A Property of the Mean Stieltjes Integral

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The purpose of this note is to consider the following problem. Suppose $[a,b]$ is an interval, f a function in $[a,b]$ and g a function which has a derivative in $[a,b]$. What conditions must be satisfied by f and g in order that the existence in the mean Stieltjes (MS) sense of one of the integrals,

$$\int_a^b fdg \text{ and } \int_a^b fg'\,dt,$$

implies that the other exists and has the same value? Lane (1954) has shown that a sufficient condition for the desired result is that f be bounded in $[a,b]$ and g' be continuous in $[a,b]$. It will be shown here that if f is quasi-continuous in $[a,b]$ and g' is bounded in $[a,b]$ with

$$g(t) - g(a) = \int_a^t g'(\alpha)d\alpha, \; t\epsilon\; [a,b],$$

then both integrals exist and have the same value.

In one sense it might be said that both Lane's theorem and the theorem to be presented here are trivial since it is well known that if $\int_a^b fdg$ exists in the Riemann-Stieltjes (RS) sense then the MS integral also exists and has the same value (Smith, 1925). Consequently any theorem for the RS integral is also valid for the MS integral, and the theorems being considered are less general than known results for the RS integral. An approach through the RS integral is not desirable however because a proper understanding of the theory of the MS integral requires a knowledge of what properties of the integral can be obtained strictly from its definition without recourse to other definitions of the integral.

The results to be presented here should be of some interest in connection with the study of Stieltjes integral equations. Furthermore, they would seem to offer some hints towards methods of obtaining further generalizations.

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THEOREM. — Suppose that $[a,b]$ is an interval, f a quasi-continuous function in $[a,b]$ and g a function whose derivative exists and is bounded in $[a,b]$. If for each number t in $[a,b]$ it is true that

$$g(t)-g(a)=\int_a^t g'(\alpha)d\alpha,$$

then each of

$$\int_a^b fdg \text{ and } \int_a^b f(t)g'(t)dt$$

exist and the two integrals have the same value.

PROOF. — The existence of

$$\int_a^b fdg$$

follows immediately from theorem 4.1 of Lane (1954) since it follows from 2.1 of Lane that g is absolutely continuous, and hence of bounded variation in $[a,b]$. If f is quasi-continuous in $[a,b]$ then it is the limit of a uniformly convergent sequence of step functions,

$$\left\{ j_n(t) \right\}_{n=1}^{\infty}.$$

It is easily shown that if h is a step function then

$$\int_a^b h(t)g'(t)dt$$

exists. Since g' is bounded, the sequence

$$\left\{ j_n(t)g'(t) \right\}_{n=1}^{\infty}$$

is uniformly convergent to fg' in $[a,b]$ and the existence of

$$\int_a^b f(t)g'(t)dt$$

follows immediately from lemma 4.1a of Lane.

To show the equality of the integrals, two cases must be considered. We shall first assume that f is of bounded variation in $[a,b]$. We shall further assume that $V_a^b f \neq 0$ and that the supremum of $|g'(t)|$ in $[a,b]$ is not zero; the proof being trivial if either of these conditions is satisfied.

If ϵ is a positive number there exists a subdivision D of $[a,b]$ such that if E is any refinement of D then

$$\left| \int_a^b fdg - \sum_{t_i \epsilon E} \frac{1}{2} [f(t_i) + f(t_{i-1})] [g(t_i) - g(t_{i-1})] \right| \tag{1}$$

$$= \left| \int_a^b fdg - \sum_{t_i \epsilon E} \int_{\alpha = t_{i-1}}^{t_i} \frac{1}{2} [f(t_i) + f(t_{i-1})] \; g'(\alpha) d\alpha \right| < \frac{\epsilon}{2}.$$

Let $\bar{E}$ be a refinement of D, whose norm defined in the usual way is less than $\epsilon / 2MV_a^b f$, where M is an upper bound for $|g'|$ in $[a,b]$. If t_j and t_{j-1} are in $\bar{E}$ and $t \epsilon [t_{j-1}, t_j]$ then $|f(t) - f(t_{j-1})| \leq V_{t_{j-1}}^{t_j} f$ and $|f(t) - f(t_j)|$ $\leq V_{t_{j-1}}^{t_j} f$. Therefore, it follows from theorem 2.1 of Lane that

$$\left| \int_{\alpha = t_{j-1}}^{t_j} [f(t) - f(t_{j-1})] g'(\alpha) d\alpha \right| \leq M(t_j - t_{j-1}) \; V_{t_{j-1}}^{t_j} f < \frac{\epsilon \, V_{t_{j-1}}^{t_j} f}{2 \, V_a^b f},$$

and likewise

$$\left| \int_{\alpha = t_{j-1}}^{t_j} [f(t) - f(t_j)] g'(\alpha) d\alpha \right| < \frac{\epsilon \, V_{t_{j-1}}^{t_j} f}{2 \, V_a^b f}.$$

Then it may be concluded that

$$\left| \int_a^b f(\alpha) g'(\alpha) d\alpha - \sum_{t_j \epsilon \bar{E}} \int_{\alpha = t_{j-1}}^{t_j} \frac{1}{2} [f(t_j) + f(t_{j-1})] g'(\alpha) d\alpha \right|$$

$$\leq \frac{1}{2} \sum_{t_j \epsilon \bar{E}} \left| \int_{\alpha = t_{j-1}}^{t_j} [f(t) - f(t_j)] g'(\alpha) d\alpha \right|$$

$$+ \left| \int_{\alpha = t_{j-1}}^{t_j} [f(t) - f(t_{j-1})] g'(\alpha) d\alpha \right|$$

$$< \frac{\epsilon}{2} \tag{2}$$

It follows from equation (1) for $E = \bar{E}$, together with equation (2) that

$$\int_a^b fdg = \int_a^b f(t)g'(t)dt \text{ if } f \text{ is of bounded variation in } [a,b].$$

If f is an arbitrary quasi-continuous function then there exists a sequence of step functions

$\left\{ j_n(t) \right\}_{n=1}^{\infty}$ uniformly convergent to f in $[a,b]$, with the property that

$\lim_{n\to\infty} \int_a^b j_n dg = \int_a^b fdg$. We have already seen that

$$\lim_{n\to\infty} \int_a^b j_n(t)g'(t)dt = \int_a^b f(t)g'(t)dt.$$

But, for each positive integer p, $j_p(t)$ is of bounded variation in $[a,b]$, hence

$$\int_a^b j_p(t)g'(t)dt = \int_a^b j_p dg.$$

Therefore

$$\int_a^b fdg = \int_a^b f(t)g'(t)dt.$$

This completes the proof.

REFERENCES

[1]Lane, R. E., 1954, The integral of a function with respect to a function; Amer. Math. Soc., Pn., vol. 5, pp. 59-66.

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