation of test procedures, we propose a permutation method with a restriction on the equality of the censored system lifetimes. We provide the computational approach using the Monte Carlo method to approximate the null distributions of the test statistics.

The proposed test procedures were compared using a Monte Carlo simulation study. The simulation results indicate that the power performance of the proposed test procedures for Type-II censored data is comparable to the test procedure based on complete system lifetime data. Based on the simulation results, we recommend the test procedure based on the maximum of the empirical likelihood ratio statistic (i.e., $Z_{A,\max}$) and the minimum of the Mann-Whitney U statistic (i.e., $U_{\min}$). However, if computation time is considered, $U_{\min}$ is preferred. The computer programs written in R (R Core Team, 2022) for executing the test procedures presented in this manuscript are available from the authors upon request.

As a potential direction for future research, extending the test procedures to handle progressive Type-II censored system-level data would be valuable. Additionally, while our work assumes known system structures, this may be unrealistic when the systems of interest are black boxes and complete information about their structures is unavailable. Therefore, developing test procedures for the homogeneity of component lifetime distributions when the system structures are unknown would be a valuable and relevant contribution to practical applications.

CHAPTER 4

Concluding Remarks and Future Research Directions

4.1. Summaries and Concluding Remarks

In this thesis, we developed and studied statistical hypothesis testing procedures for testing the homogeneity of component lifetime distributions based on system lifetimes from multiple data sources. Testing the homogeneity of component lifetime distributions based on system lifetimes is of practical interest in identifying the most reliable component supplier based on system lifetime data, determining if the performance of the is different in different systems, or the actual performance of the components in a system is different from that obtained when the components are out of the system. Several distribution-free hypothesis testing procedures for testing the homogeneity of component lifetime distributions based on complete and Type-II censored system lifetime data are proposed by assuming known system structures.

4.1.1. Summary of Chapter 2

In Chapter 2, we studied the problem of testing the homogeneity of component lifetime distributions based on system-level data, which can be applied to many practical situations in life testing procedures involving systems with known structures. We showed that those existing parametric test procedures might suffer from the inflation of Type-I error rates when the underlying probability distributions of the component lifetimes are misspecified. To address this issue, we focus on developing nonparametric statistical test procedures for the homogeneity of component lifetime distributions based on system-level data. We proposed two empirical likelihood ratio tests based on the empirical likelihood ratio and the nonparametric estimation of component lifetime distributions. We provided the computational algorithms for obtaining the null distributions of the test statistics using the Monte Carlo method. Our simulation results show that the proposed nonparametric procedures provide comparative power values with those existing tests. These proposed test procedures have advantages in power performance when the two systems are very different.

4.1.2. Summary of Chapter 3

As described in Chapter 2, researchers are interested in testing the equality of component lifetime distributions based on various system settings. However, in the process of collecting system lifetime data, it is possible that complete information on the time to failure of the systems may not be available due to cost and time constraints. Censoring is often adopted to reduce the length and expenses of a life-testing experiment or lifetime data collection process.

Based on Type-II censored system-level data with known structures, we proposed different nonparametric statistical testing procedures for testing the homogeneity of component lifetime distributions. The empirical likelihood ratio statistics and Mann-Whitney U statistic developed for testing the homogeneity of component lifetime distributions based on complete system-level data are considered. The proposed test procedures are based on the permutations of the unobserved censored system lifetimes, which can be considered a generalization of the test procedures based on complete system-level data. In order to reduce the number of permutations required in the computation of test procedures, we propose a permutation method with a restriction on the equality of the censored system lifetimes. We provide the computational approach using the Monte Carlo method to approximate the null distributions of the test statistics.

The proposed test procedures were compared using a Monte Carlo simulation study. The simulation results demonstrate that the proposed nonparametric procedures have reasonable power values. We recommend the test procedure based on the minimum of the Mann-Whitney U statistic (i.e., $U_{\min}$).

4.2. Future Research Directions

In this thesis, we proposed using empirical likelihood ratio statistics for assessing the homogeneity of component lifetime distributions using complete and Type-II censored system-level data. For future research on this topic, the following research directions can be explored.

4.2.1. Extensions to Other Censoring Schemes

This dissertation exclusively focuses on complete and Type-II censored system lifetime data. There are other kinds of censoring schemes such as Type-I censoring and progressive Type-II censoring, have been used for life-testing experiments.

For Type-I censoring, the life-testing experiments for System 1 and System 2 are terminated at a prefixed time C. In this situation, the numbers of observed system failures from System 1 and System 2 are random variables while the time for the experiments is prefixed. The test procedures developed for Type-II censored system lifetime data may not be suitable for Type-I censored system lifetime data because there is a positive probability that there are no observed system failures under the Type-I censoring scheme, especially when the two systems are very different (e.g., series system against parallel systems). For this reason, suitable adjustments of the test procedures developed for Type-II censored data are needed.

For the progressive Type-II censoring scheme, which is a generalization of Type-II censoring, an experiment is conducted with n units placed on a lifetime testing. At the time of the first failure, R_1 units are randomly removed from the remaining n-1 surviving units. At the time of the second failure, R_2 units are randomly removed from the remaining $n-2-R_1$ units, and so on until the m-th failure. At this point, all remaining units ($R_m = n-m-R_1-R_2-...-R_{m-1}$) are removed. The values of R_i are predetermined before the study and remain fixed throughout. Note that the complete sample and Type-II right censored sample are special cases of the progressive Type-II censored sample.

Based on the progressive Type-II censored system lifetime data from System 1 and System 2, we are interested in testing the homogeneity of the component lifetime distributions. Kaplan-Meier estimate of survival function can be applied to progressive Type-II right-censored data. The following steps can be used to estimate the survival function of the component lifetime using Kaplan-Meier estimation for progressive Type-II right-censored data:

- Order the observed failure times of System i ($i = 1, 2$) with progressive Type-II censoring scheme $(R_{i1}, R_{i2}, \dots, R_{ir_i})$ in ascending order, and denote them by $\boldsymbol{T}_i = (T_{i1} < T_{i2} < \dots < T_{ir_i})$. For $j = 1, 2, \dots, r_i$, define d_{ij} as the number of failures at time T_{ij}, and n_{ij} as the number of units at risk just before time T_{ij}. At time T_{i1}, $n_{i1} = n_i$, and at subsequent times, $n_{ij} = n_{i,j-1} - d_{i,j-1} - R_{i,j-1}$.

- Compute the Kaplan-Meier product limit estimator of the survival function as follows:

$$S_i(t) = \prod_{j:T_{ij} \leq t} \left(1 - \frac{d_{ij}}{n_{ij}}\right)^{(1-R_{ij})}$$

  where the product is taken over all failure times T_{ij} such that $T_{ij} \leq t$, and the exponents $(1 - R_{ij})$ account for the progressive Type-II right-censoring scheme.

- To analyze the reliability of a system with signature s_i, we calculate the reliability polynomial $h_i(p)$, as defined in Eq. (1.2). We can obtain the estimated survival function of the components, denoted by $\hat{p}_i(t) = h_i^{-1}(S_i(t))$. Using these component survival functions and censored system data, we can then construct the likelihood function for both systems.

According to Hall et al. (2015), the empirical likelihood function for System i based on progressive Type-II censored sample for System i, $\boldsymbol{T}_i = (T_{i1}, \ldots, T_{ir_i})$ ($i = 1, 2$), can be expressed as

$$L_{T_i}(t) = \binom{M_i}{Y_i(t)} h_i(\hat{p}_i(t))^{Y_i(t)}[1 - h_i(\hat{p}_i(t))]^{M_i - Y_i(t)},$$

where $Y_i(t) = \sum_{j=1}^{M_i} I_{(t,\infty)}(T_{ij}),\ i = 1, 2$, and $I_{(t,\infty)}(T_{ij})$ is an indicator function defined as

$$I_A(b) = \begin{cases} 1, & \text{if } b \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Here, M_i is the total number of systems (including both observed failures and censored systems) for System i, $Y_i(t)$ counts the number of systems for System i that have failed by time t, and $\hat{p}_i(t)$ is the estimated survival function of System i based on the reliability polynomial $h_i(p)$.

When we have progressive Type-II censored system lifetime data, a similar idea of going through all the permutations of the censored system lifetimes proposed in Chapter 3 can be considered. However, considering all the required permutations for progressive Type-II censoring is more complicated than conventional Type-II censored data, and it needs to be executed with great care.

4.2.2. Extension to Situations with Unknown System Structures

In this dissertation, we assumed that the system structures are known for complete and Type-II censored system lifetime data. However, this assumption may not be realistic in some situations, especially when the systems of interest are black boxes. Therefore, it would be interesting to develop test procedures for the homogeneity problem without complete information about the system structure.

It is necessary to estimate the system signatures from the available data to handle the nonparametric estimation of the component lifetime distributions when the system structures are unknown. Jin et al. (2017) assumed that auxiliary data in the form of a variable K, representing the number of failed components at the time of system failure, is available in addition to the system lifetime data. The information on the variable K is typically obtained from a subsequent autopsy of a failed system. The data (T_1, K_1), $(T_2, K_2), \ldots, (T_N, K_N)$ can be used to estimate the reliability polynomial h, which is calculated using system signatures as described in Eq. (1.2). Based on the work by Jin et al. (2017), we can define the test statistics Z_K, Z_A, and U, and construct the simulated null distributions for these statistics when the system structures of System 1 and System 2 are unknown.

4.2.3. Consider Other Test Statistics

In this dissertation, we consider testing the homogeneity of component lifetime distributions using system lifetime data based on two empirical likelihood ratio statistics, namely Z_K and Z_A, originally proposed by Zhang (2006). Zhang (2006) also introduced other test statistics based on the empirical likelihood ratio, including test statistics obtained by employing a different weight function $w(t)$ in the empirical likelihood ratio.

For future research, we can explore other test statistics beyond Z_K and Z_A by using different weight functions $w(t)$ and examine the power performance of those test statistics.

APPENDIX A

APPENDIX of CHAPTER 2

Appendix A: Additional Simulation Results for Different Distributions and Settings

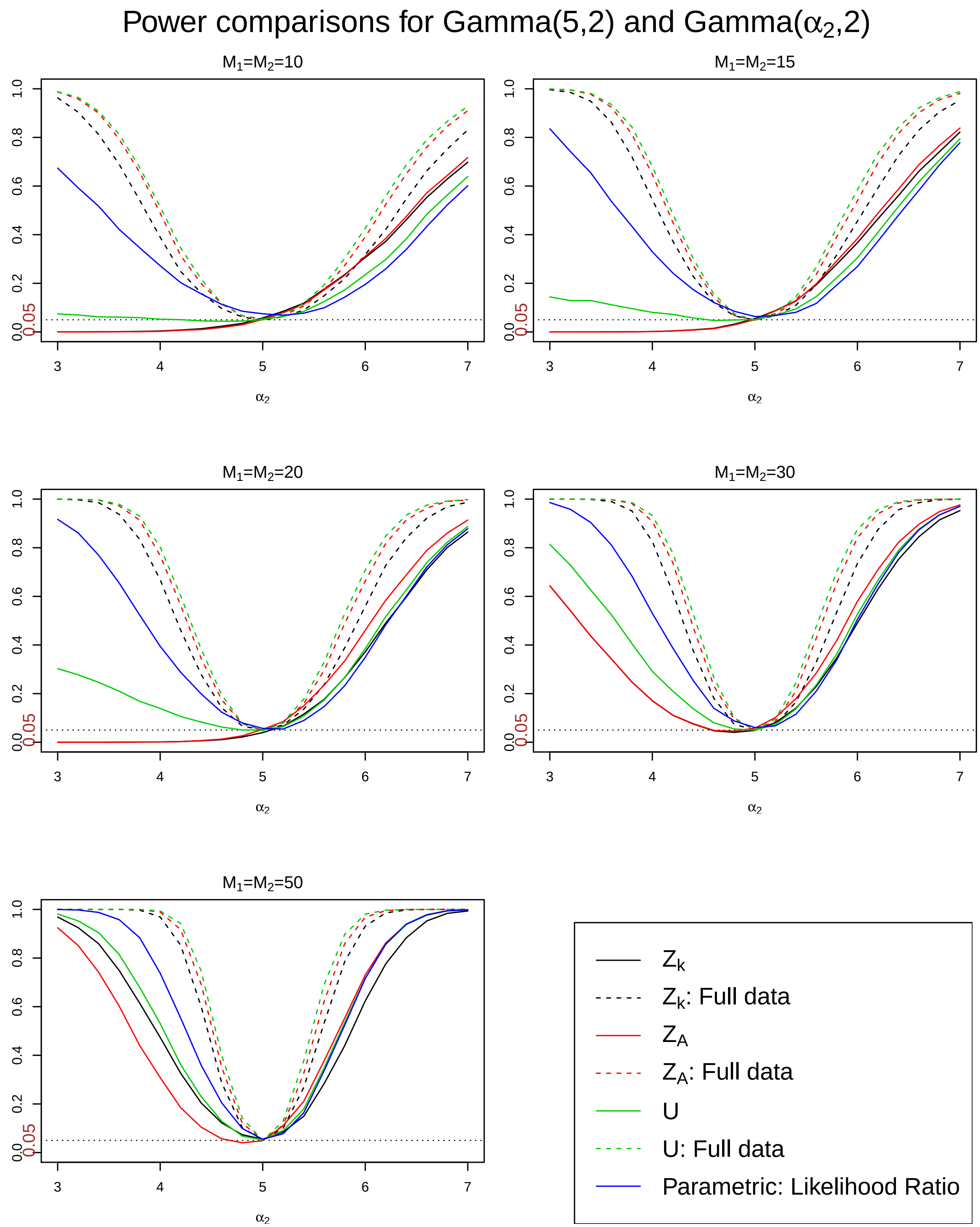

Figure A.1: Simulated power curves of the parametric and nonparametric test for $\alpha_1 = 5$ and $\alpha_2 = 3\ (0.2)\ 7$, $\beta_1 = \beta_2 = 2$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$ (i.e., [**D5**] with system structures [**S1**]).

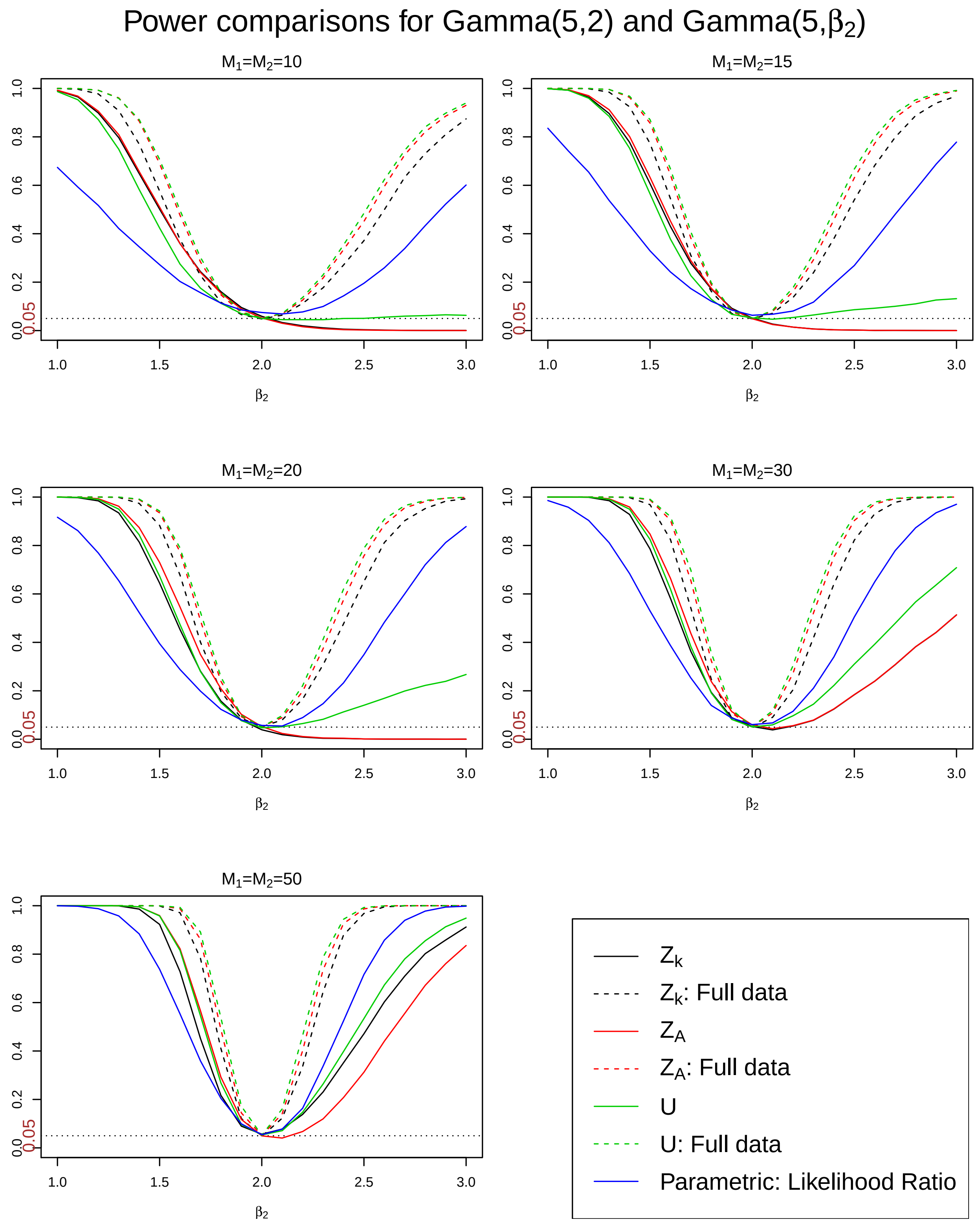

Figure A.2: Simulated power curves of the parametric and nonparametric test for $\beta_1 = 2$ and $\beta_2 = 1\ (0.1)\ 3$, $\alpha_1 = \alpha_2 = 5$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 0, 0, 1)$ and $\mathbf{s}_2 = (1, 0, 0)$ (i.e., [**D6**] with system structures [**S1**]).

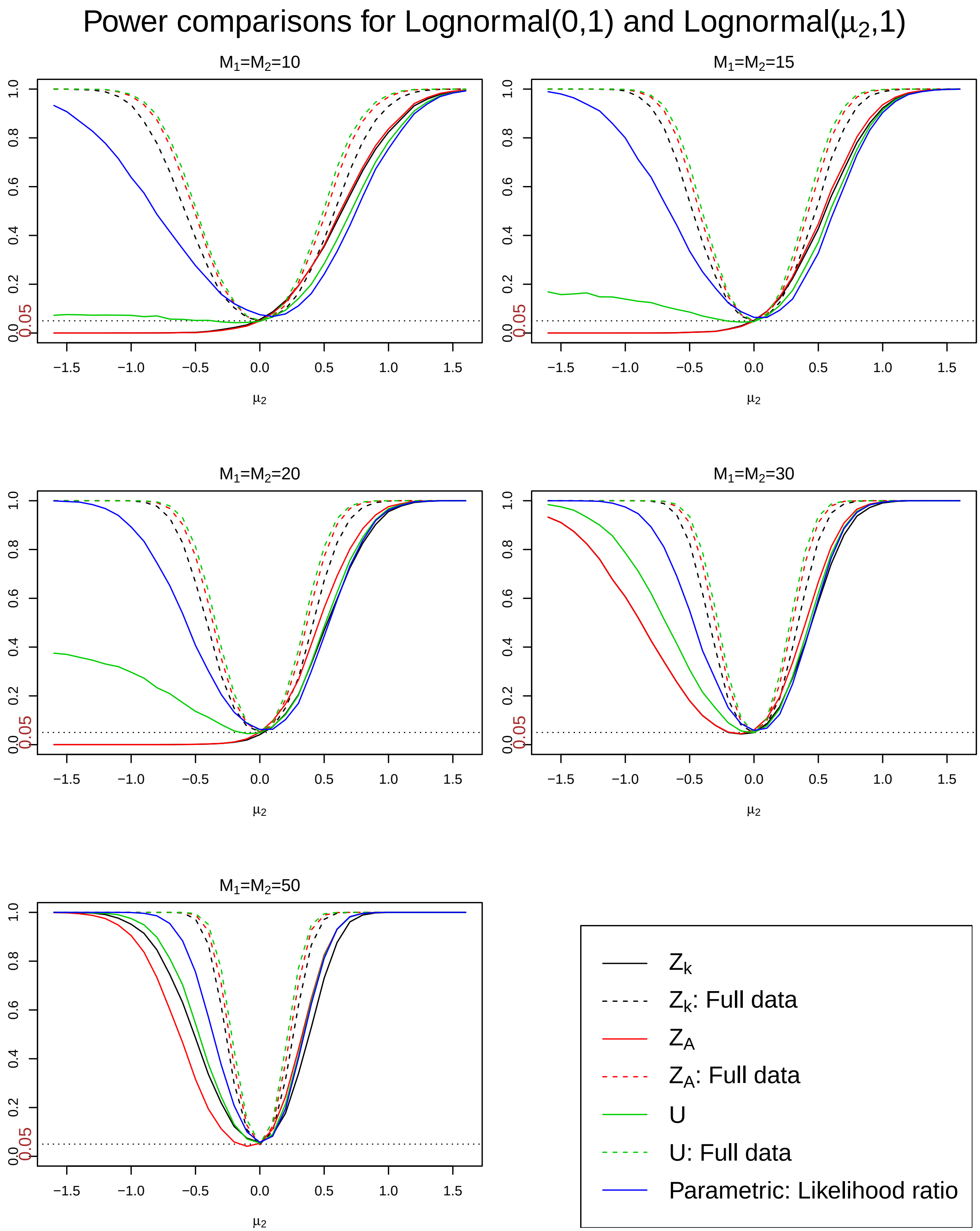

Figure A.3: Simulated power curves of the parametric and nonparametric test for $\mu_1 = 0$ and $\mu_2 = -1.6\ (0.1)\ 1.6$, $\sigma_1 = \sigma_2 = 1$ with $M_1 = M_2$ for lognormal components from systems $\mathrm{s}_1 = (0, 0, 0, 1)$ and $\mathrm{s}_2 = (1, 0, 0)$ (i.e., [**D7**] with system structures [**S1**]).

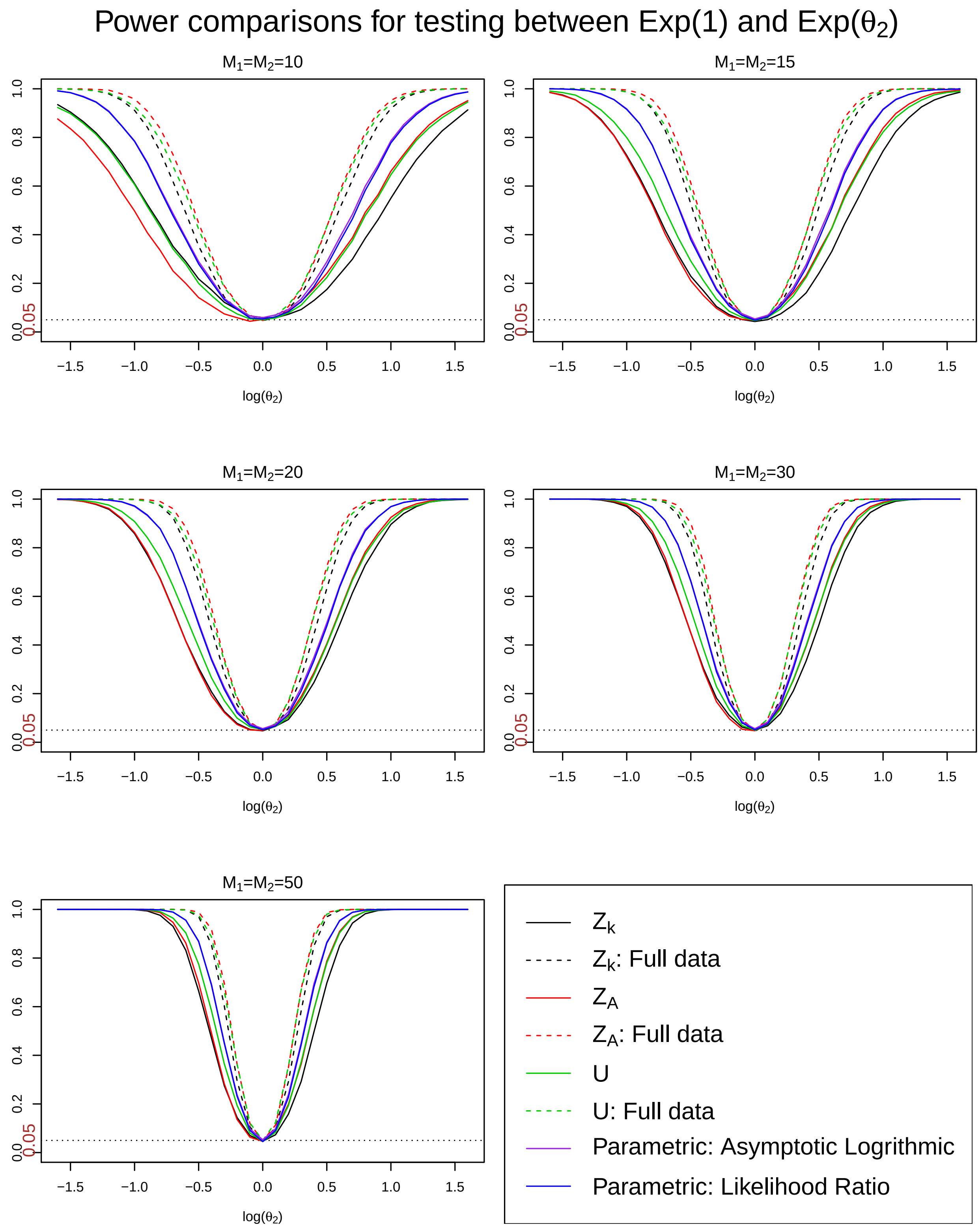

Figure A.4: Simulated power curves of the parametric and nonparametric test for $\ln\theta_1 = 0$ and $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ with $M_1 = M_2$ for exponential components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D1**] with system structures [**S2**]).

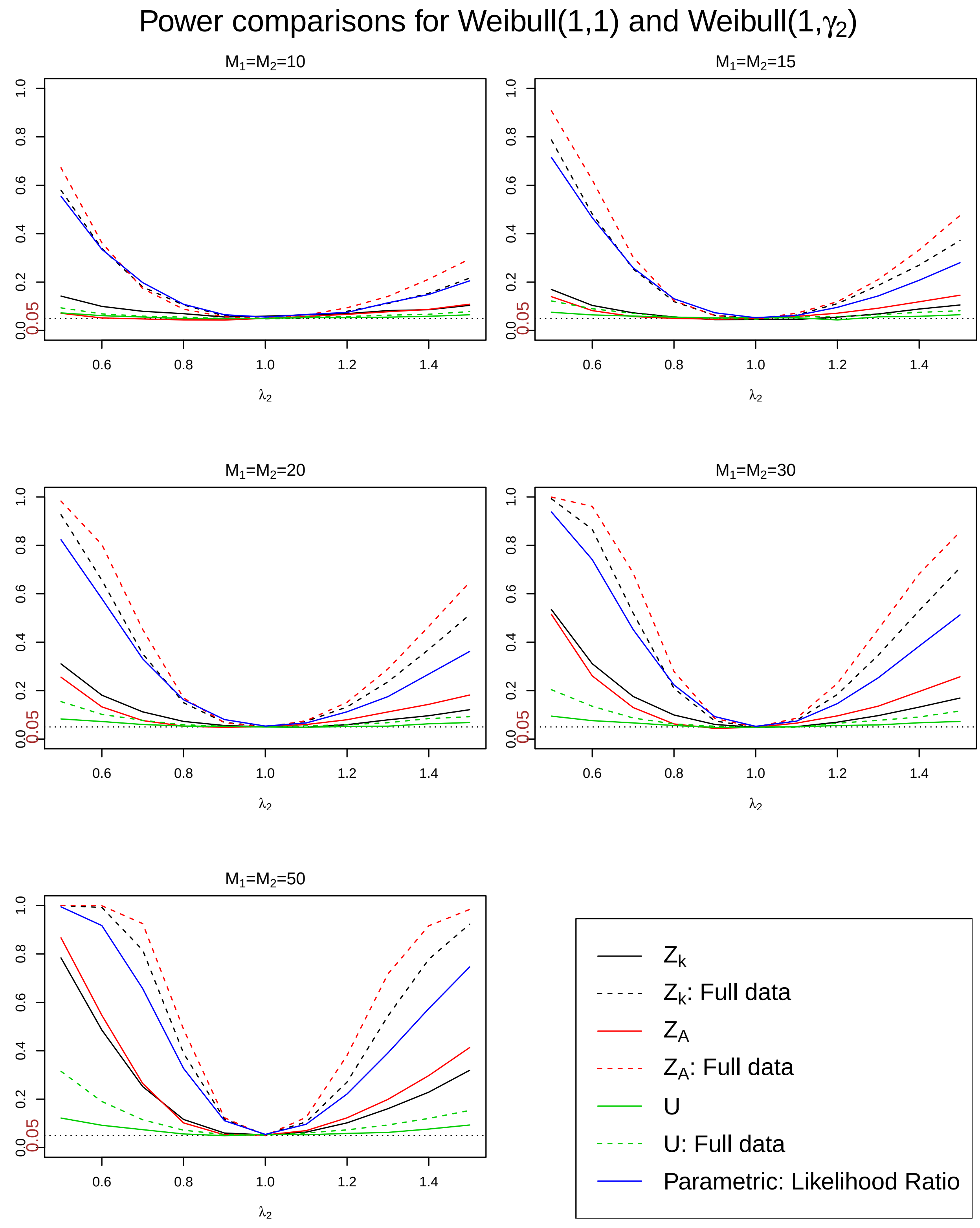

Figure A.5: Simulated power curves of the parametric and nonparametric test for $\gamma_1 = 1$ and $\gamma_2 = 0.5\ (0.1)\ 1.5$, $\lambda_1 = \lambda_2 = 1$ with $M_1 = M_2$ for Weibull components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D2**] with system structures [**S2**]).

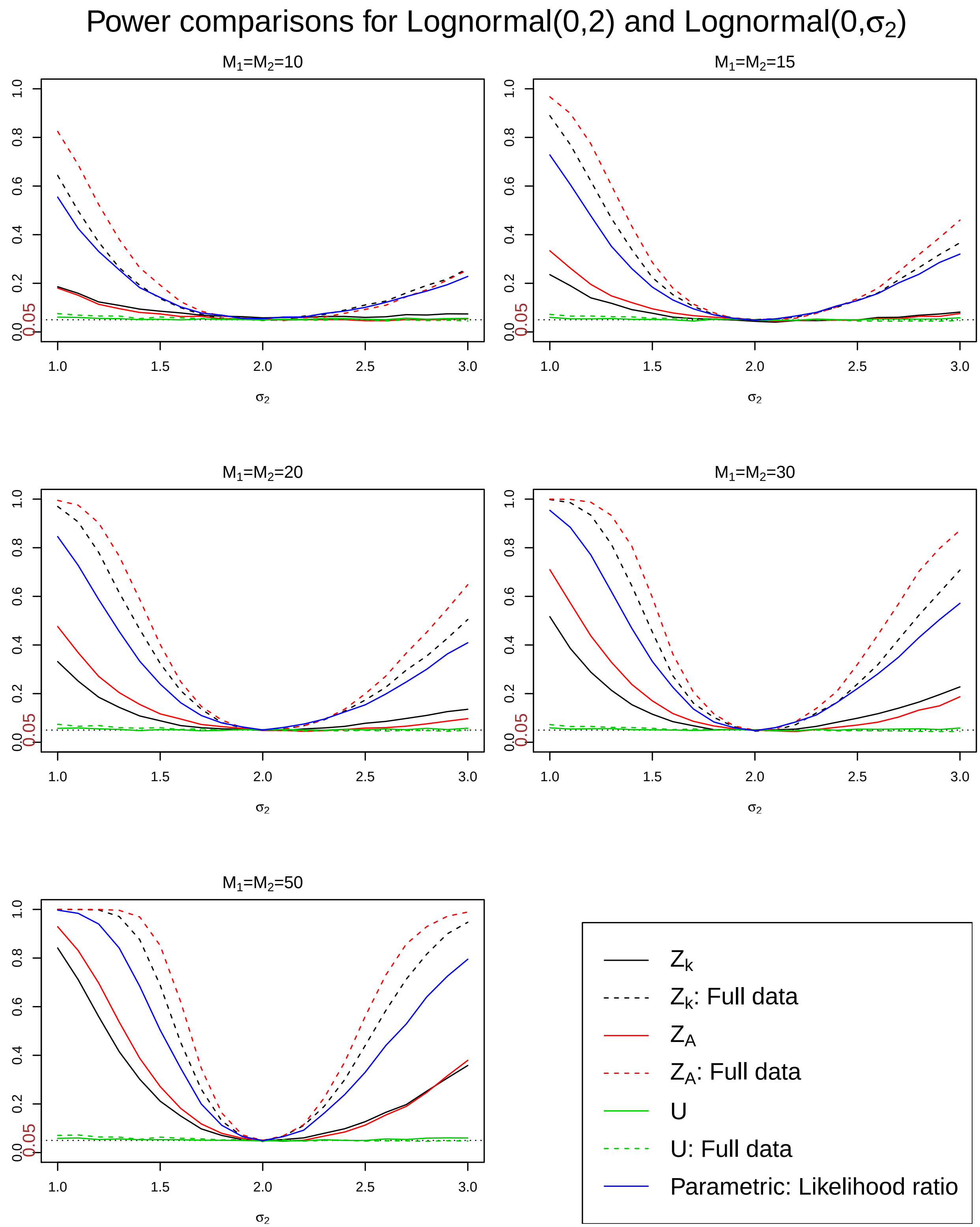

Figure A.6: Simulated power curves of the parametric and nonparametric test for $\mu_1 = \mu_2 = 0$, $\sigma_1 = 2$ and $\sigma_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for lognormal components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., **[D3]** with system structures **[S2]**).

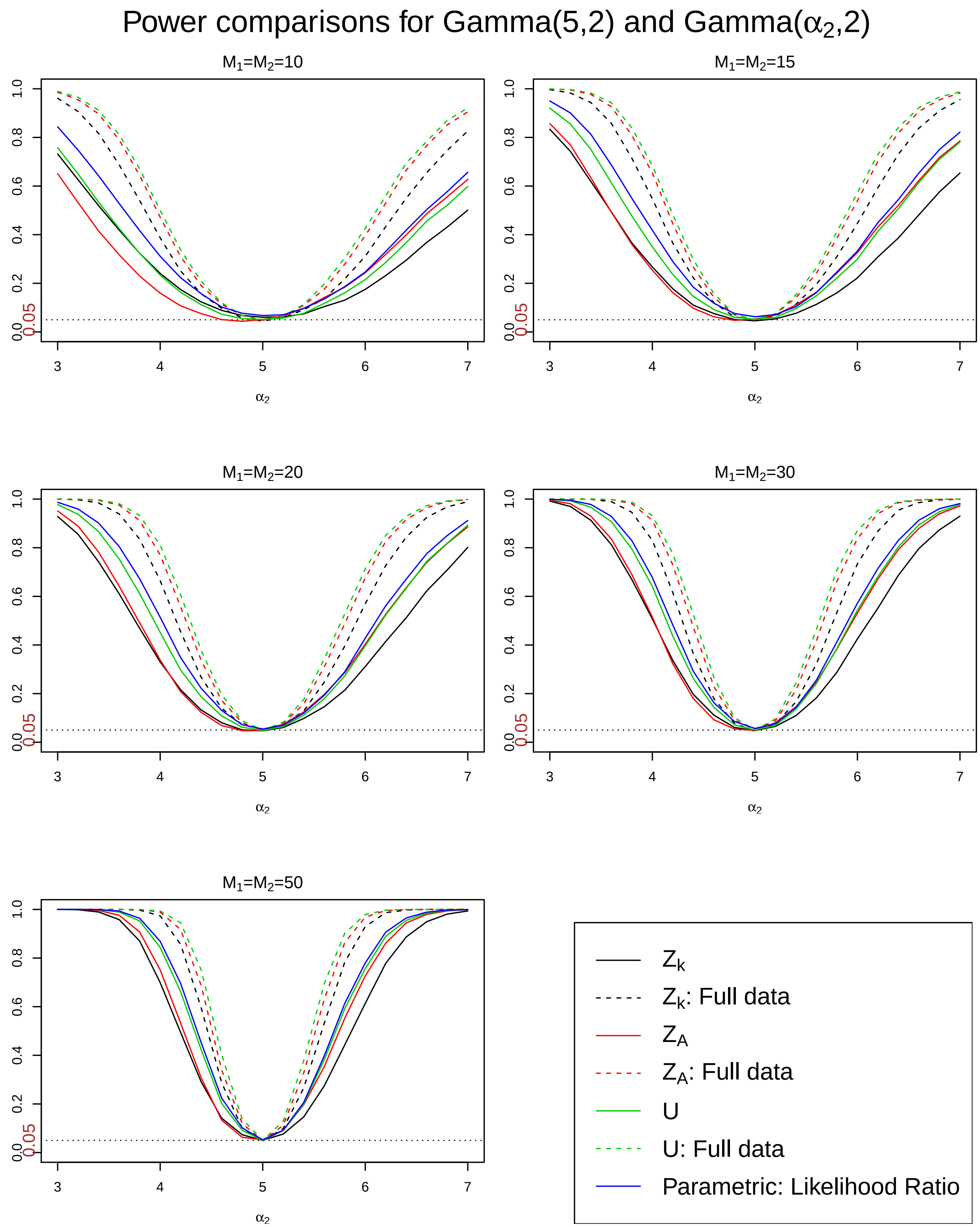

Figure A.7: Simulated power curves of the parametric and nonparametric test for $\alpha_1 = 5$ and $\alpha_2 = 3\ (0.2)\ 7$, $\beta_1 = \beta_2 = 2$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D5**] with system structures [**S2**]).

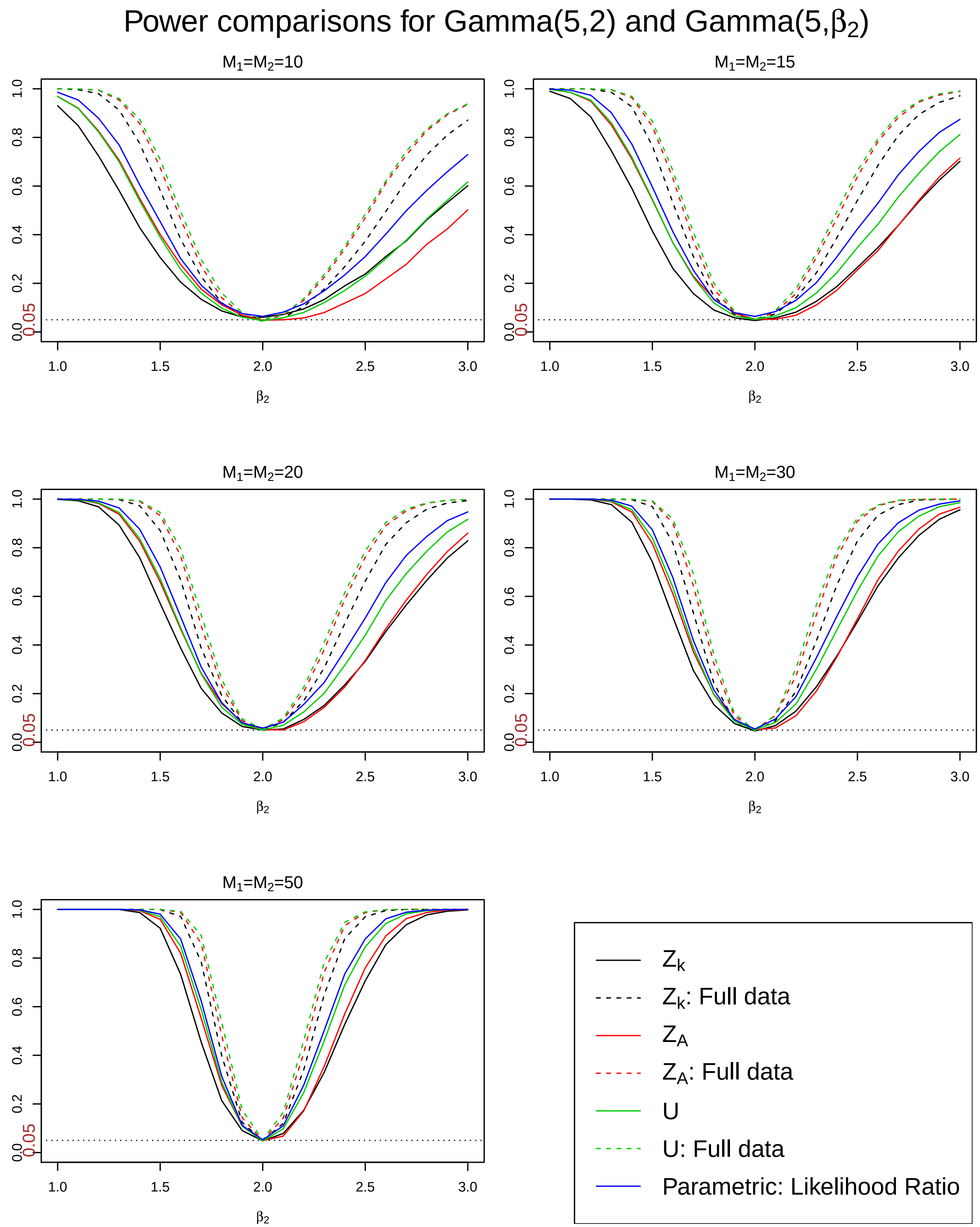

Figure A.8: Simulated power curves of the parametric and nonparametric test for $\alpha_1 = \alpha_2 = 5$, $\beta_1 = 2$ and $\beta_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D6**] with system structures [**S2**]).

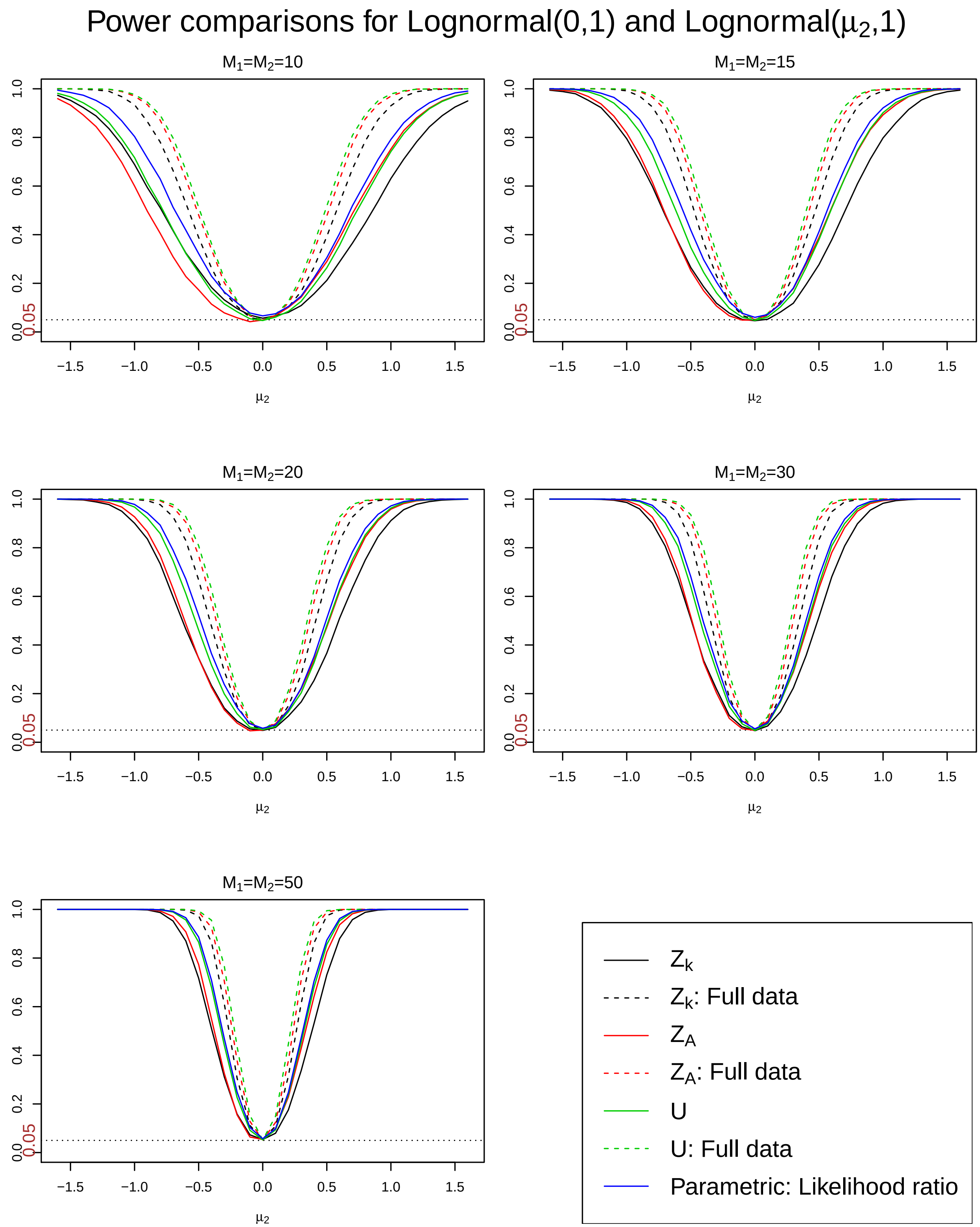

Figure A.9: Simulated power curves of the parametric and nonparametric test for $\mu_1 = 0$ and $\mu_2 = -1.6\ (0.1)\ 1.6$, $\sigma_1 = \sigma_2 = 1$ with $M_1 = M_2$ for lognormal components from systems $\mathrm{s}_1 = (0, 2/3, 1/3)$ and $\mathrm{s}_2 = (1/4, 1/4, 1/2, 0)$ (i.e., [**D7**] with system structures [**S2**]).

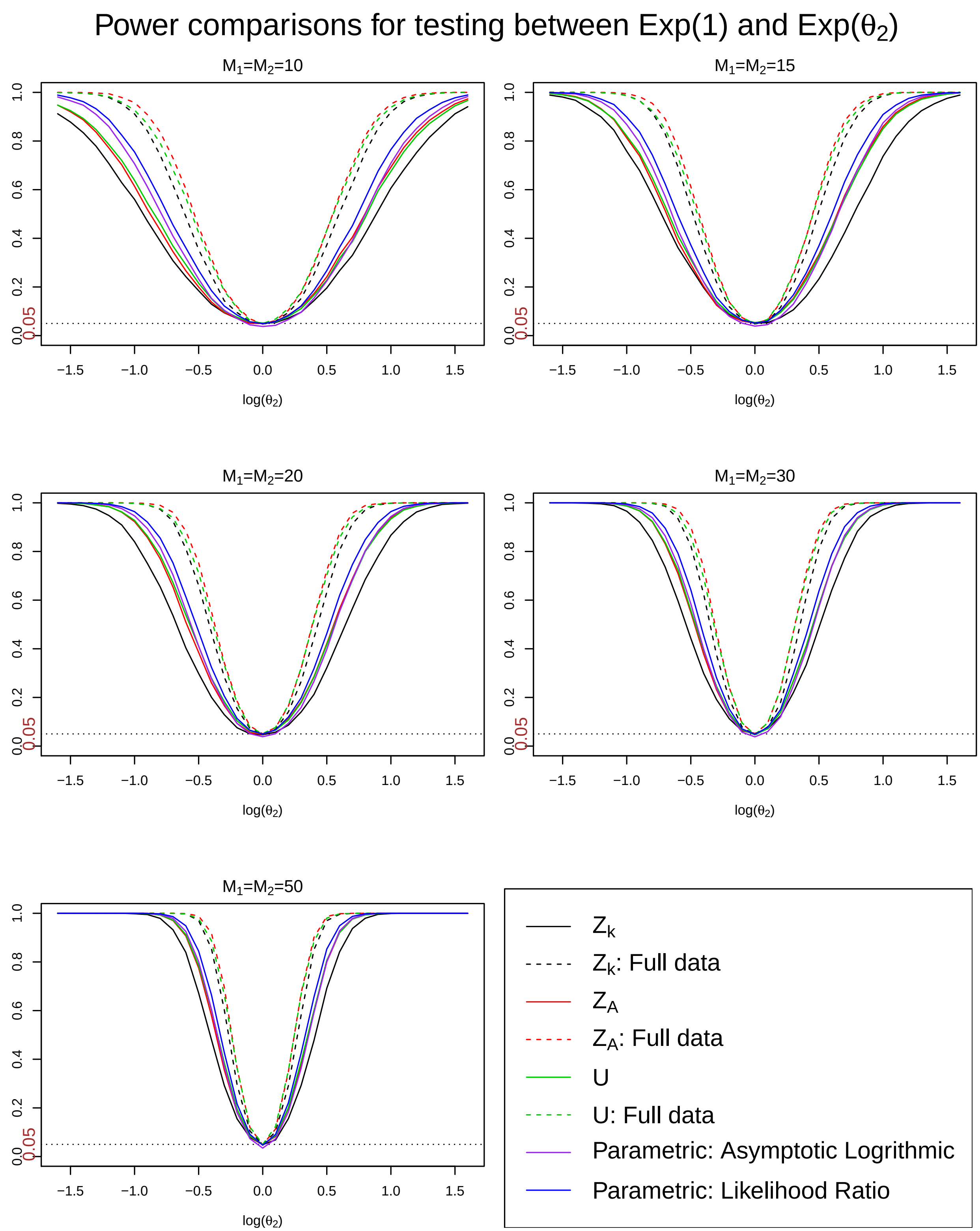

Figure A.10: Simulated power curves of the parametric and nonparametric test for $\ln\theta_1 = 0$ and $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ with $M_1 = M_2$ for exponential components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D1**] with system structures [**S3**]).

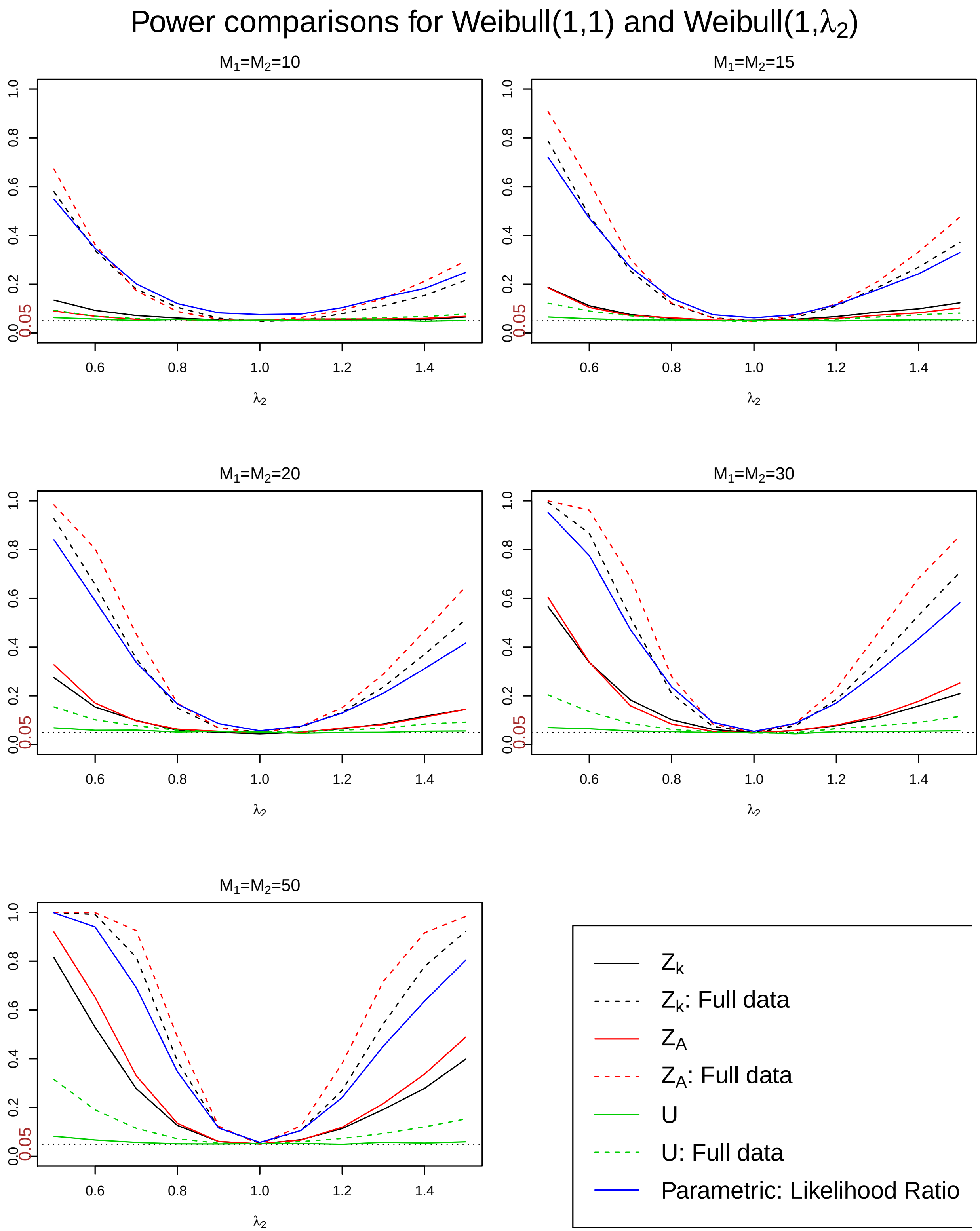

Figure A.11: Simulated power curves of the parametric and nonparametric test for $\gamma_1 = 1$ and $\gamma_2 = 0.5\ (0.1)\ 1.5$, $\lambda_1 = \lambda_2 = 1$ with $M_1 = M_2$ for Weibull components from systems $\mathrm{s}_1 = (0, 2/3, 1/3)$ and $\mathrm{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D2**] with system structures [**S3**]).

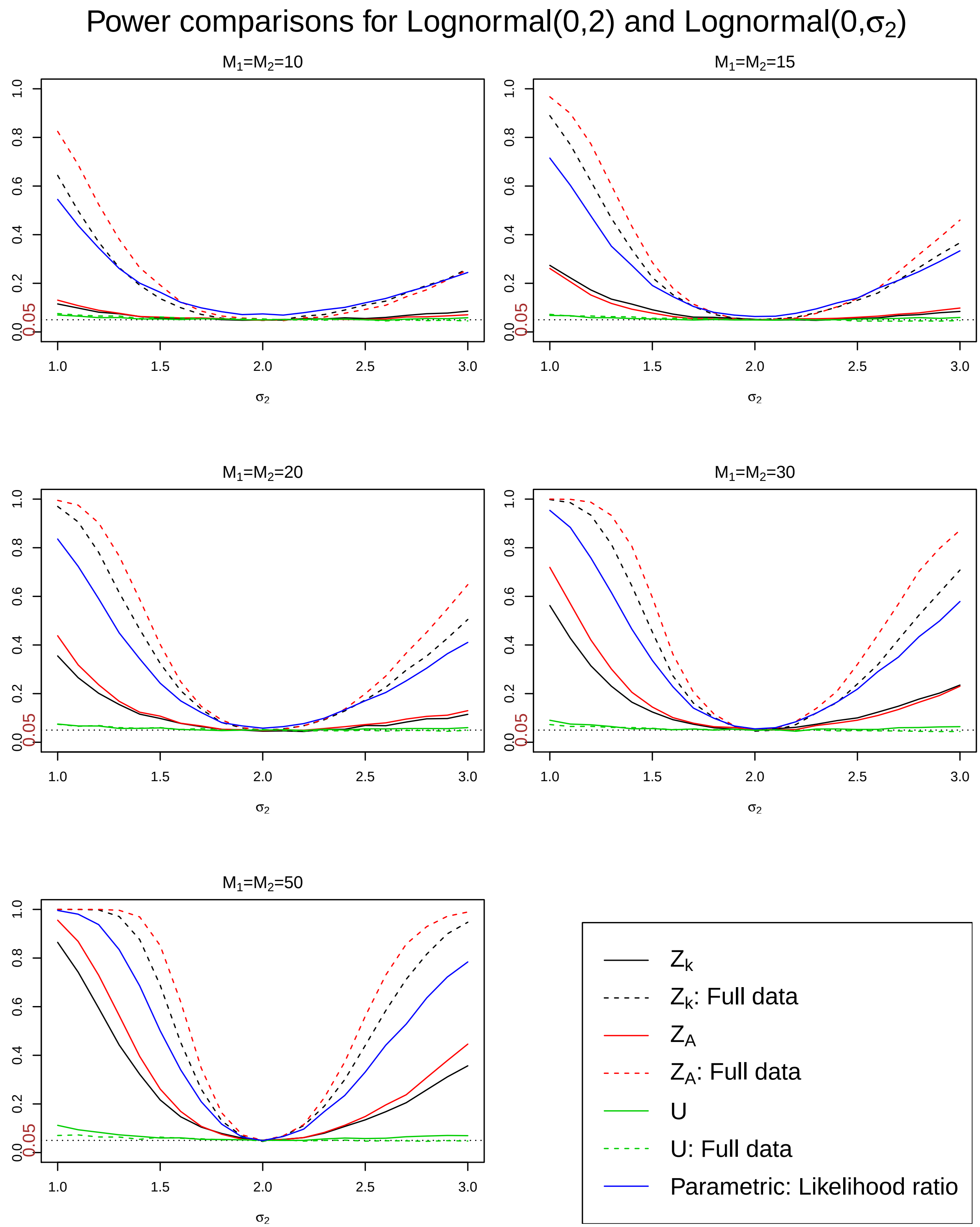

Figure A.12: Simulated power curves of the parametric and nonparametric test for $\mu_1 = \mu_2 = 0$, $\sigma_1 = 2$ and $\sigma_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for lognormal components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D3**] with system structures [**S3**]).

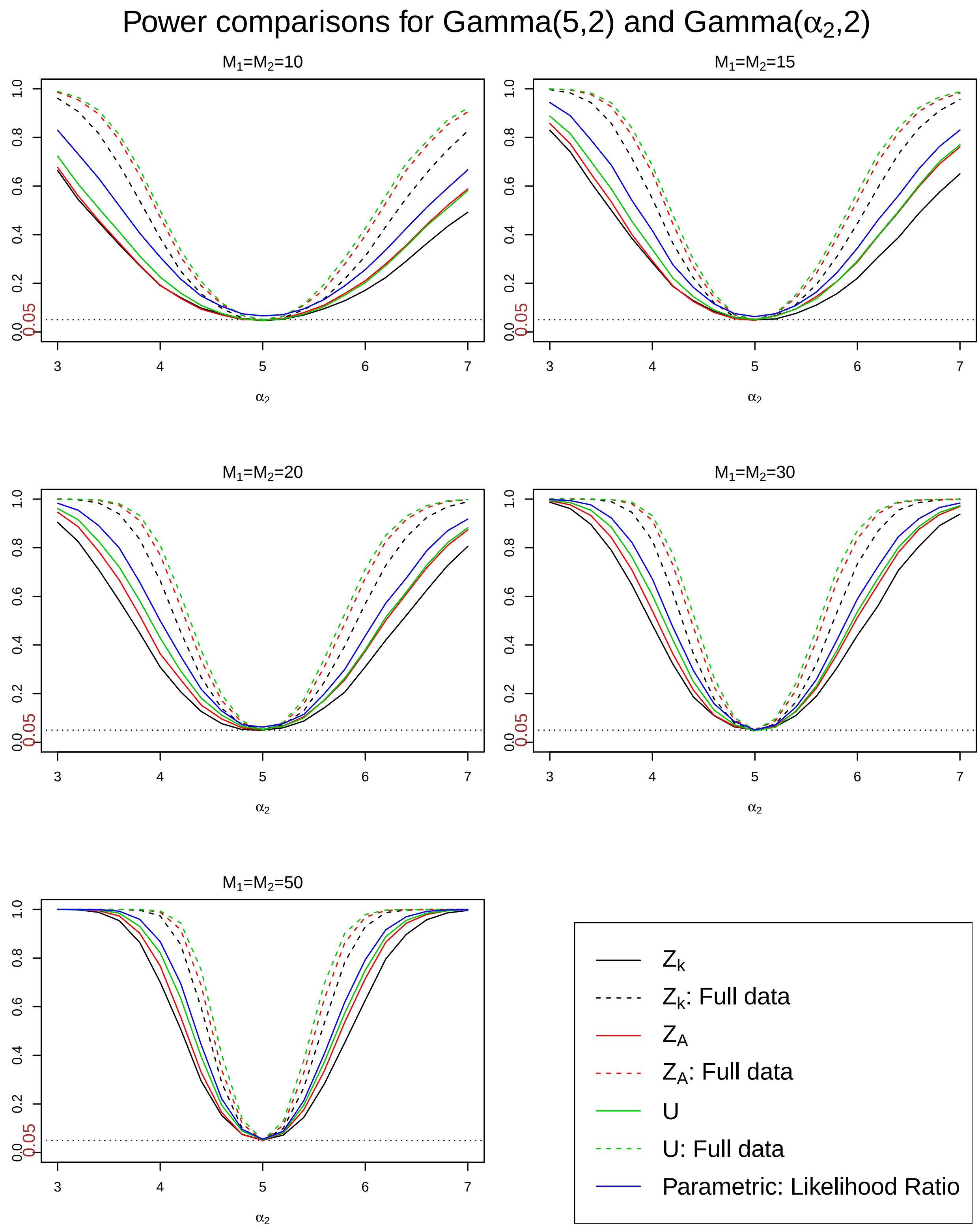

Figure A.13: Simulated power curves of the parametric and nonparametric test for $\alpha_1 = 5$ and $\alpha_2 = 3\ (0.2)\ 7$, $\beta_1 = \beta_2 = 2$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D5**] with system structures [**S3**]).

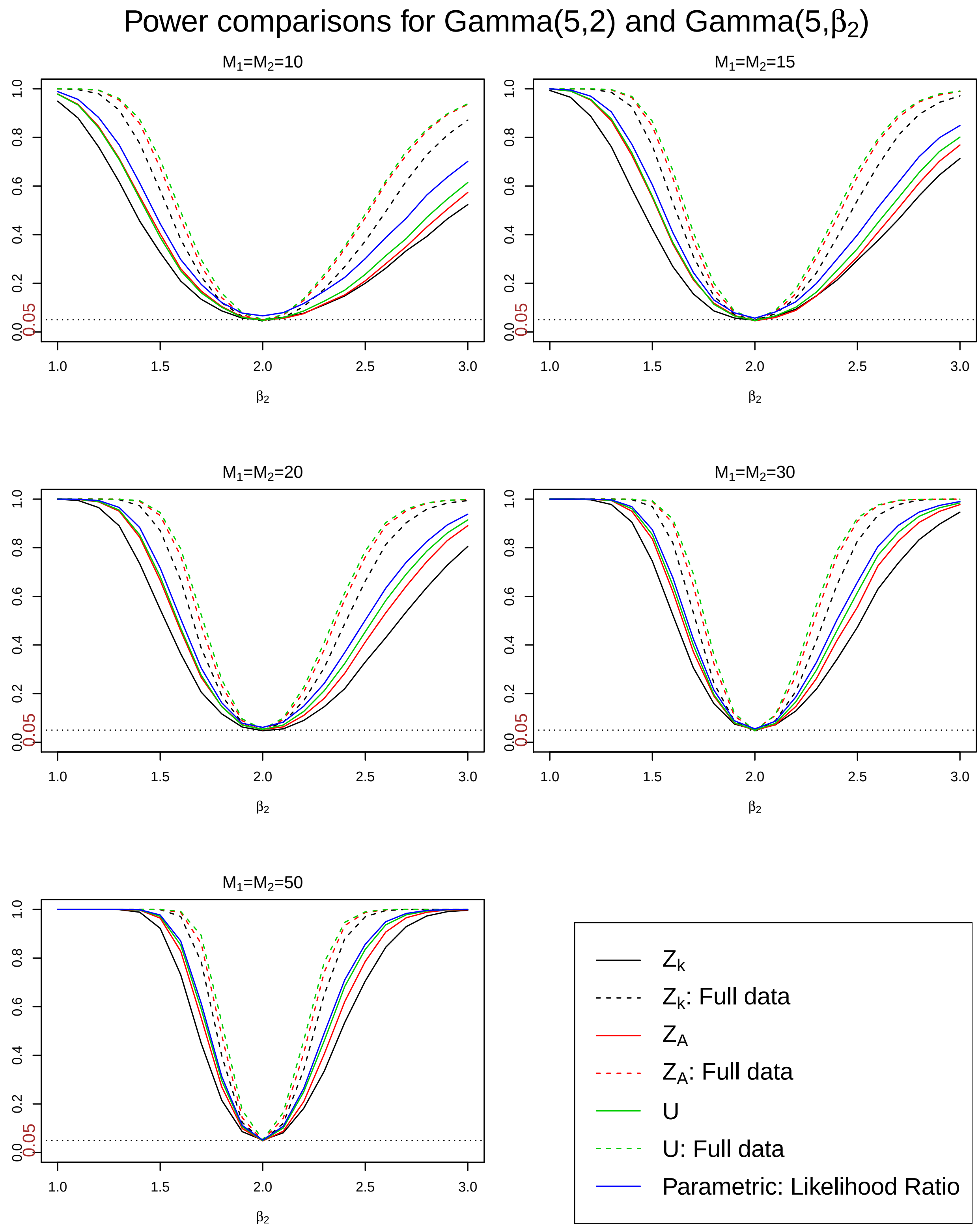

Figure A.14: Simulated power curves of the parametric and nonparametric test for $\alpha_1 = \alpha_2 = 5$ and $\beta_1 = 2, \beta_2 = 1\ (0.1)\ 3$ with $M_1 = M_2$ for gamma components from systems $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D6**] with system structures [**S3**]).

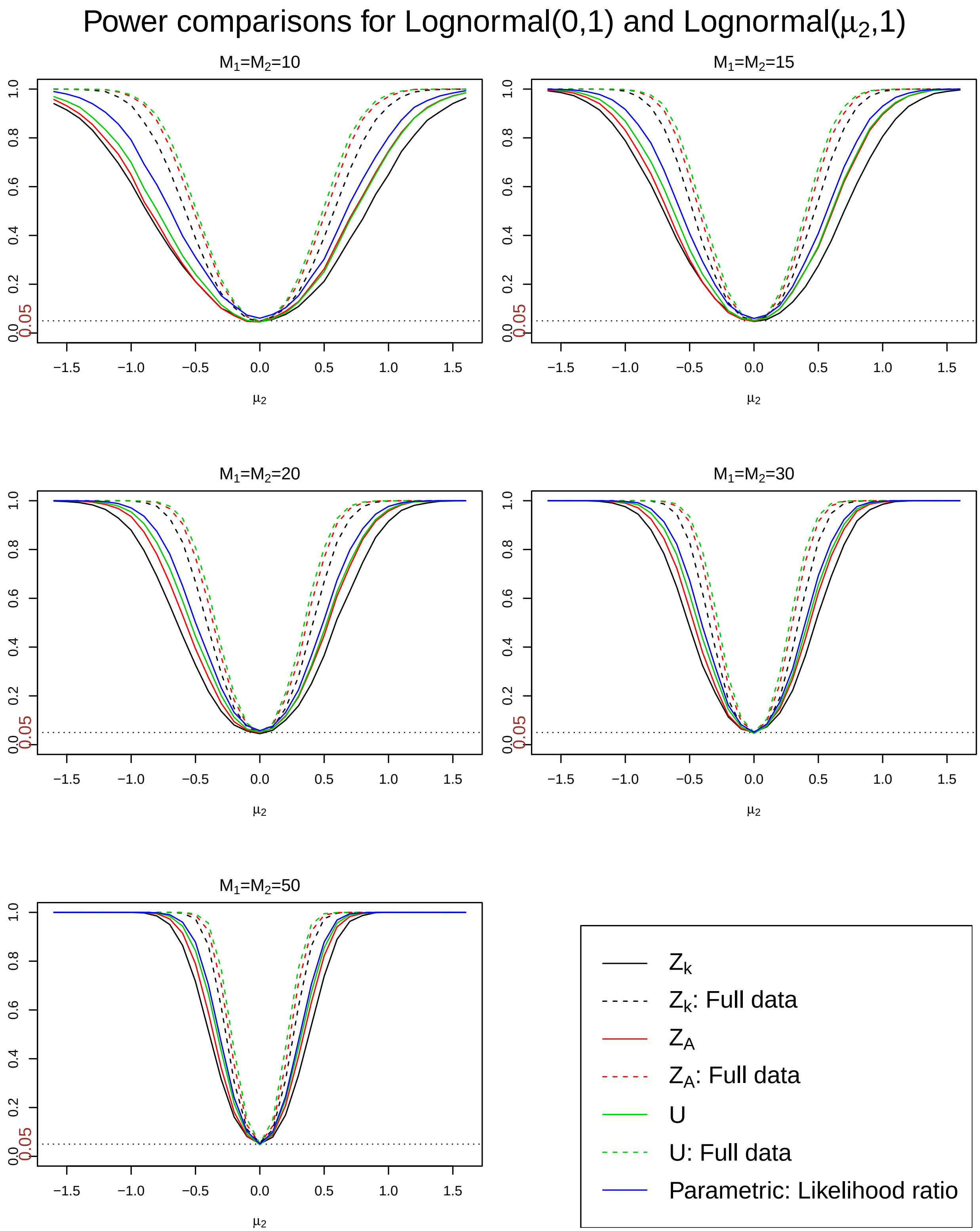

Figure A.15: Simulated power curves of the parametric and nonparametric test for $\mu_1 = 0$ and $\mu_2 = -1.6\ (0.1)\ 1.6$, $\sigma_1 = \sigma_2 = 1$ with $M_1 = M_2$ for lognormal components from systems $\mathrm{s}_1 = (0, 2/3, 1/3)$ and $\mathrm{s}_2 = (0, 1/2, 1/4, 1/4)$ (i.e., [**D7**] with system structures [**S3**]).

APPENDIX B

APPENDIX of CHAPTER 3

Appendix B: Simulated Power Curves for Different Settings with Sample Sizes $M_1 = M_2 = 10$ and 30 for Type-II Censored Lifetime Data

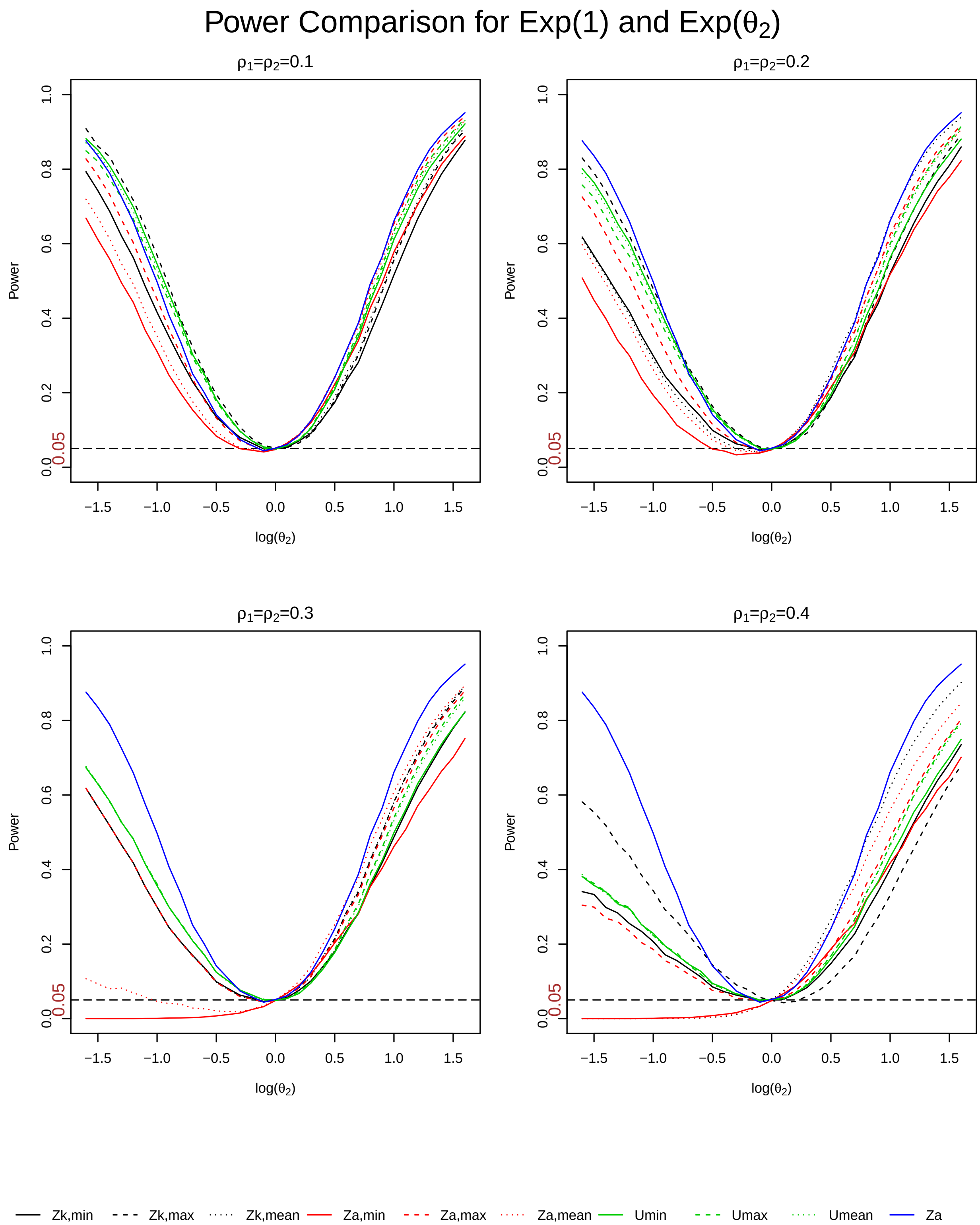

Figure B.1: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying component lifetime distributions are $Exp(1)$ for System 1 and $Exp(\theta_2)$ for System 2, where $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S1**]).

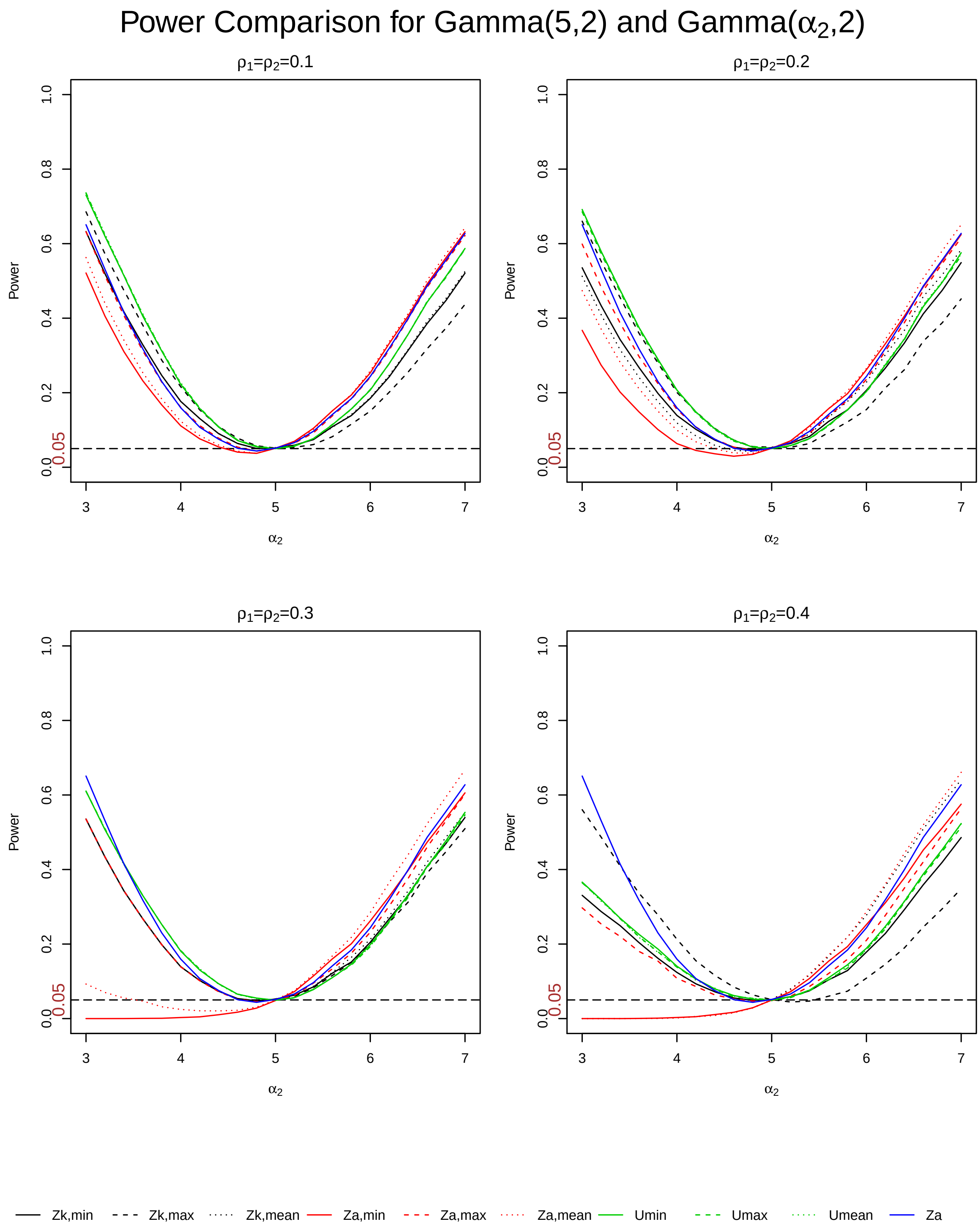

Figure B.2: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying component lifetime distributions are $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where $\alpha_2 = 3\ (0.2)\ 7$ (i.e., [**D2**] with system structures [**S1**]).

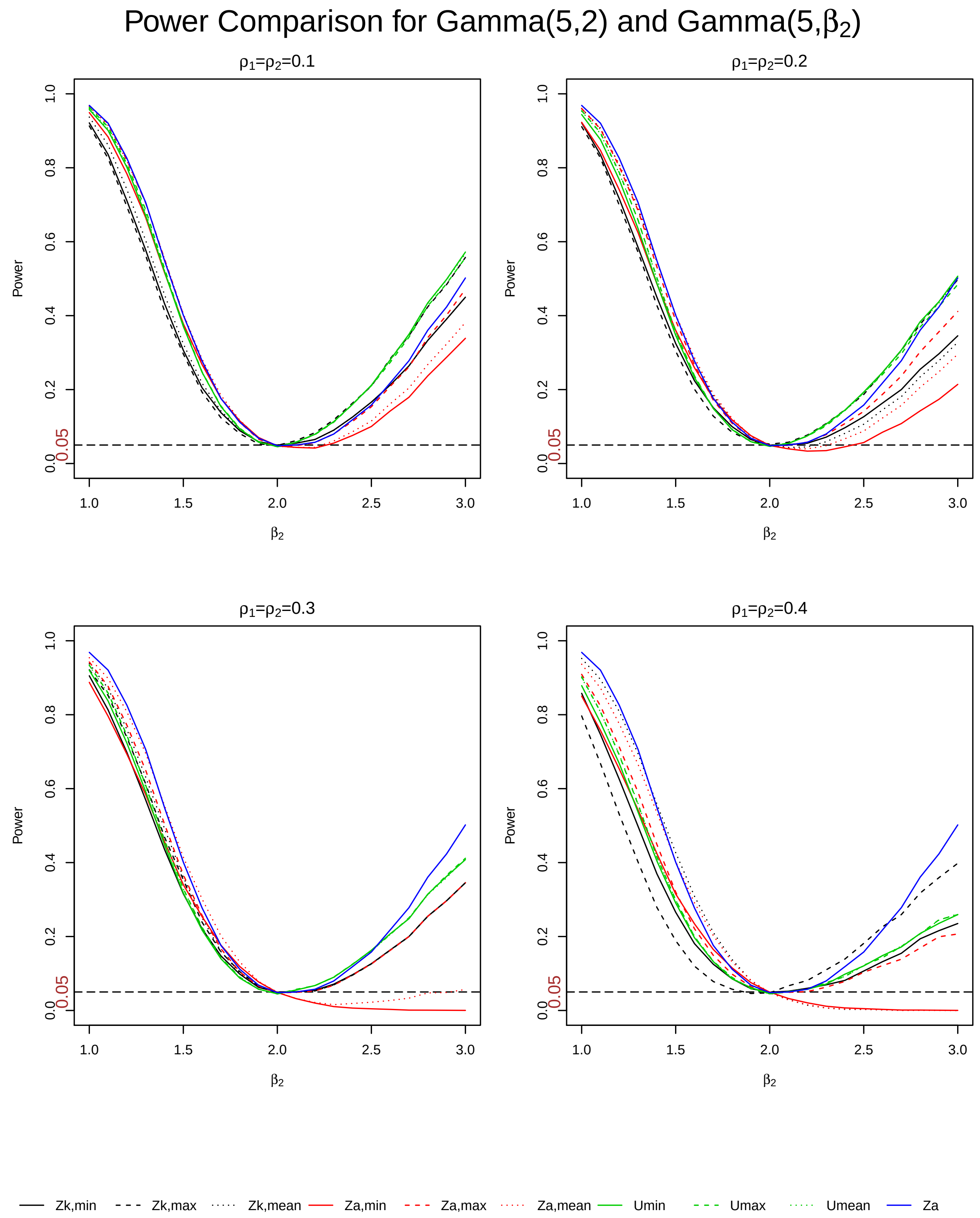

Figure B.3: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying component lifetime distributions are $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2, where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S1**]).

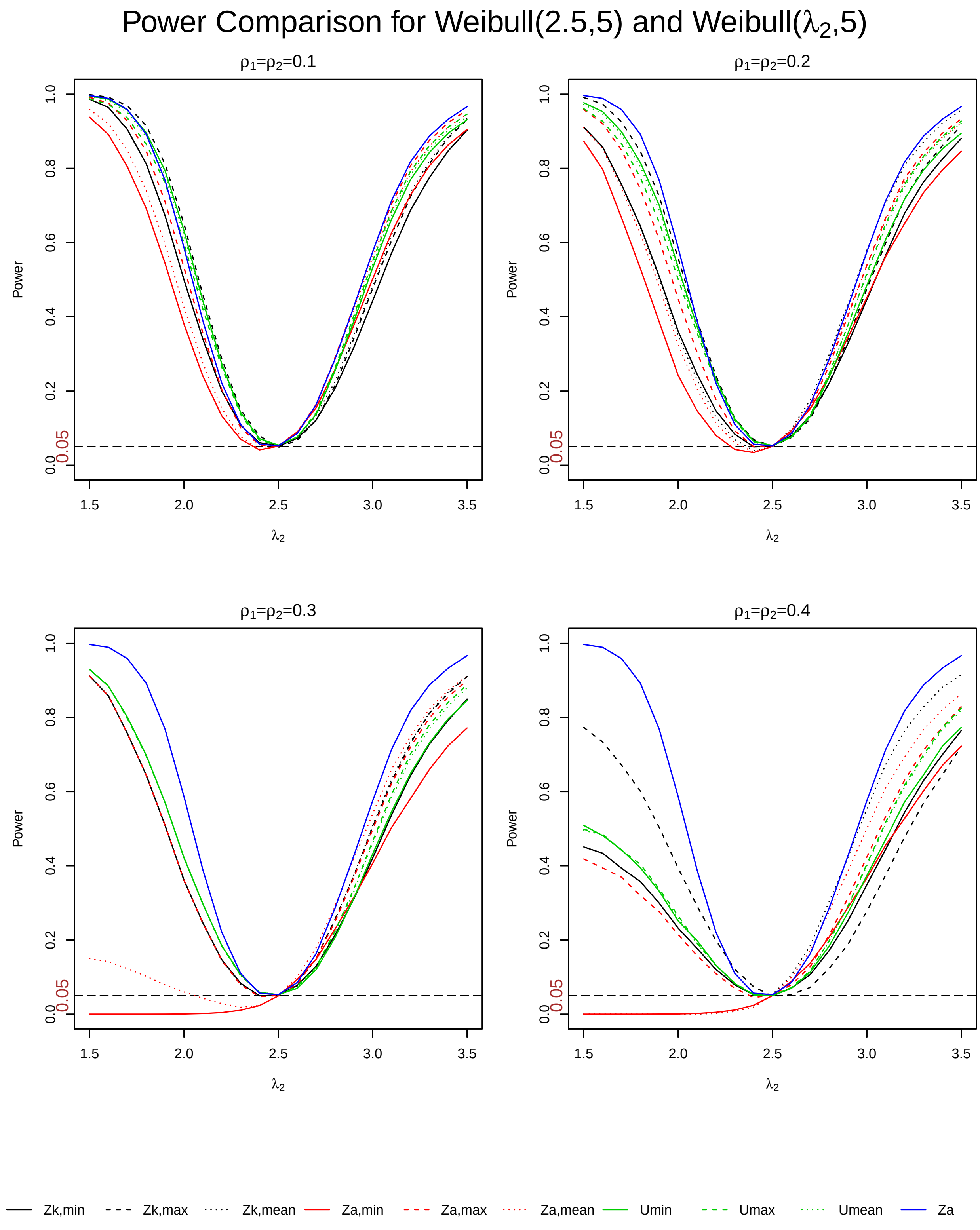

Figure B.4: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2, where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D4**] with system structures [**S1**]).

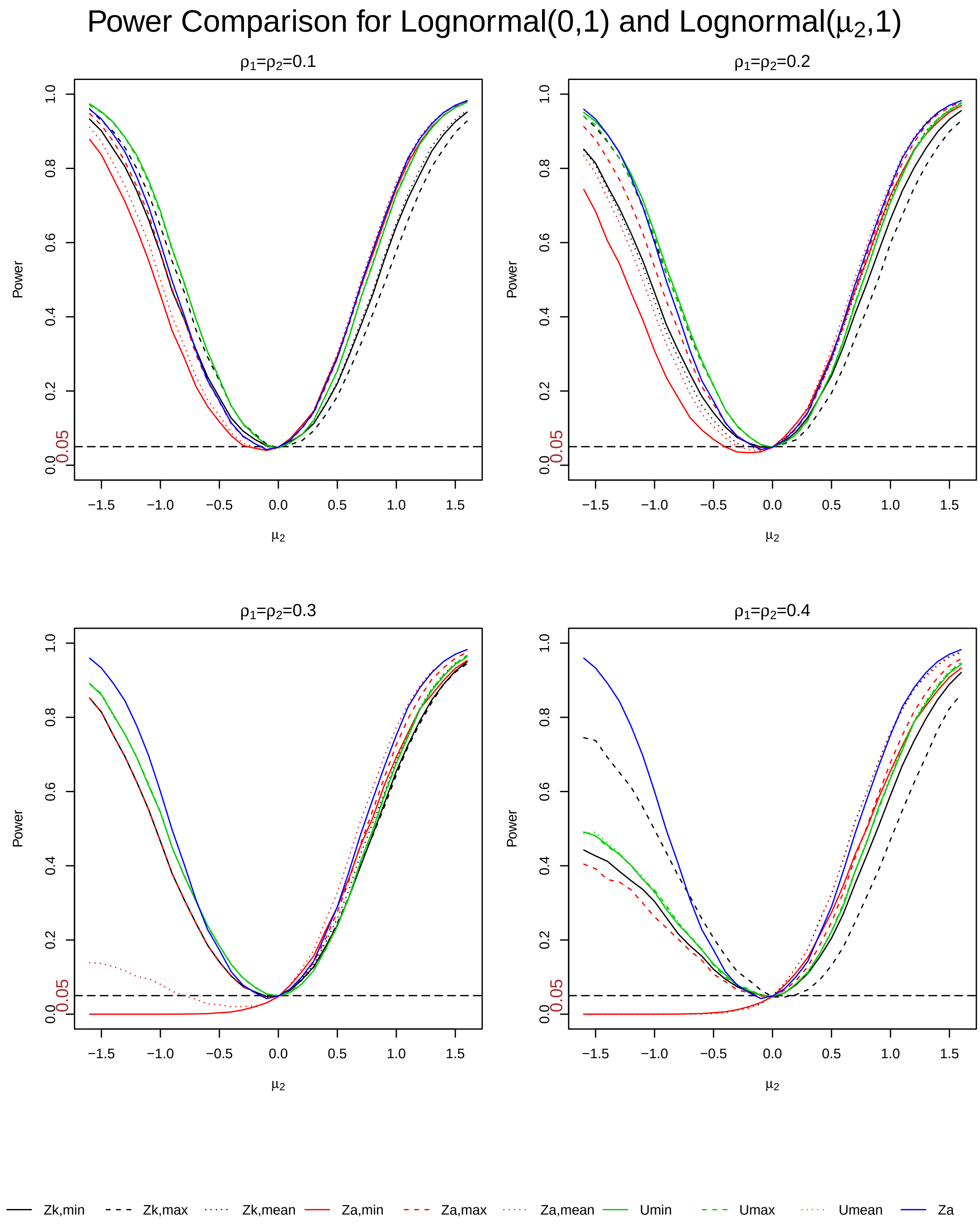

Figure B.5: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions are $Lognormal(0, 1)$ for System 1 and $Lognormal(\mu_2, 1)$ for System 2, where $\mu_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D5**] with system structures [**S1**]).

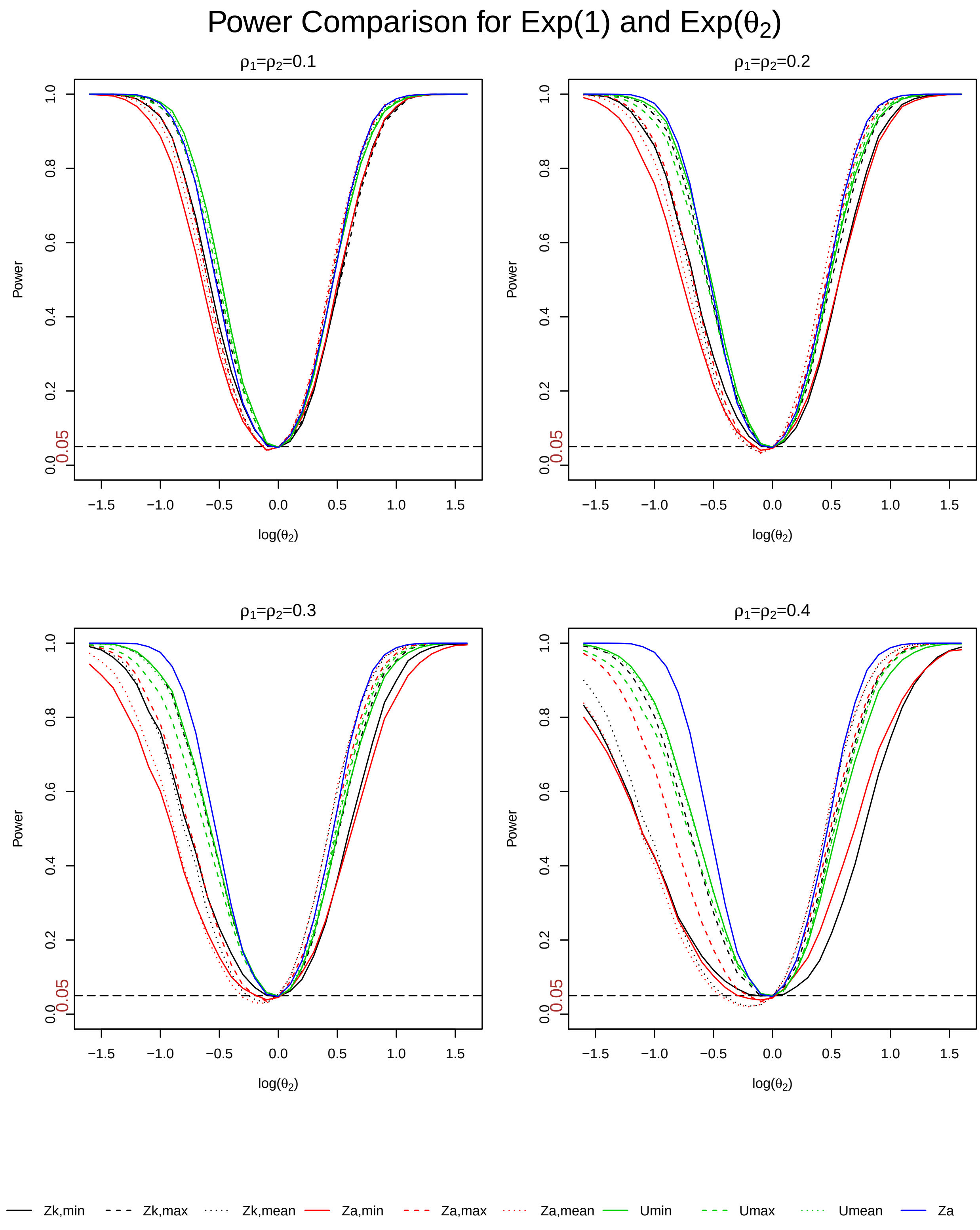

Figure B.6: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Exp(1)$ for System 1 and $Exp(\theta_2)$ for System 2, where $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S1**]).

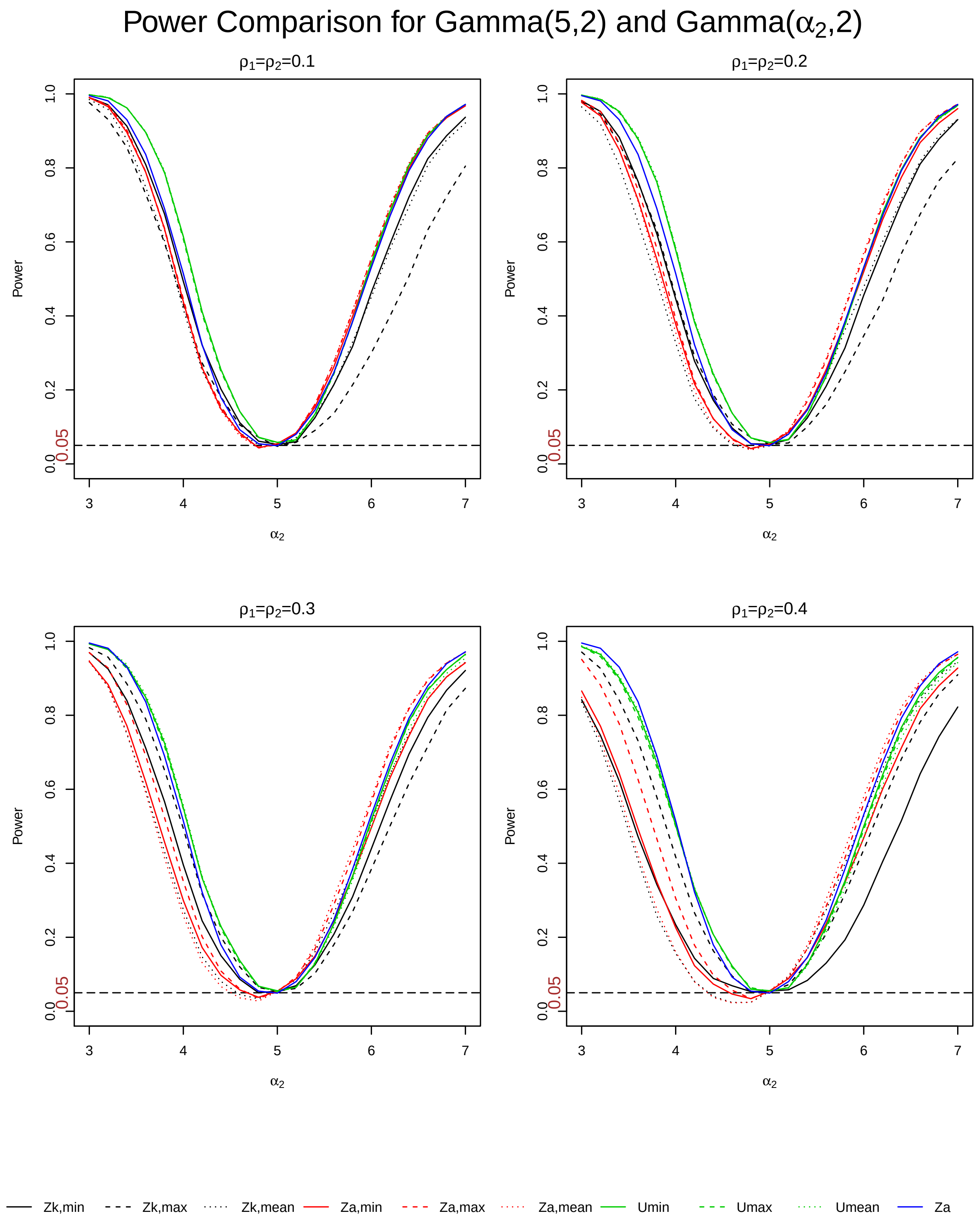

Figure B.7: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where $\alpha_2 = 3\ (0.2)\ 7$ (i.e., [**D2**] with system structures [**S1**]).

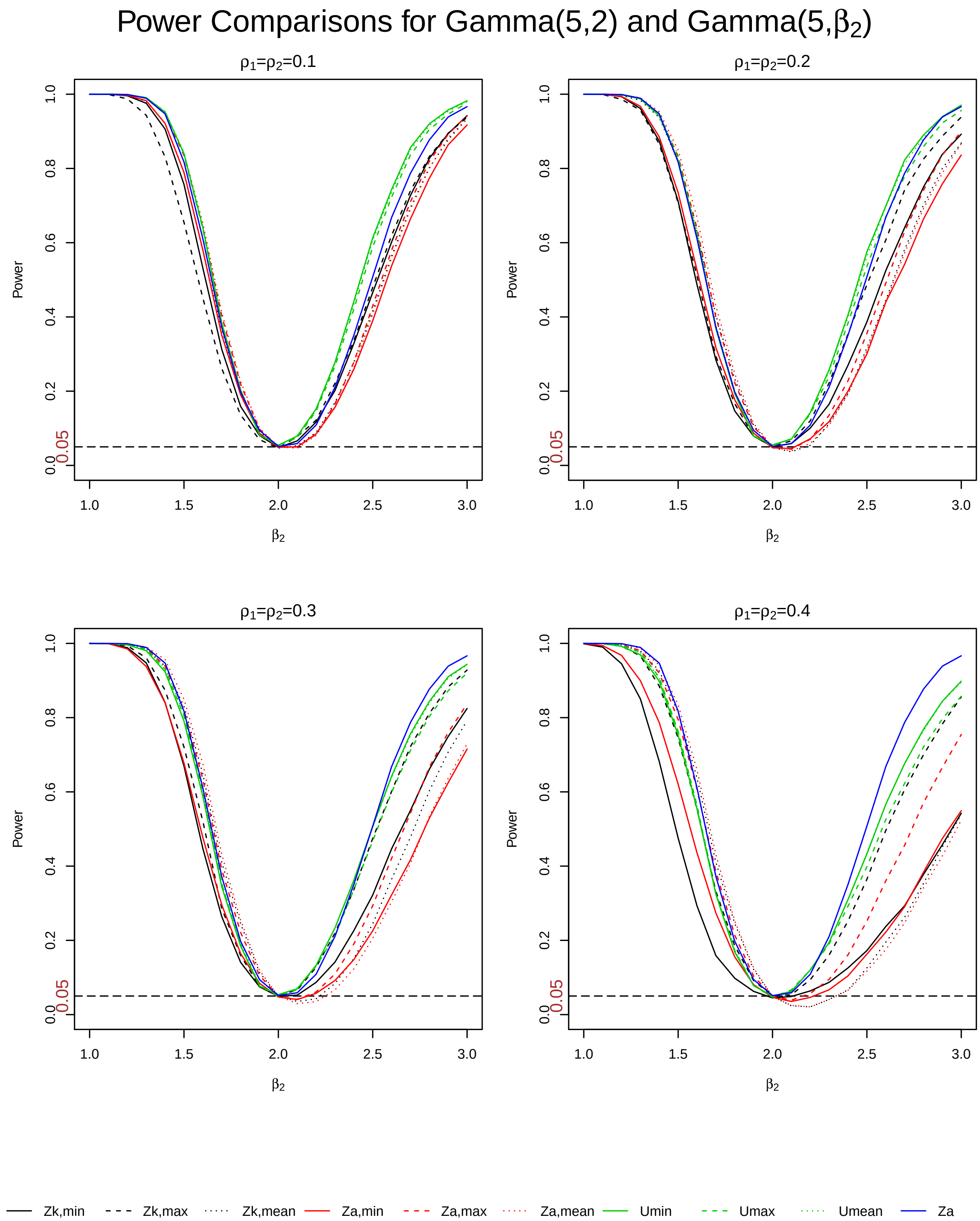

Figure B.8: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2, where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S1**]).

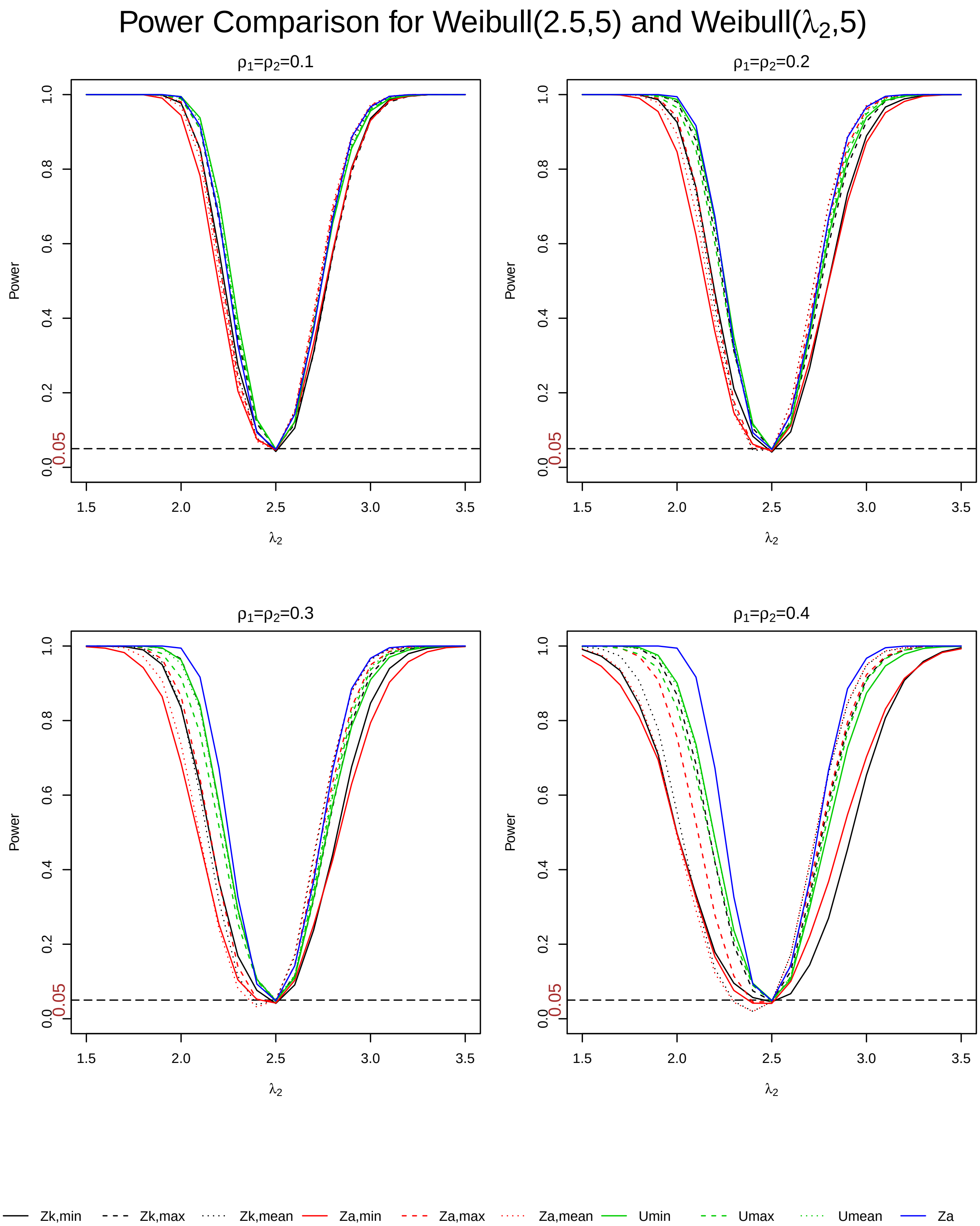

Figure B.9: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2, where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D4**] with system structures [**S1**]).

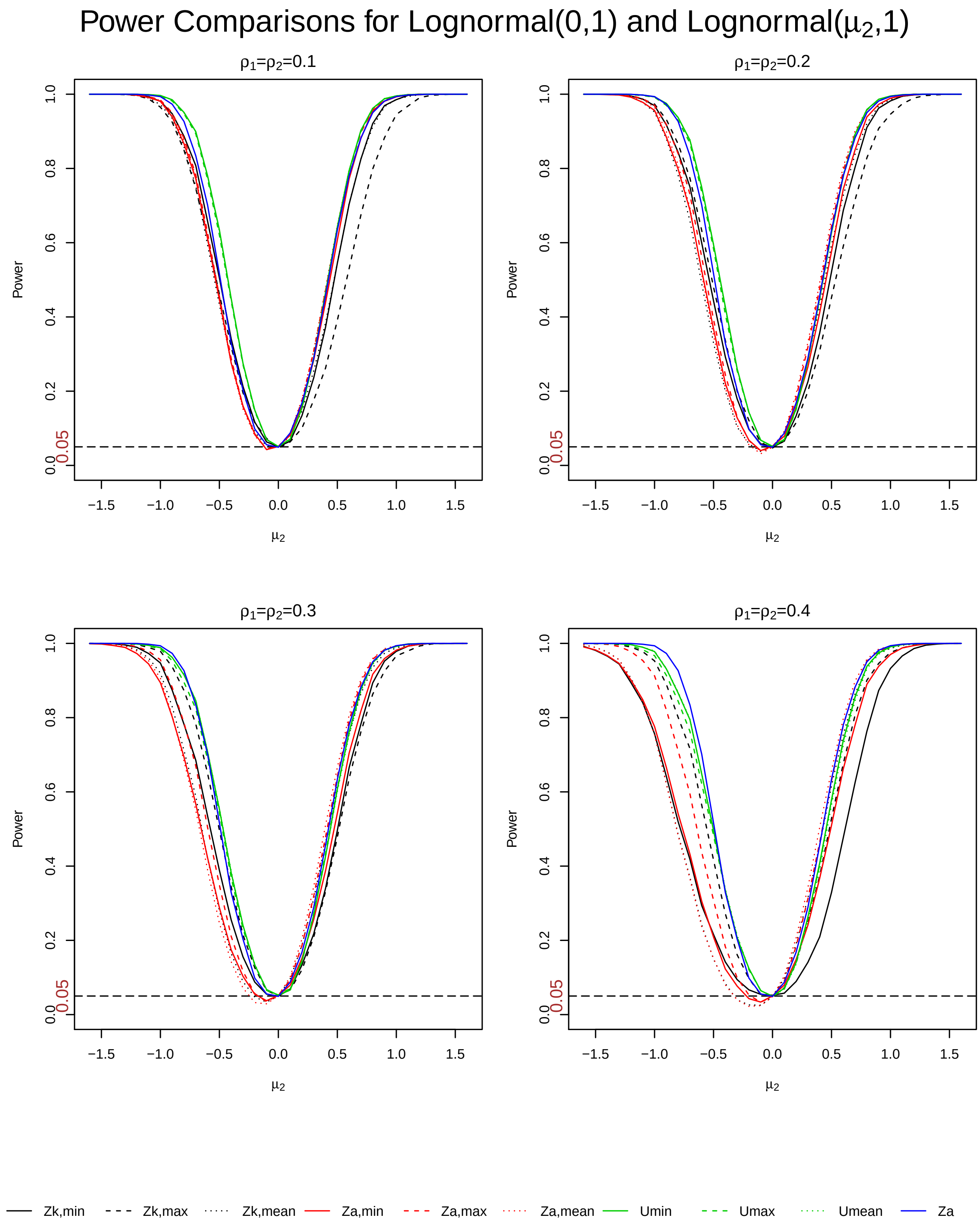

Figure B.10: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (1/4, 1/4, 1/2, 0)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Lognormal(0, 1)$ for System 1 and $Lognormal(\mu_2, 1)$ for System 2, where $\mu_2 = -1.6\ (0.1)\ 1.6$ (i.e., **[D5]** with system structures **[S1]**).

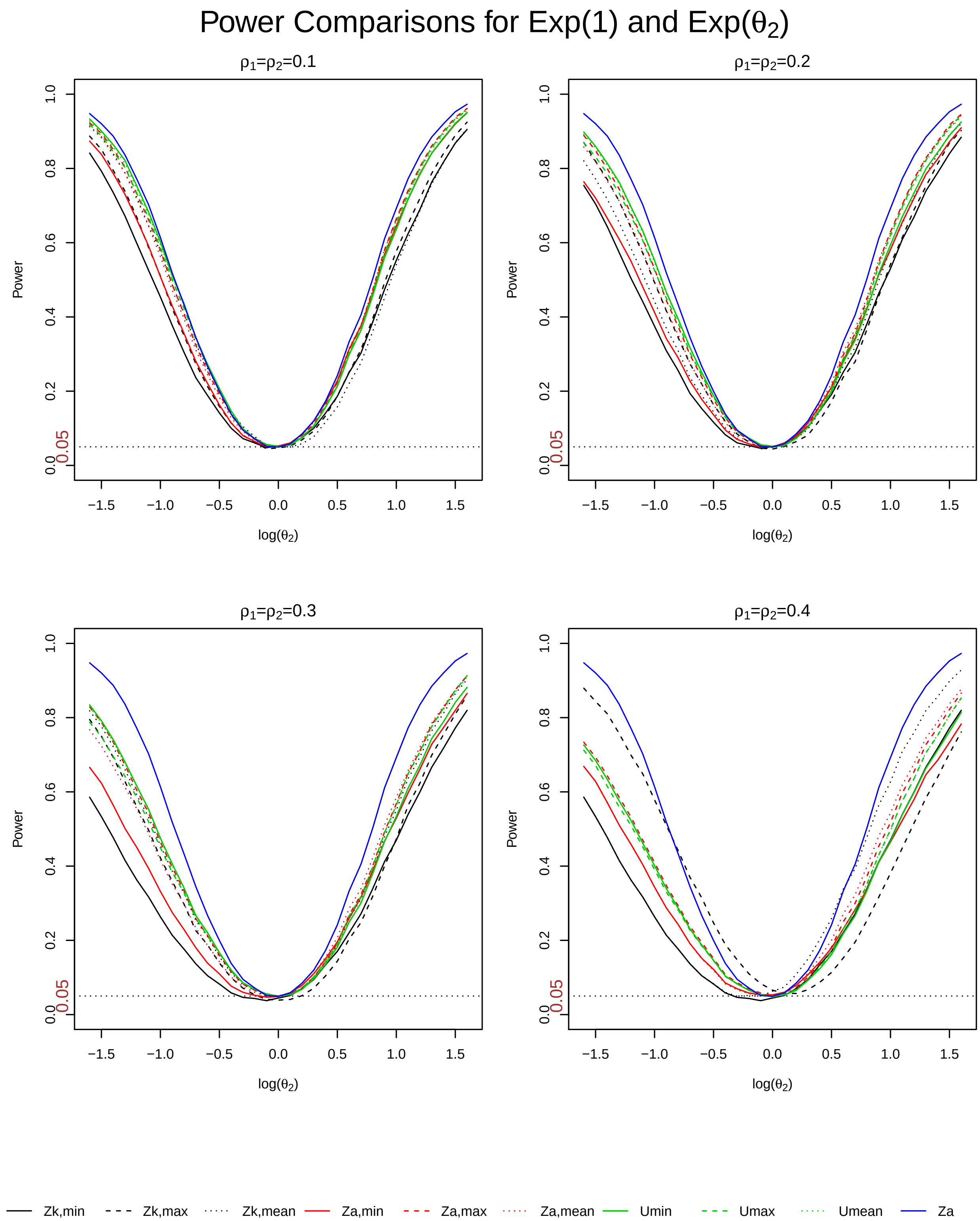

Figure B.11: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Exp(1)$ for System 1 and $Exp(\theta_2)$ for System 2, where $\ln\theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S2**]).

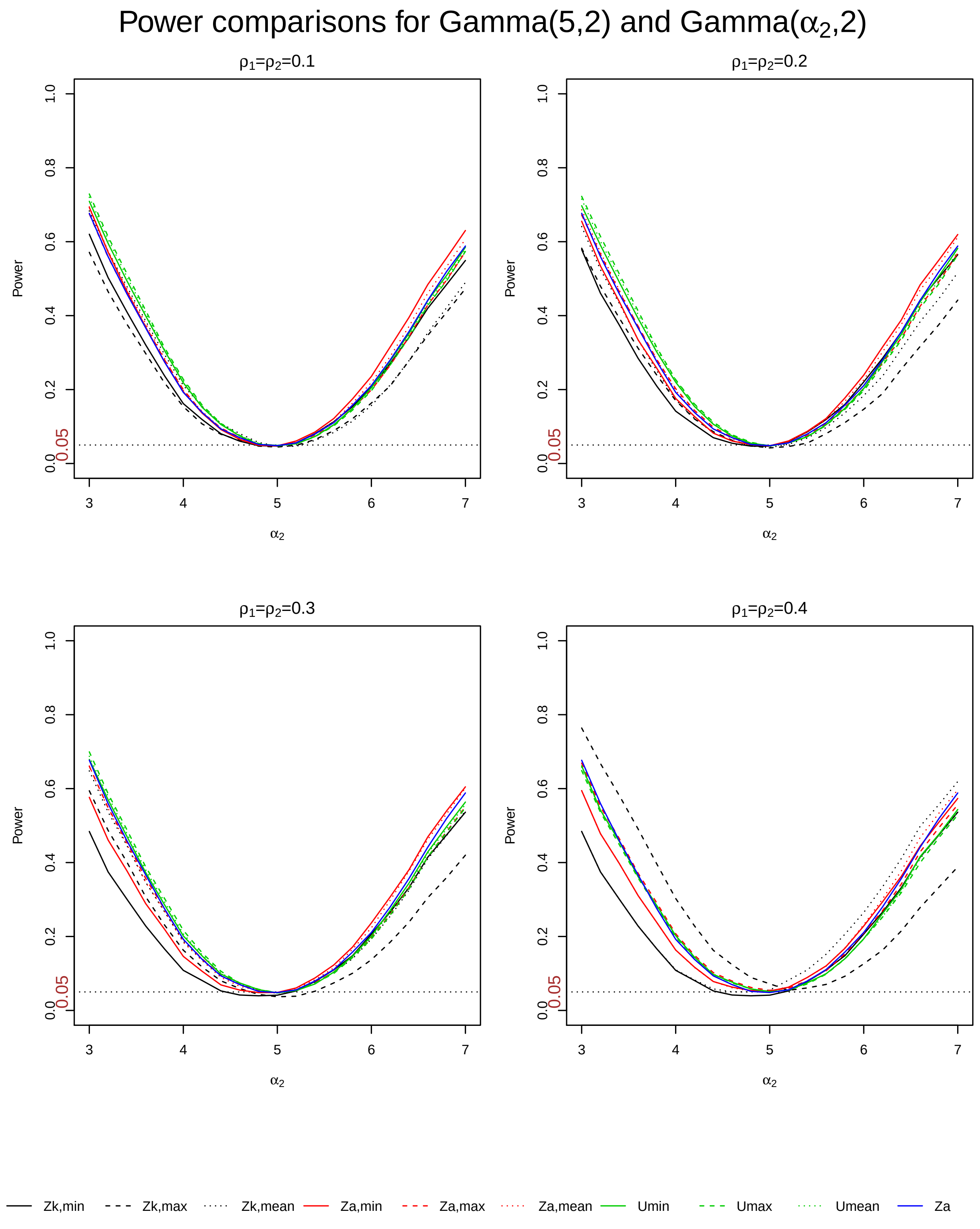

Figure B.12: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where $\alpha_2 = 3\ (0.2)\ 7$ (i.e., [**D2**] with system structures [**S2**]).

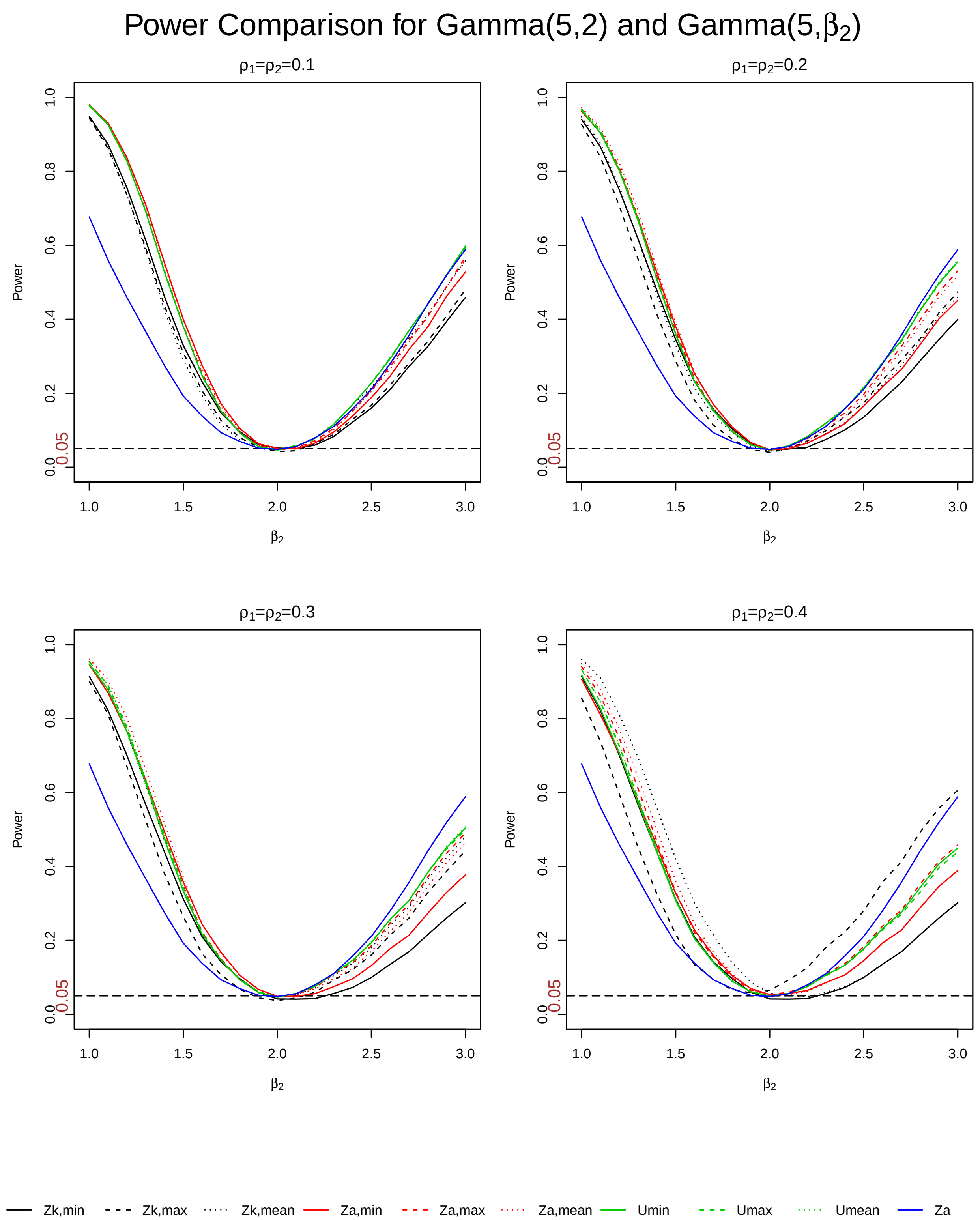

Figure B.13: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2 where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S2**]).

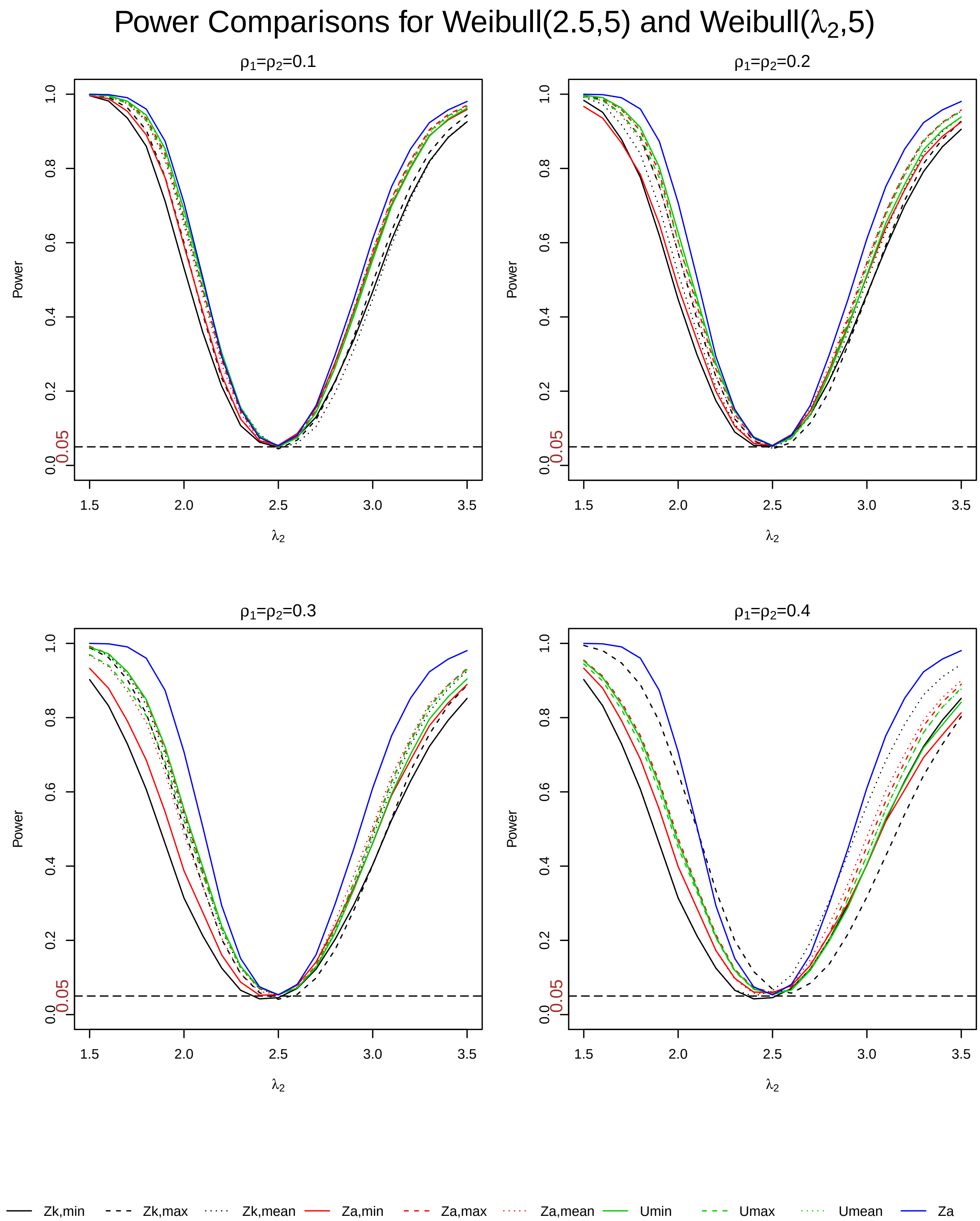

Figure B.14: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2, where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D4**] with system structures [**S2**]).

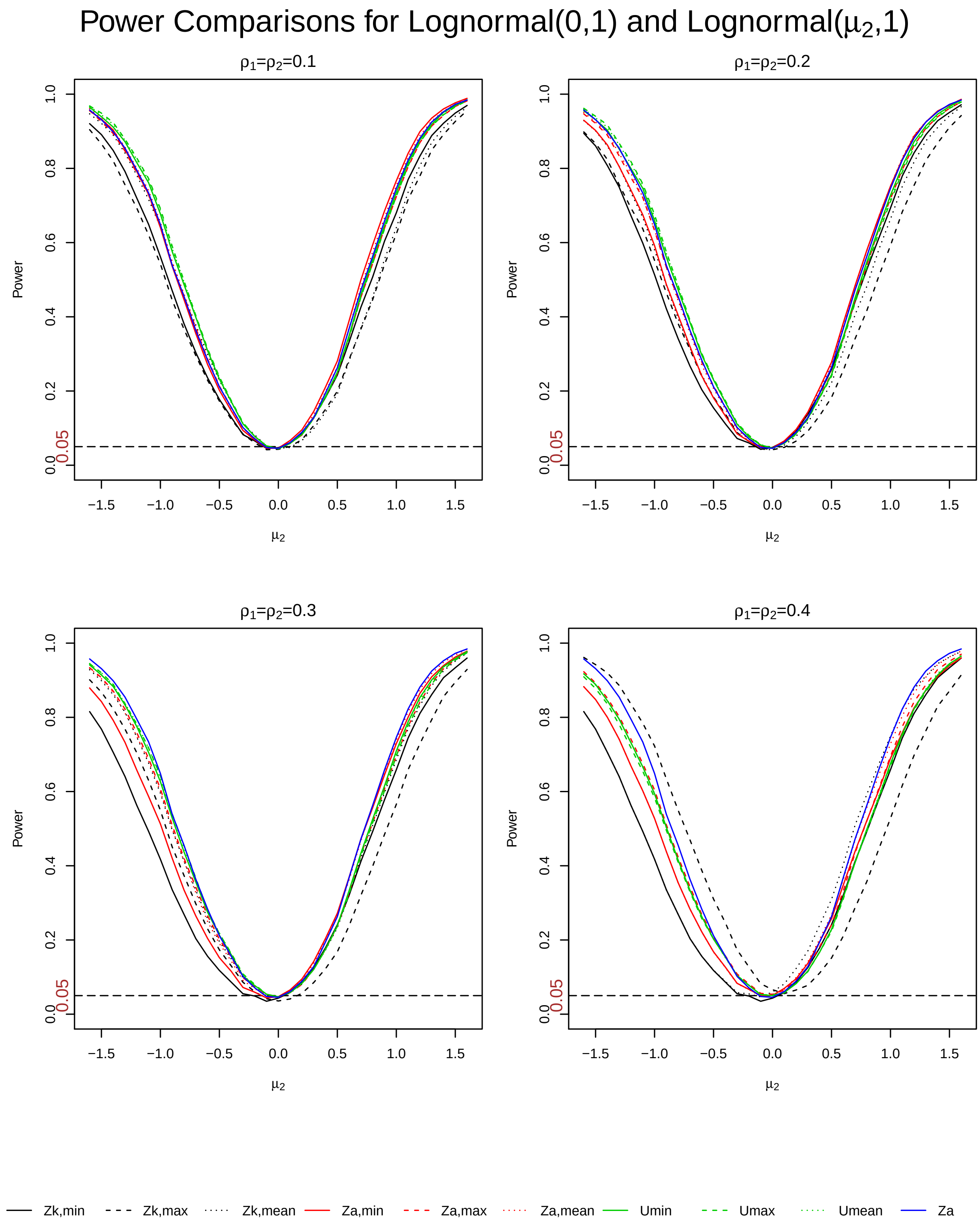

Figure B.15: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $s_1 = (0, 2/3, 1/3)$ and $s_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 10$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Lognormal(0, 1)$ for System 1 and $Lognormal(\mu_2, 1)$ for System 2, where μ_2= $-1.6\ (0.1)\ 1.6$ (i.e., [**D5**] with system structures [**S2**]).

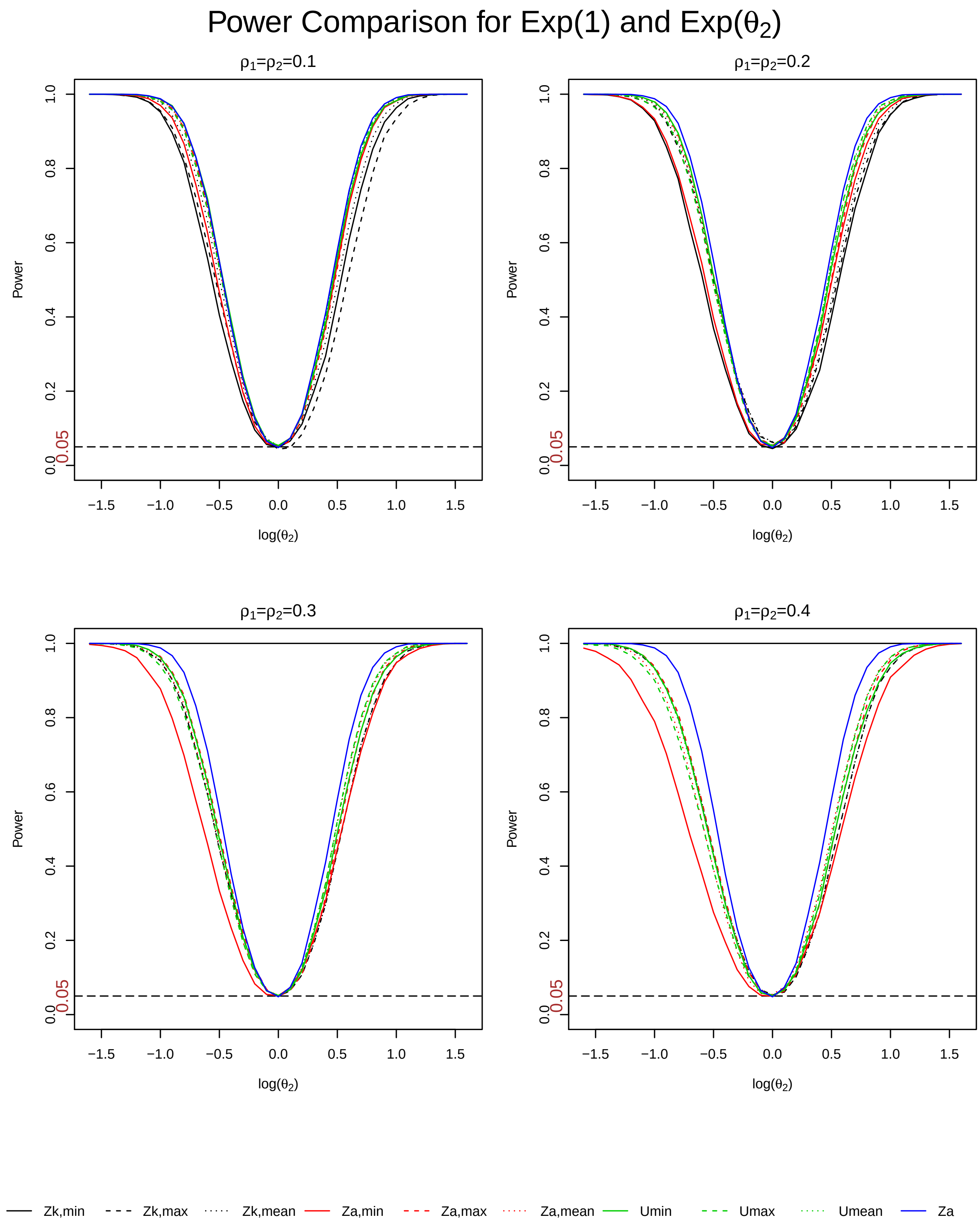

Figure B.16: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions distributions are $Exp(1)$ for System 1 and $Exp(\theta_2)$ for System 2, where $\ln \theta_2 = -1.6\ (0.1)\ 1.6$ (i.e., [**D1**] with system structures [**S2**]).

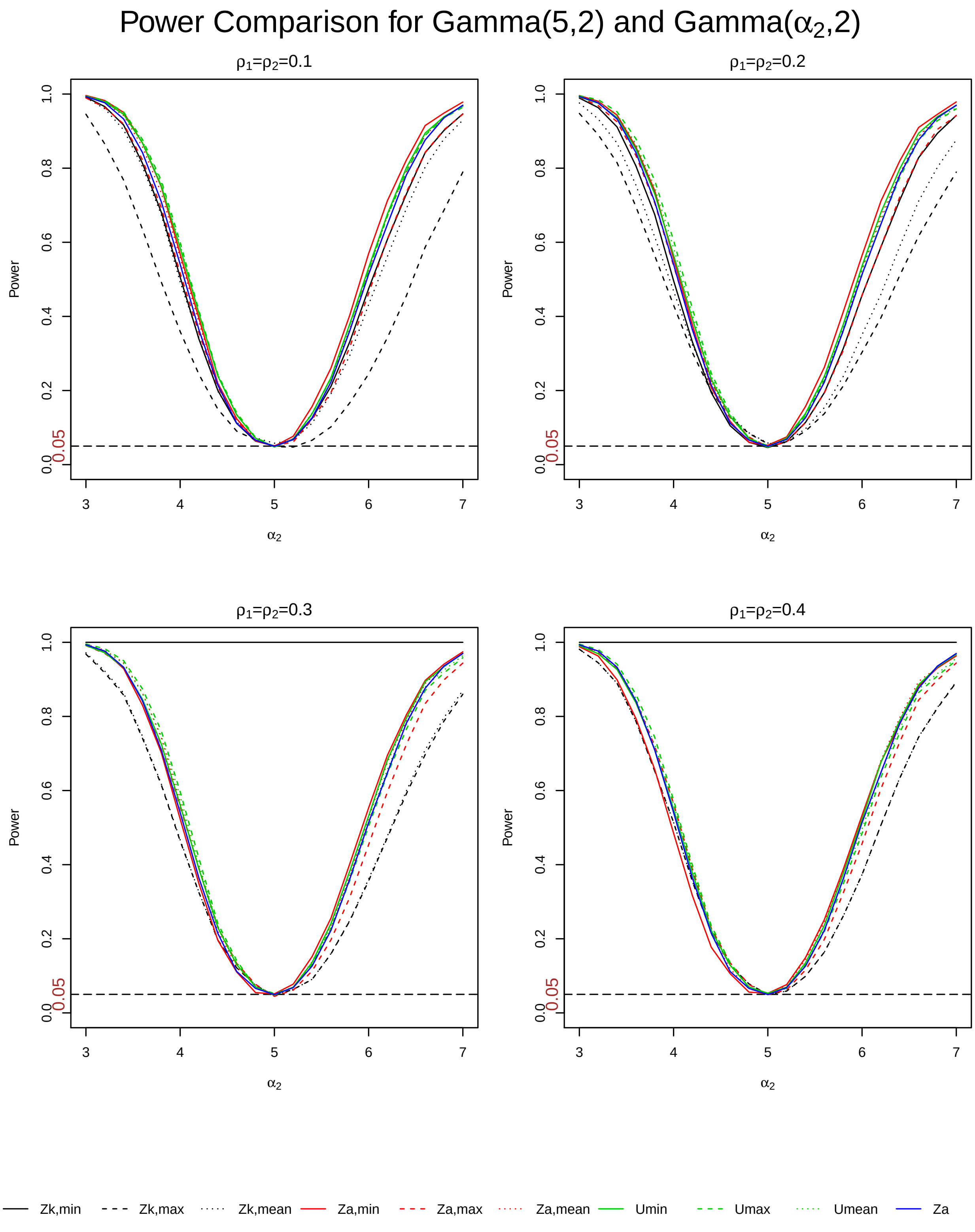

Figure B.17: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(\alpha_2, 2)$ for System 2, where $\alpha_2 = 3\ (0.2)\ 7$. (i.e., [**D2**] with system structures [**S2**]).

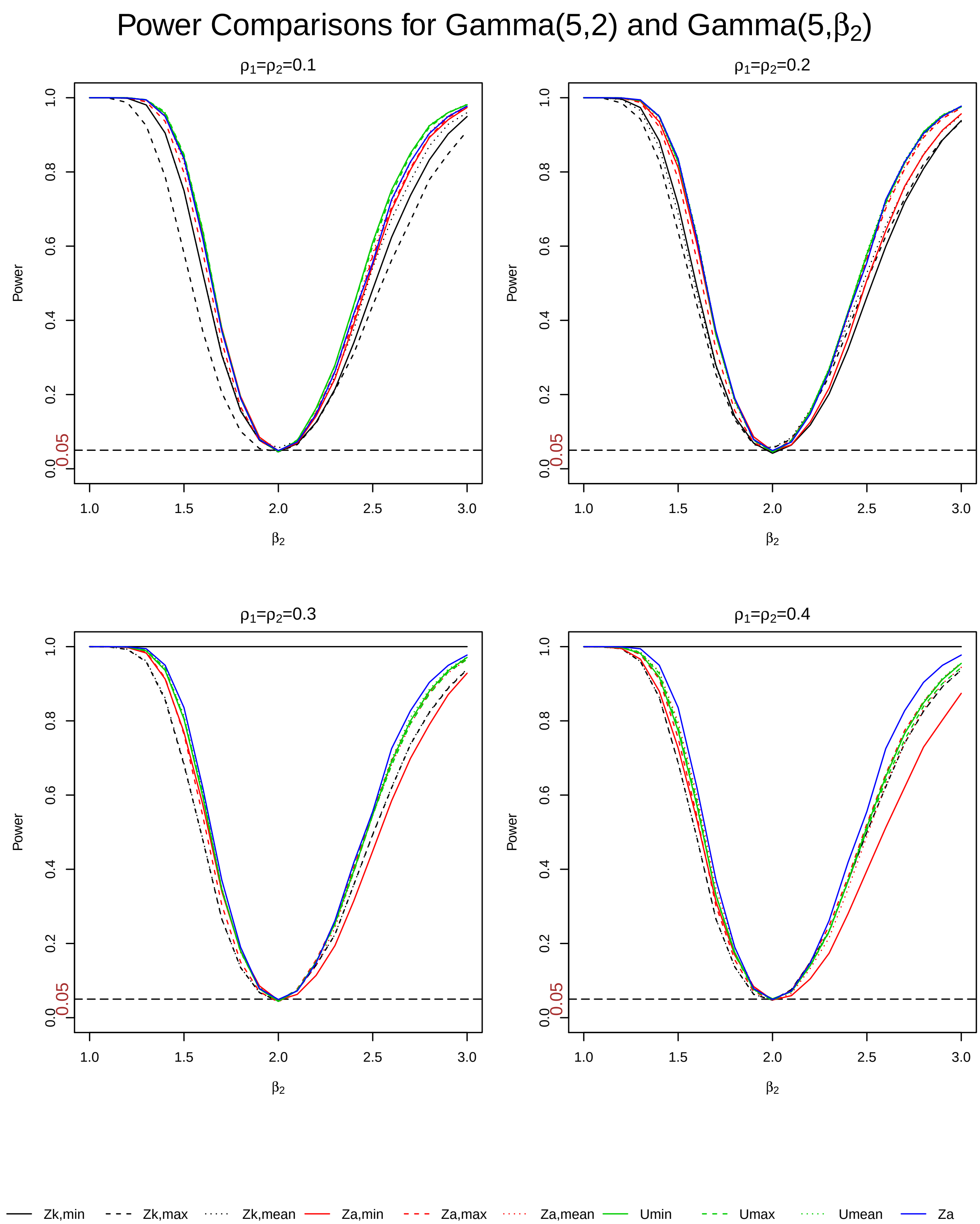

Figure B.18: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Gamma(5, 2)$ for System 1 and $Gamma(5, \beta_2)$ for System 2 where $\beta_2 = 1\ (0.1)\ 3$ (i.e., [**D3**] with system structures [**S2**]).

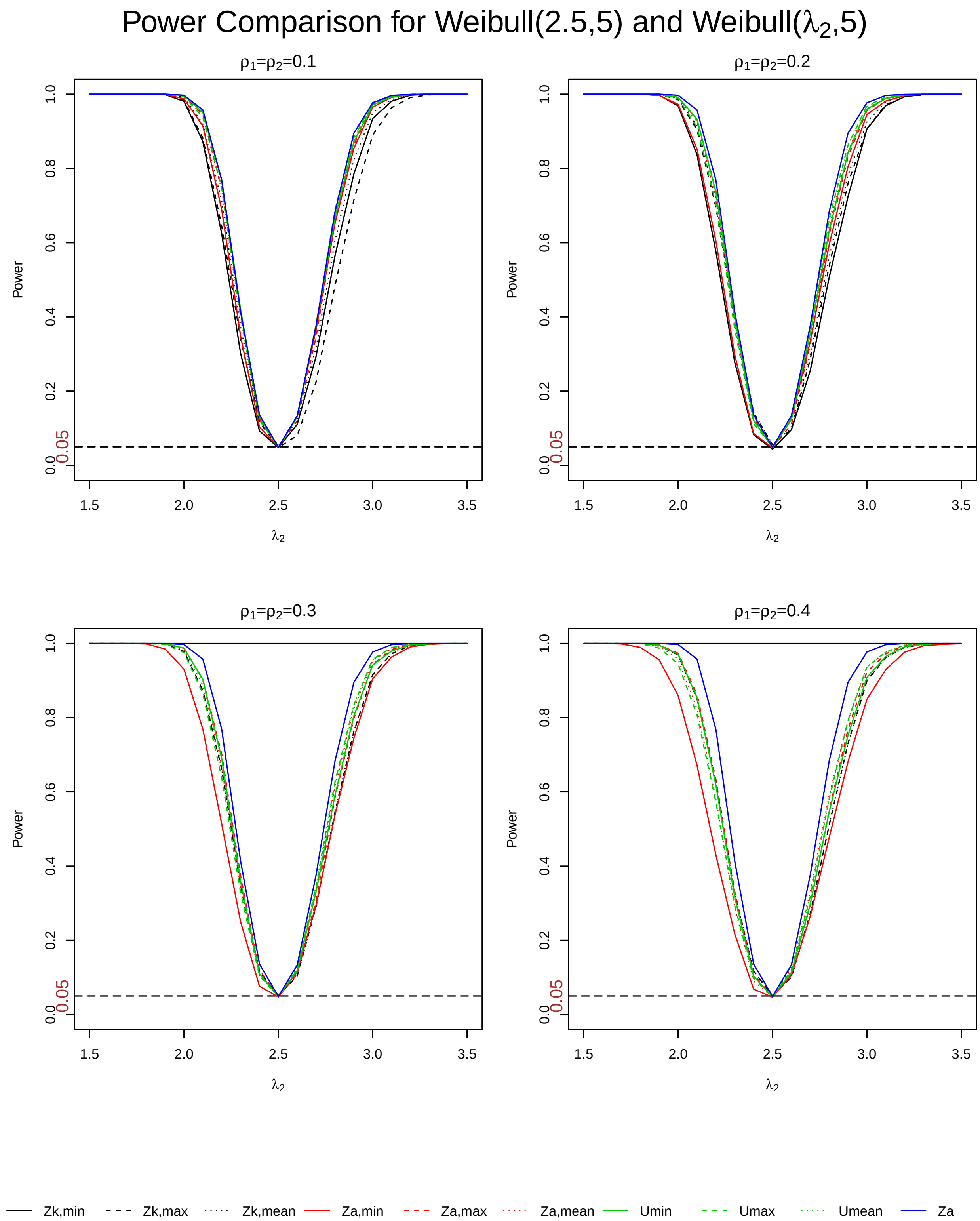

Figure B.19: Simulated power curves for the proposed test procedures based on restricted permutations with system signatures $\mathbf{s}_1 = (0, 2/3, 1/3)$ and $\mathbf{s}_2 = (0, 1/2, 1/4, 1/4)$, sample sizes $M_1 = M_2 = 30$, censoring proportions $\rho_1 = \rho_2 = 0.1$, 0.2, 0.3, and 0.4, and the underlying distributions $Weibull(2.5, 5)$ for System 1 and $Weibull(\lambda_2, 5)$ for System 2 where $\lambda_2 = 1.5\ (0.1)\ 3.5$ (i.e., [**D3**] with system structures [**S2**]).

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