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Advertising, Pricing and Stability in Oligopolistic Markets for New Products

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ADVERTISING, PRICING AND STABILITY IN OLIGOPOLISTIC MARKETS
FOR NEW PRODUCTS

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by

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Abstract

In an oligopolistic market for a new product, individual competing firms may possess a differential advantage in terms of the initial goodwill. From a business policy perspective, the question is whether the competitive market equilibrium is affected by the initial goodwill of each firm. That is, do the final market shares depend on the initial stocks of goodwill of competing firms?

For a consumer nondurable or a highly depreciable durable product in an oligopolistic market, we formulate this problem as a non-zero-sum, open loop, noncooperative differential game. It is assumed that the goodwills of the competing firms adjust according to the Nerlove-Arrow capital accumulation equation, i.e., $\dot{x}_i = u_i - \delta x_i$, where $x_i$ is the goodwill level of firm $i$, $u_i$ is its advertising level and $\delta$ is its goodwill decay constant. A solution to such a game (a Nash Equilibrium) is a choice of price and advertising by the competing firms such that each maximizes his own discounted profits subject to his own goodwill accumulation and given the path of the rival.

We prove existence of a Nash Equilibrium and we show the conditions under which the game converges to a particular stationary point regardless of the initial conditions. That is, we show the conditions under which the final market shares do not depend on the initial stocks of goodwill of competing firms.
1. **Introduction**

The success of any marketing strategy depends on the strengths of the competitive analysis on which it is based (Henderson 1983). Competitive analysis defines the arena in which a company's marketing strategy is conceived, planned and executed. It delineates the web that details the interface between the competitors and defines the variables that model a company's competitive strategy. In the presence of active competitors, the objective of competitive strategy is to provide the firm a unique differential advantage over its competitors for long run survival and profitability. Hence, the development of a firm's competitive behavior, i.e., assessment of the competitors' likely reactions to its strategic moves and their possible impact on its profitability.

In assessing the competitive behavior, at one extreme we can assume that a firm operates with competitive independence as, for example, is the case in pure competition or monopolistic competition. In a purely competitive market, the firm has no problem of marketing strategy. All firms produce the same product and no firm charges more than the market price. All buyers have perfect information and advertising cannot be used as a competitive vehicle since buyers purchase strictly according to price. In the presence of monopolistic competition, although the buyers may perceive the various competitors offering a differentiated product in terms of certain attributes such as quality, service, etc., because of the large number of competitors, the competitive strategy developed by a firm may not provoke reaction from its competitors.

The explicit consideration of competitive behavior becomes a must in oligopolistic markets where there are few firms (or duopolistic markets where they are two firms) since the competitive strategy developed by one firm can
impact the profitability posture of its competitors. As articulated by Kotler (1971, p. 99), in the presence of active competitors, the competitors can be assumed to be very sensitive to the company's marketing plan, particularly its price and advertising. Hence, the firm must consider for any contemplated plan $i$, the competitors' most probably response, the firm's reaction, the competitor's next move, the firm's move and ad infinitum. The series of moves and countermoves, however, may eventually result in a competitive market equilibrium. That is, a state of the market where none of the oligopolists makes any further changes in its marketing plan since, given the equilibrium strategy of the competitors, no other marketing plan can yield a better performance.

In developing a competitive strategy, a firm can match the distinctive advantages of its competitors in terms of certain controllable strategic variables such as price and advertising. The one factor, however, that it cannot match is their distinct advantage in terms of their initial stocks of goodwill. In a number of emerging oligopolistic markets for new products (e.g., telephone equipment and services, personal computers), often a firm may be competing with other firms who because of their name awareness, general favorable image and past track record in related products, may have an initial distinct advantage in approaching the market. From a business policy perspective, the question is whether the competitive market equilibrium is affected by the initial goodwill of the individual firms. That is, do the final market shares depend upon the initial stocks of goodwill of competing firms? Or, is the initial goodwill really a strategic variable undermining the competitive market equilibrium?

In order to investigate this question for a consumer nondurable or a highly depreciable durable product in an oligopolistic market, we formulate
the competitive strategy problem as a non-zero-sum, noncooperative
differential game (Isaacs 1965; Case 1979). For the sake of analytical
tractibility, the formulation is confined to the case of duopoly. It is
assumed that: (a) the key strategic or control variables defining the
marketing plan of the individual firms are price and advertising, (b) the
products offered by the two competing firms are close substitutes, and (c) the
goodwills of the competing firms adjust according to the Nerlove-Arrow capital
accumulation model (Nerlove and Arrow 1962).

The representation of competitive encounters between the firms by the
means of differential games explicitly assumes that the competitors are
rational and seek a known performance objective, e.g., discounted profits.
Further assuming that the competing firms are in a situation of conflict or do
not cooperate with each other, the differential game theory formulation lends
itself to the possible determination of competitive equilibrium or Nash
equilibrium. For the proposed competitive strategy model, we prove existence
of a Nash equilibrium and show the conditions under which the game converges
to a particular stationary point regardless of the initial goodwill
conditions. That is, we delineate conditions under which the final market
shares of the competing firms do not depend upon their initial stocks of
goodwill.

The organization of this paper is as follows: Section 2 presents the
model formulation. Section 3 develops necessary conditions for Nash
equilibrium solutions. Section 4 addresses the question of equilibrium
stability or the existence of a stationary equilibrium. The paper concludes
with Section 5 summarizing the significance of the model, its relation to
other works on the subject in marketing, and possible extensions.
2. Model Formulation

Consider a model where there are two firms operating on a single market. The goodwills of the two firms at time \( t \) are denoted by \( x(t) \) and \( y(t) \) respectively. For simplicity, we model the change over time of the goodwills to behave according to the well known Nerlove-Arrow (1962) goodwill accumulation equation.

\[
\begin{align*}
\dot{x} & = u_1 - \delta_1 x; \quad x(0) = x_0 \\
\dot{y} & = u_2 - \delta_2 y; \quad y(0) = y_0
\end{align*}
\]

where \( \delta_i \) is the goodwill depreciation parameter of firm \( i \), a dot above a variable represents differentiation with respect to time and \( u_i \) is the advertising effects of firm \( i \). The cost of having the effects of \( u_i \) is given by \( C_i(u_i) \) for some convex cost function \( C \).

Equation (1) can be thought of as a parsimonious representation of a diffusion process without the word-of-mouth effect. (See Dodson and Muller (1978), where the model was shown to be a special case of the model by Bass (1969).) As in Horsky (1977), we assume that the market shares are functions of the respective goodwills. The difference in the formulation is due to the fact that we wish to introduce pricing and their effect in combination with advertising on profits and market shares.

Moreover, we are interested in formulating the competitive effect as a game in which the two players compete in the market place using both price and advertising as strategic variables.

Since advertising has multiperiod cumulative effect, the game has a dynamic structure. The decision of the firm today affects its position and
its rival's position (as captured by goodwill levels or market shares) not only today but in the future as well. Thus the natural technique which describes such a business environment is a differential game.

Denote the highest achievable goodwill level as $N$. If the goodwill is interpreted, for example, as the number of people who are potential (or current) customers, then $N$ is the total number of people in the market. If, however, the goodwill is interpreted as a stock of advertising goodwill, then the ceiling will be a natural result of the ceiling we shall shortly impose on the rate of advertising $u$.

Consider the following probabilities of purchase and Figure 1. With probability $(1-x/N)$ the consumer is not a potential customer of firm 1 (e.g., is not aware of firm 1 or the brand of firm 1 is not in his evoked set or he is loyal to firm 2). With probability $x/N$ he is a potential customer and then either a) he is not a customer of firm 2 (with probability $(1-y/N)$) and purchases at a rate of $a(p_1)$ or b) he is a customer of firm 2 (with probability $y/N$) and purchases at a rate of $b(p_1, p_2)$, where $p_1$ and $p_2$ are the prices of firms 1 and 2 respectively.

Given that there are $N$ number of potential customers, expected sales for firm 1 at any time can be written as:

$$ S_1 = \frac{x}{N} (1-y/N) a(p_1) + \frac{x}{N} y/N b(p_1, p_2) N $$

For convenience, we can multiply equation (2) by $N$ to achieve the following revenue function:

$$ \Pi_1(p_1, p_2, x, y) = \left( x(N-y)a(p_1) + xyb_1(p_1, p_2) \right)(p_1 - c) $$
\( \text{probability } = 1 - \frac{x}{N} \)

No purchase

\( \text{probability } = \frac{x}{N} \)

\( \text{probability } = 1 - \frac{y}{N} \)

\( \text{probability } = \frac{y}{N} \)

Purchase value \( a(p_1) \)

Purchase rate \( b(p_1, p_2) \)

\( x \) and \( y \) are the goodwill levels of firm 1 and 2 and \( p_i \) is the price of firm \( i = 1, 2 \).

Figure 1
where \( c \) is the production cost. We assume that the production costs for both firms are the same and also they do not change over time. Each firm now wishes to maximize its own discounted profits by employing the optimal paths of pricing and advertising given the path of its rival. This, formally, is a non-zero-sum, noncooperative, open loop differential game whose solution can be achieved by using control theory. In order to define the game formally, we need to define the payoffs and the strategy sets. Let the payoff for firm \( i \) be defined as discounted profits, i.e.,

\[
J_i = \int_0^\infty e^{-rt} \{ \Pi_i(p_1, p_2, x, y) - C_i(u_i) \} dt
\]

Let the strategy set \( S_i \) be all piecewise continuous functions defined on \([0, \infty)\), that take their values in a compact set \([0, u_i]\). For example, the cost \( C_i(u_i) \) which is convex and satisfies that \( \lim_{u_i \to \infty} C_i = \infty \) will induce a control function as desired. Equation (4) assumes that the discounting rate, \( r \), for both the firms is the same.

For every initial stock of goodwills \( x_0 \) and \( y_0 \), define the game \( G(x_0, y_0) \) as the game with strategy set \( S_i \), payoff functions \( J_i, i = 1, 2 \); and at \( t = 0 \), the game starts at the initial stocks of \( x(0) = x_0 \) and \( y(0) = y_0 \). A Nash Equilibrium for the game \( G(x_0, y_0) \) is a set of functions \( u_1^*(t), p_1^*(t), u_2^*(t), p_2^*(t) \), such that \( u_1^*(t), p_1^*(t) \) maximizes \( J_i \) subject to (1.1) given \( u_j^*(t), p_j^*(t) \) for \( j \neq i \).

A stationary Nash Equilibrium is a pair of values \( u_1^*, x^*, p_1^*, u_2^*, y^*, p_2^* \) such that \( u_1^* = \delta_1 x^* \), \( u_2^* = \delta_2 y^* \) and \( u_1^*, p_1^*, u_2^*, p_2^* \) is a Nash equilibrium for the game \( G(x^*, y^*) \).

Note that in a stationary Nash equilibrium, prices, advertising and goodwills do not change overtime.
2.1 Purchase Rate Specification

In order to establish Nash solutions to the differential game specified by equations (4), (1.1) and (1.2), it is first necessary to specify the purchase rates \( a(p_1) \) and \( b(p_1, p_2) \) in equation (3). Following Eliashberg and Jeuland (1982), Wolf and Shubik (1978), and McGuire and Stailen (1982), we assume a linear purchase rate equation. That is,

\[
(5) \quad b_i(p_1, p_2) = a_i(1-kp_1) + \gamma(p_j-P_i), \quad j \neq i
\]

When both firms charge the same price, i.e., \( p_1 = p_2 \), we assume that they distribute the market according to \( a_1 \) and thus the purchase rate function for the firm in the market in which it is a monopolist is given by

\[
(6) \quad a_1(p_1) = (a_1+a_2)(1-kp_1)
\]

Note, for example, that if \( a_1 = a_2 \), then the difference between the market for the monopolist and the market for the oligopolist is that in the latter, when prices are qual, the oligopolists split the monopolist's market into two equal shares. The firm thus can be thought of as having two types of market. In one, it is a monopolist facing the demand of \( a_1(p_1) \) and in the other, it is an oligopolist facing the demand given by \( b_1(p_1,p_2) \). The firm cannot price discriminate. However, if it could, it would have chosen a different price for its monopolist and oligopolist markets denoted by \( p_1^m \) and \( p_1^c \) respectively. To determine these prices, note that if the firm could price discriminate, it would have chosen \( p_1^m \) so that it maximizes \( (p_1-c)a_1(p_1) \), in the monopolistic market and it would have chosen \( p_1^c \) so that it maximizes \( (p_1-c)b(p_1,p_2) \) in its oligopolistic market, where \( c \) is the production cost.
Performing the differentiation with respect to price, and solving for $p_1^m$ and $p_1^c$, we get that these prices are given by:

\[ p_1^m = \frac{1}{2} k + c/2; \quad p_1^c = \frac{(a_1 + \gamma p_2)}{2(a_1 k + \gamma)} + c/2 \]  

The optimal price for the firm denoted by $p_1^*$ is bounded between these two prices, i.e.,

\[ p_1^c < p_1^* < p_1^m \]

Since $p_1^c + p_1^m$ as $\gamma \to 0$, the optimal price of the firm at each period of time will be closer to the monopolist price as the effect of competition (i.e., $\gamma$) lessens.

The two profit functions are depicted in Figure 2. At $p_1^*$, the monopolist profits are still increasing while the oligopolist is decreasing. Thus, when the firm sets a single price, it does so because it cannot price differentiate. If it could, it would have differentiated the price to its two markets. This fact will help to indicate the effect of increase on either $x$ or $y$ on the price.

3. Market Equilibrium

3.1 Price Equilibrium

Substitution of equations (5) and (6) into equation (3) for each of two firms and further differentiation with respect to $p_1$ and $p_2$ yields the following first order condition for the maximization of profit, given by equation (3), with respect to price:

\[ (N-y)\phi_1(p_1) + y\theta_1(p_1, p_2) = 0 \]
Note that the $\phi_i$ and $\theta_i$ represent the marginal profit of the monopolist and the oligopolist market respectively. Thus $\phi_i > 0$ and $\theta_i < 0$ (see Figure 2).

A decrease in $y$, for example, will cause the monopolist part of the market of firm 1 to expand which will cause the price to move towards the monopolist price, i.e., to increase. An increase in $x$ will cause in much the same way a decrease in the price of firm 2 which will trigger a decrease in firm 1's price as well. Thus both prices are decreasing in $x$ and $y$. To show this formally, define the solutions of (8.1) and (8.2) to be $p_1^* = f(x,y)$ and $p_2^* = g(x,y)$. What we show in Appendix 1 is that $f_x$ and $f_y$ are negative and similarly for $g_x$ and $g_y$.

The price competition in the industry can be described by a reaction function analysis. Equations (8.1) and (8.2) form a system of reaction function which can be depicted in a $p_1 \times p_2$ plane as in Figure 3. For example, equation (8.1) can be thought of as setting the price $p_1$ for firm 1 given the price $p_2$ of the competitor. Thus it yields the reaction of firm 1 to any price changes of firm 2.

Total differentiation of (8.1) yields

$$\frac{dp_1}{dp_2} = -\frac{\Pi_1}{\Pi_1} \frac{p_1 p_2}{p_1 p_2} > 0$$
which is positive since $\Pi_1 P_1 P_2 < 0$ and $\Pi_1 P_1 P_2 > 0$.

Thus we can conclude that the reaction functions have positive slopes.

The condition $\Pi_1 P_1 P_2 P_2 > \Pi_1 P_1 P_2 P_2$ guarantees that there is a unique intersection point of the two reaction functions. This intersection point defines the equilibrium prices. Thus at every time $t$ the market is in a short run equilibrium in prices. This equilibrium depends upon the goodwill levels $x$ and $y$. Changes of these goodwill levels will cause a change in reaction functions which will result in a different equilibrium price. For example, if $y$ increases, the reaction function of the second firm (equation 8.2) does not change. But, as it depicted in figure (3), the reaction function of firm 1 shifts leftward which will be followed by lower equilibrium prices.

Since for every $t$ we can find the equilibrium prices as a function of the state variables, and since prices do not affect the goodwill variables, we can divide the competition in the market into two phases. At every time $t$, for a given $x$ and $y$, the firms compete via prices. Given the result of this competition, the firms' payoffs can be described as a function just of the goodwill levels, and thus firms will be engaged in a dynamic competition via their investments in their respective goodwills, namely, advertising.

Prices thus are adjusted instantaneously, and are dependent on the levels of goodwills at the time they are adjusted. The revenue function of firm $i$ can now be written as

\begin{equation}
R_i(x,y) = \Pi_i(f(x,y), g(x,y), x, y)
\end{equation}

The maximization of equation (4) can now be done with respect to the advertising level only.
Concavity of the revenue function is needed for sufficiency; i.e., concavity implies that the necessary conditions are also sufficient and we are assured that the policy we designate as "optimal" is indeed a maximum.

Thus for firm 1 we need concavity of $R_i$ with respect to $x$. In Appendix 1 we show that if $\gamma < k(\alpha_1 + \alpha_2)/2$, then indeed $R_1$ is concave in $x$. The interpretation of this condition is as follows: Note that $|\partial b_1/\partial p_1| = k\alpha_1 + \gamma$ and that $|\partial a_1/\partial p_1| = k(\alpha_1 + \alpha_2)$. Thus the slope of the demand of the monopolist will be larger than the one by the oligopolist if $|\partial a_1/\partial p_1| > |\partial b_1/\partial p_1|$ which will hold if $\gamma < \alpha_1 k$. Similarly for firm 2 we have $\gamma < \alpha_2 k$. These two conditions clearly imply that $\gamma$ is smaller than the mean of $(\alpha_2 + \alpha_2)k$ as required. These conditions imply that both the elasticity of the monopolist will be smaller than the oligopolist and that the demand for the product in the monopolistic market will always be larger (for a given price) than the demand for the product in the oligopolistic market. This clearly is an intuitive requirement since we can expect the price sensitivity of individuals to be larger in the oligopolistic situation.

3.2 Nash Equilibrium

The choice of pricing and advertising is done simultaneously as discussed in the last section, however, since the choice of price is time autonomous, prices depend on the level of goodwills only. Solving for $P_1$ and $P_2$ as functions of goodwills and substituting the result into the objective function $J_1$ defined in equation (4), we get:

$$J_1 = \int_0^\infty e^{-rt} [R_i(x, y) - C_i(u_i)] \, dt$$

Each firm now maximizes $J_1$ with respect to $u_i$, subject to the state constraint (1.1).
Fershtman and Muller (1983) have proved that if $R_i$ is a concave function, $R_i^{xx}$ and $R_i^{xy}$ are bounded and $C_i$ is bounded from below then the differential game $G(x_0, y_0)$ associated with $J_i$ and (1.1) has a Nash equilibrium solution for any initial conditions $x_0, y_0$. In Appendix 1 we show that $R_i^{xx} < 0$ and thus $R_i$ is a concave function of $x$ if $2\gamma < (a_1 + a_2)k$. Under the same condition, we show in Appendix 2 that $R_i^{xy}$ and $R_i^{xx}$ are bounded. Thus to guarantee the existence of a solution to the game we need just the two conditions that $2\gamma < (a_1 + a_2)k$ and that $C'' > \varepsilon$ for some $\varepsilon > 0$. The first assumption was elaborated upon in the previous section. The second assumption, because of the assumed convexity of $C$ implies that at zero level of advertising $C'' \neq 0$. For example, a quadratic function satisfies this condition. The way to show existence in the finite horizon case is to form a set of functions $B_i$ of all possible goodwill paths of player $i$. Since $u_i$ is bounded, this family is closed and equicontinuous and thus it is compact.

For each goodwill path of player 2, there exists a goodwill path of player 1. This is simply done by solving the control problem of player 1 given the path of player 2 and observing that the solution of the control problem is indeed a maximum since the Hamiltonian is concave in $x$ by our assumption that $2\gamma < (a_1 + a_2)k$. The function that assigns a path of $x$ given the path of $y$, and vice versa is continuous from a compact convex set into itself and thus we can use a fixed point theorem to prove existence of an equilibrium.

In the infinite horizon case we multiply $B$ by an exponential decay factor of $e^{-rt}$ to achieve compactness and the rest of the proof follows much the same way.

In the one player case, for each initial condition $x_0$ there exists a unique optimal path. Moreover this path converges to a steady state, i.e., a
point at which all three variables, price, advertising and goodwill are stationary (see Gould (1970)). In the two players case, i.e. the game situation, there is no guaranteed uniqueness. Indeed for any initial condition $x_0, y_0$ there might be several Nash equilibria. The possibility that one of these will converge to a stationary point will be discussed in the next section. Clearly it is possible to have a solution which does not converge to any stationary point. For example, if one player decides to employ a cyclical advertising and consequently cyclical goodwill policy, it is straightforward to show that the best reply for the second firm is to employ cyclical policy as well.

Recall that the result of the optimization with respect to price can be written as $P_1^* = f(x,y), P_2^* = g(x,y)$ where the derivatives of both functions with respect to the two variables are negative. Differentiating with respect to time yields:

$$\dot{P}_1^* = f_x^* x + f_y^* y$$

(10.1)

$$\dot{P}_2^* = g_x^* x + g_y^* y$$

(10.2)

Thus when both goodwills are increasing, i.e. the market expands, prices will fall. Since the market size in dollar terms i.e. $\pi(P_1, P_2, x, y)$ depends not only on the level of goodwills but on prices as well, this will cause a further expansion in the market size. The period of expansion is thus a combination of goodwill accumulation which is a result of high levels of advertising and a decline of prices through time. In case that both level of goodwills decline, the reverse happens and prices increase. If one variable, say $x$, increases and $y$ decreases, then we cannot determine a priori the price
movement over time.

Suppose the game starts with no accumulation of goodwill, then initially $\dot{x}$ and $\dot{y}$ are positive and thus both prices decline. It is important to note that this is due to the effect of competition. In the one player case, this monopolist will change a fixed (monopoly) price and though it will accumulate goodwill so as to increase his market size and thus profits, the price will remain constant throughout the planning horizon. The effect of goodwill accumulation of his rival in the two player case initially causes the firm to counteract this goodwill growth by a price decline. At a later date when the goodwill level is high enough, it is possible to counteract his rival's goodwill growth by increasing the price and decreasing his own goodwill level. Since the firm overcapitalized initially, it now reduces the stock to its optimal level.

4. Stability

As discussed in the previous section, the problem does not have a unique solution and there are solutions which are cyclical. We are interested, however, in the existence of solutions which are more stable than the above, i.e. solutions which approach a stationary equilibrium. A stationary equilibrium is defined as in the one player case as an equilibrium of the game in which all variables are stationary. Once the path of the solution reaches the stationary point, it stays there forever. Note that for every stationary level of $y$, there exists a level of $x$ such that both will remain stationary. What we show, however, is that even when they do not start at the stationary point, the system will converge towards it as time tends to infinity. Given that the current value Hamiltonian for firm 1 is:

$$H = R_1(x, y) - C_1(u_1) + \lambda u_1 - \lambda \delta_1 x$$
the necessary conditions for player 1 are as follows:

(11) \[ \dot{\lambda}^* - r\lambda = -\partial R_1/\partial x + \delta_1 \lambda \]

(12) \[ \lambda = C_1'(u_1) \]

(1.1) \[ \dot{x} = u_1 - \delta_1 X, \ x(0) = x_0 \]

Differentiating equation (12) with respect to time, and substituting \( \lambda \) and \( \lambda^* \) from (12) and (11) respectively yield the following equation:

(13) \[ C_1'' \dot{u}_1 = (r + \delta_1)C_1' - R_1 \]

When the system is stationary, i.e. \( \dot{u} = \dot{x} = 0 \), the stationary equilibrium is a solution of the following equations:

(14) \[ (\gamma + \delta_1)C_1'(\delta_1 x) = R_1 \]

(15) \[ (\gamma + \delta_2)C_2'(\delta_2 y) = R_2 \]

where in equation (14), \( \delta_1 x \) was substituted for \( u_1 \) and similarly for (15). This forms a nonlinear system of two equations with two unknowns \( x^* \) and \( y^* \) - the stationary equilibrium point.

Note that the system, since it is nonlinear, might have multiple solutions. None of the solutions depend on the initial conditions \( x_0 \) and
The uniqueness of the stationary equilibrium point is guaranteed if the following condition holds (see appendix 3):

\[(16) \quad (\delta_1(y + \delta_1)C_1'' - R_{1x}) (\delta_2(y + \delta_2)C_2'' - R_{2y}) > R_{xy}^2 R_{xy} \]

The likelihood of this condition to hold is greater the more convex the cost function is ($C'' > 0$) measures the convexity of $C$), the more concave the revenue function is ($R_{1xx}$ and $R_{2xy}$). The first two conditions are rather standard in that they correspond to the regular requirements of strict convexity/concavity. The fact that more than just concavity (convexity) is needed for the uniqueness of the convergence point can be found for example in the economic literature on global stability. See for example Cass and Shell (1976). The requirement on the absolute value of $R_{xy}$ should be elaborated upon: Note that if $|R_{1xx}| > |R_{xy}|$, then condition (16) holds. This, however, can be interpreted as follows: $R_{1x}$ is the marginal revenue of firm 1. The effect of a change in $y$ on $R_{1x}$ should be smaller than the effect of 1's own goodwill $x$ on $R_{1x}$. Suppose it is not, then when 2 changes its goodwill level, and 1 reacts, then to counteract the change in $y$, $x$ has to make a large change in $x$ since $R_{1xx}$ is smaller. But this large change will induce an even larger change in $y$ since $R_{2yy}$ is smaller than $R_{2yx}$ and so a large change in $y$ is necessitated to counteract the change in $x$. Clearly such a case, when $|R_{1xx}| < |R_{1xy}|$ is very unstable.

In our example of linear demand function it is straightforward to show that if $\gamma$ is small enough and $C$ convex enough than (16) holds (note that $R_{1xx}$, $R_{2xy}$ are continuous in $\gamma$ and compute (16) when $\gamma = 0$).

The way to show the existence of a Nash Equilibrium that converges to the (unique) stationary point is to define the family $B$ (as in section 4) as the
set of goodwill paths that converge to the stationary point. The set is compact (with the appropriate decay factor) as a subset of the previous class, and convex. Since the best response for a convergent goodwill path is to converge as well (to the same point since it is unique) we can use the fixed point theorem and therefore for every initial conditions $x_0$ and $y_0$, there exists a Nash equilibrium which converges to the stationary point. The stationary point, as mentioned earlier, is independent of the initial conditions.

Since prices are determined at each period, prices at the stationary point are independent of the initial stock of goodwill as well. Thus market shares at the stationary equilibrium will be independent of the initial market shares at the start of the game!

5. Conclusions

In recent years, a number of analytical approaches have been suggested in marketing to model competitive encounters among competing firms (for a brief review, see Dolan (1981)). Examples of such approaches include judgmental models, reaction matrix approach, simulation and game theory. Dolan (1981) cites an example of a competitive pricing model developed by General Electric Corporation Consulting Services where deterministic management judgments are used to model competitive reaction. Developed by Lambin, Naert and Bultez (1975) and extended by Hanssens (1980), reaction matrix approach estimates competitive reaction elasticities for the various marketing mix elements based on historical data.

The differential game theory framework for competitive analysis explicitly considers two critical aspects of market behavior—competition and dynamics. In this respect, it is a powerful tool to examine the diffusion of new products in a competitive market (Mahajan and Muller 1979).
Since the seminal work of Bass (1969), recent extensions of diffusion models dealing with the nature of diffusion models (Easingwood, Mahajan and Muller 1983), estimation issues (Bretschneider and Mahajan 1980; Lilien, Rao and Kalish 1981; Schmittlein and Mahajan 1982), pricing strategies (Robinson and Lakhani 1975; Bass 1980; Dolan and Jeuland 1981; Clarke, Darrough and Heineke 1982; Bass and Bultez 1983; Kalish 1983), advertising strategies (Dodson and Muller 1978; Horsky and Simon 1983), product interdependence (Peterson and Mahajan 1978), market size (Mahajan and Peterson 1978), repeat purchase (Dodson and Muller 1978; Lilien, Rao and Kalish 1981; Mahajan, Wind, and Sharma 1983) and time-space integration (Haynes, Mahajan and White 1977; Mahajan and Peterson 1978) have implicitly assumed the presence of monopolistic competition. However, for a number of products this assumption is unrealistic and myopic.

Assuming that competitors use explicit decision rules to set their marketing plans that are known to the firm, the medium of simulation has been used to investigate competitive behavior. (For an example of a simulation model for the evaluation of pricing strategies in a duopolistic market, see Clarke and Dolan (1983)). Recent years, however, have seen a burgeoning interest in using game theory to model the dynamic competitive behavior for new products. Mate (1982), for example, has modeled the competition between two firms as a two-person, non-zero-sum, noncooperative Markovian game to study optimal advertising behavior. Deal (1979) and Eliashberg and Jeuland (1982) have used a non-zero-sum differential game framework to examine optimal advertising expenditures and pricing strategies, respectively, in a duopolistic market over a finite planning horizon. Teng and Thompson (1980) and Thompson and Teng (1980) have developed differential game oligopoly models to derive optimal advertising and pricing policies.
The work reported here complements the efforts of Eliashberg and Jeuland (1982), Teng and Thompson (1980) and Rao (1982) in examining the diffusion of a new product in the realistic setting of market competition. The main difference between these works and ours is in the objective of the papers. Eliashberg and Jeuland are mainly interested in the effects of entry on pricing strategies. Teng and Thompson are mainly investigating the effects of competition on advertising policies when the oligopolists "learn by doing." Rao investigates the problem of investment on goodwill in a model which is similar to ours and looks for conditions which will guarantee local stability of the steady state.

There are differences in the technical aspects of the papers which are worthwhile highlighting here. Eliashberg and Jeuland have a finite time differential game for which they show existence of a Nash equilibrium by a simulation analysis for various values of the parameters. We could not have used such a method since convergence to a stationary equilibrium occurs at infinity and clearly simulation cannot be held for infinite horizon. The numerical method (for a quadratic cost function) was also employed by Teng and Thompson. Rao, on the other hand, has tried to prove existence for the general case but his attempt is somewhat troublesome. The main technical difference, however, is that he investigates local stability properties and not global ones. Thus for small perturbations around the stationary equilibrium point, the paths will return this stationary point. This does not address the question we pose in this paper of the total independence of the stationary point from the initial starting point, since there is no a priori reason to believe that the starting point will be in a small neighborhood around the stationary point.

The implication of the proposed model should be of interest to both
practitioners and researchers. In a number of emerging oligopolistic markets for new products, the market may be dominated by a single or multiple competitors in terms of their initial goodwills. (Examples include IBM, Apple and Texas Instruments in the market for personal computers and Bell telephone for equipment and services). The key question is whether the initial goodwill enjoyed by a competitor gives a distinctive strategic advantage and can it effect the market equilibrium? The results reported in this paper delineate the conditions under which the market equilibrium is not affected by the initial stocks of goodwill.

The proposed model is not without shortcomings. However, these shortcomings should be considered as possible avenues for further extensions. First, the model assumes that a firm's goodwill is primarily generated through advertising and ignores the effect of word-of-mouth communication. Second, we assume that the new product under consideration is a consumer nondurable or a highly depreciable durable product. Third, it is assumed that the production costs of the competing firms are the same and remain constant over the entire diffusion period, ignoring learning or experience effects. Fourth, we assume that in discounting their profits, competing firms use the same discounting rate. Finally, it is assumed that all the competing firms have a single known objective and ignore the possibility of multiple or different objectives.

Future research should strive towards the relaxation of these assumptions to examine innovation diffusion in more realistic competitive settings.
Appendix 1

Let $P_1 = f(x,y)$ and $P_2 = g(x,y)$. We wish to show that $R_{1xx}$ and $R_{2xx}$ are negative if $2\gamma < (\alpha_1 + \alpha_2)k$. We show that $R_{1xx} < 0$. The proof that $R_{2xx} < 0$ follows symmetrically. Substituting the first order conditions into the objective function $\pi_1$ we get:

$$ R_1 = x(f-c)^2n \text{ where } n = (\alpha_1 + \alpha_2)k(N-y) + (\gamma + \alpha_1k)y $$

$$ R_{1xx} = 4(f-c)n f_x + 2xn [f_x^2 + (f-c)f_{xx}] $$

From the first order conditions, define $L$ as the solution for $P_2$ as a function of $P_1$ i.e. $P_2 = L(P_1)$. $L$ is thus given by

$$ L = -\phi(P_1)(N-y)/\gamma - \theta(P_1)/\gamma \text{ where} $$

$$ \phi_1(P_1) = (\alpha_1 + \alpha_2)(1-kP_1) - (P_1 - c)(\gamma_1 + \alpha_2)k $$

$$ \theta_1(P_1) = \alpha_1(1-kP_1) - \gamma P_1 - (P_1-c)(\gamma + \alpha_1k) $$

From the discussion in section (3) it is evident that $\phi_1$ is positive and $\theta_1$ negative.

$$ \partial L/\partial P_1 = L_p = 2(\alpha_1+\alpha_2)k(N-y)/\gamma y + 2\alpha_1 k/\gamma + 2 $$

Substitute $L$ into the first order condition for firm 2 to achieve the
following

\[ 0 = F(x,y) = (N-x) \phi_2(L) + x \theta_2(L), \]\n
where

\[ \phi_2(L) = (\alpha_1 + \alpha_2)(1-kL) - (L-c)(\alpha_1+\alpha_2)k \]

\[ \theta_2(L) = \alpha_1(1-kL) + \gamma(P_1-L) - (L-c)(\gamma + \alpha_2k) \]

As before \( \phi_2 > 0 \) and \( \theta_2 < 0 \). It is straightforward to check that

\[ f_x = -\frac{\partial F}{\partial x} \]

is negative and thus we are left to show that

\[ f_x^2 + (f-c)f_{xx} \]

is negative. \( f_{xx} \) can be computed to be:

\[ f_{xx} = \frac{[(\alpha_1 + \alpha_2)k - 2\gamma)L_p + \gamma][2(\theta_2 - \phi_2)]}{(\partial F / \partial \gamma)^2} \]

Since \( (\alpha_1 + \alpha_2)k - 2\gamma > 0 \) (by assumption), \( f_{xx} < 0 \).

\[ f_x^2 + (f-c)f_{xx} = \left(\frac{1}{\partial F / \partial \gamma}\right)^2 A \]

where

\[ A = \left(\theta_2 - \phi_2\right)[\theta_2 - \phi_2 + 2(f-c)((\alpha_1 + \alpha_2)k - 2\gamma)L_p + 2(f-c)\gamma] \]

In order to show that \( A < 0 \), first it is straightforward to show that \( \partial A / \partial \gamma > 0 \) and \( A \) is negative as \( \gamma \to 0 \). Since \( P_2^c < P_1 < P_1^m \) (defined in section 3) it is easily shown that \( P_1 > 1/3k + 2c/3 \). Substituting this lower bound it is evident that where \( 2\gamma = (\alpha_1 + \alpha_2)k, A < 0 \). Thus as long as \( 2\gamma < (\alpha_1 + \alpha_2)k, R_{1xx} < 0 \).

Q. E. D.
Following Appendix 1, $R_{1}^{xx}$ and $R_{1}^{xy}$ can be written as:

$$R_{1}^{xx} = 4(f-c)n_{f_{x}} + 2x_{n} \left[ f_{x}^{2} + (f-c)f_{xx} \right]$$

$$R_{1}^{xy} = 2(f-c)n_{f_{y}} + 2x_{n}f_{xy} + 2x_{n}(f-c)f_{xy} + ((f-c)^{2} + 2x(f-c)f_{x})n_{y}$$

Since both $x$ and $y$ are bounded between zero and $N$, in order to show the boundedness of the above expressions, we only have to show the boundedness of $f_{x}$, $f_{y}$, $f_{xx}$, and $f_{xy}$.

$$\frac{\partial F}{\partial p_{1}} = -2(N-x)(\alpha_{1}+\alpha_{2})kL_{p} + xy - x(\alpha_{1}+\alpha_{2})kL_{p}$$

First, note that $L$ is bounded as $y$ tends to zero since $\phi(p_{1}) \to 0$ as $y \to 0$. The limit of their ratio can be found to be finite by using L'Hospital rule since $y(0) > 0$. Clearly $L_{p}$ is bounded from below. Thus, $\frac{\partial F}{\partial p}$ is bounded from below and $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ from above. Thus $f_{x}$ and $f_{y}$ are bounded. With respect to $f_{xx}$ (and similarly $f_{xy}$) divide nominator and denominator by $L_{p}$. As $y \to 0$ $(1/L_{p})\frac{\partial F}{\partial p_{1}}$ is nonzero and the nominator is finite. Thus $f_{xx}$ is bounded.
Define \( x = \phi_1(y) \) and \( y = \phi_2(x) \) as the solutions of (14) and (15) respectively. Since \( R_{xx}^1 \) and \( R_{yy}^2 \) are negative, the sign of \( \phi_1' \) is the same as the sign of \( R_{xy}^1 \). If \( R_{xy}^1 \) and \( R_{xy}^2 \) have opposite signs, the stationary point is unique.

If \( R_{xy}^1 > 0 \), then it is sufficient to prove that at every stationary point, \((\phi_1^{-1})' > \phi_2'\). If \( R_{xy}^1 < 0 \), it suffices to show that \((\phi_1^{-1})' < \phi_2'\). In both cases, computing these derivatives yield condition (16). Q.E.D.
NOTE

Rao's proof is problematic. He bases his proof on an existence theorem by Friedman (1977, theorem 7.1) which is based on the Brower fixed point theorem. This theorem cannot be used here since in Rao's case a strategy is a sequence with infinite number of elements. Rao also claims that his strategy set is compact and convex with respect to a specific metric he defines. This complicates his proof considerably, since now he has to prove any claim he makes in his existence proof with respect to the new metric. Moreover, compactization is not that difficult to achieve. What is difficult is to gain compactness without losing continuity. Thus an argument is needed to show that the best response function is continuous. This is the core of his proof which is simply missing.
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