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BIDDING FOR OFFSHORE OIL

TOWARD AN OPTIMAL STRATEGY

KEITH C. BROWN

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Keith C. Brown

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Introduction

SINCE 1950 petroleum companies have paid over one and one-half billion dollars for leases off the Louisiana coast. The leases are sold at a competitive auction conducted by the Bureau of Land Management of the Department of the Interior. The bids submitted by individual companies for a tract often differ by startling amounts. Clearly, the uncertainty connected with what might be found when a tract was drilled was a major cause of the variation of bids among companies bidding on a given tract.

The research efforts which led to this monograph began with the discovery of the summary listing of all bids on tracts offered for lease by the New Orleans office of the Bureau of Land Management. The large amounts of money spent and the wide variation of many tracts promised a fascinating study. The work reported here had two main goals: (1) to examine the bidding records to see what patterns and regularities existed which would be of help to a firm in deciding how much to bid for a tract lease and (2) to develop an optimal bidding strategy for a firm based on the observed bidding behavior. A statistical analysis of the bid records did lend support to several important hypotheses about the behavior of bidders; these hypotheses were then used for the development of an optimal strategy model. From this model a formula was derived for calculating the expected profit-maximizing bid as a function of the expected value of the tract, the predicted standard deviation of the bids of competitors, and the estimated number of competing bidders. It is further shown that the expected profit is usually not very sensitive to a small misestimation in the number of bidders or to the standard deviation of the competitor bid distribution. The bidding rule developed is novel, but its implications are not at variance with commonly held ideas about the behavior of firms in imperfectly competitive markets.

It was not necessary in building the model to distinguish between the search for oil and that for gas. Petroleum (or oil) is here used as a convenient collective noun to stand for valuable liquid or gaseous hydrocarbons which are produced by drilling into a permeable formation which contains them.

There are two kinds of competitive bidding situations. One is closed or sealed bidding, in which the participants independently submit offers to a judge who accepts the highest according to established rules. The other is open bidding, at which the participants publicly make offers until no one is willing to raise his offer, the high bid being the winner. While this monograph is focused on the specific problem of optimal sealed bidding for petroleum leases in the Louisiana offshore area, some of the results may be applicable to other bidding situations.

Without the stimulation and helpfulness offered by many people and organizations this study could not have been completed. Dr. Wallace F. Lovejoy was not only an invaluable source of knowledge about the petroleum industry, but also a patient and sympathetic counselor. Professors David Huang, Carter Murphy, John Spratt, and Paul Minton each assisted in a number of ways. Harold Rudel of Sun Oil Company, Warren Davis of Gulf, John Arps, petroleum consultant, John Rankin of the New Orleans office of the Bureau of Land Management, and C. J. Bonnecarrere, secretary to the Louisiana State Mineral Bond, granted valuable interviews. My wife, Deborah, prepared the index; her patience and help were great comforts while the manuscript was being revised. All responsibility for any errors that remain lies, of course, solely with the author.

Bidding for Offshore Oil

I

Uncertainty and the Selection of Investments

THE OUTCOME of the drilling of a wildcat well on a lease is almost never known precisely at the time the well is drilled. In fact, even a probability density function of the present values of the possible outcomes is typically not known with certainty. But even wildcat wells are not drilled blindly; before a rational entrepreneur undertakes such a project, he must have enough information to convince him that the possibilities of gain are sufficient to justify the drilling costs. Thus, it can be persuasively argued that, although a potential investor may not be able to specify an *objective* probability density function, he, in effect, does crudely specify a *subjective* probability density function in the process of deciding whether or not to undertake the investment.[1]

Many petroleum companies, of course, are not limited to undertaking one investment at a time. If more than one investment is being considered, then the rational investor ought to be more concerned about the overall prospects of a possible investment portfolio for gain or for loss than about the outcomes of the individual investments in isolation. If the outcome of each project in a portfolio is not completely independent of the outcome of every other project in the portfolio, then the relationships between the investments may be important in determining the overall prospects. This introduces no theoretical problems so long as these relationships to each other are known; then the overall prospects of the portfolio can be determined, at least in theory, from the individual investment prospects and their relationships to each other.

Interrelationships between possible returns are of at least two distinct kinds: (1) possible future events may affect the outcomes of some investments in a similar manner and (2) making one investment may substantially affect the distribution of outcomes for one or more other investments. For example, a change in the market price of crude oil would affect the outcomes of many, many petroleum company investments. Or an investment in developing some novel production technique might affect the outcome of investments in areas where this technique is applicable. If the investor is indeed interested in the probability density functions of the outcomes of possible portfolios and if the outcomes of individual investments are not independent of each other, then the most desirable portfolio cannot, in general, be selected by decision rules applicable to each possible investment in isolation. Rather, the desirability of including an investment in a portfolio must be evaluated on the basis of its effect on the portfolio frequency distribution of outcomes. Indeed, a diversified portfolio may appear very much more desirable to an investor than some multiple of any of the single investments available. This concept underlies Markowitz's explanation of the general desirability of diversification.[2]

Now, if the outcome of investing in a given portfolio can be stated only in probabilistic terms, how does the typical investor react to the parameters of the probability distribution associated with the portfolio in appraising the desirability of owning it? This question has formed the basis for an enormous amount of recent theoretical and empirical research. It is not feasible to attempt here to summarize this research except to say that there is general agreement that investors prefer a higher to a lower expected return, everything else being equal, and dislike the possibility that the return may be lower than expected. Thus, in comparing the desirability of portfolio outcome frequency distributions, not only will the means of the two distributions be considered, but also their shapes. Among the possible portfolios that can be formed from a typical set of potential investments, it is usually possible to increase the expected return from the portfolio only at the expense of introducing a greater variability of return. Different investors appear to have different tradeoffs between expected return and return variability. For example, one petroleum firm may like a high aggregate risk portfolio from which it expects to earn high returns. Another firm may accept lower expected profits in return for a greater probability of earning at least the expected return rate.

The relative variability of the outcomes of a portfolio as compared to the variabilities of the individual component investments can be made low if there exist enough investments with relatively independent outcomes or if some investments can be found which will more likely turn out well if others turn out poorly and vice versa. If, however, the outcomes of all available investments exhibit a strong positive dependence it may not be possible to form a portfolio from these investments which has substantially less expected relative variance than any of the individual investments. If, in addition, the relative variabilities of all the available investment outcomes are about the same, then the decision process of an investor is simplified —he simply adds investments in descending order until either his capital or the set of available investments is exhausted. It is important to emphasize that this will be the case only where the outcomes of all investments are so strongly correlated and the relative variabilities of the outcomes so nearly the same that diversification cannot significantly reduce return variability. In all other cases the optimal portfolio cannot be chosen in such a simple manner.

There are a number of factors which may reduce the potential benefits available from diversification. The general level of business activity may affect a very large fraction of available investments in much the same manner; investments with outcomes independent of or negatively related to the general level of business activity may either be unavailable or have such high prices (due to the strong demand and apparently limited supply) as to lose much of their attractiveness. Similarly, systematic bias in estimating the frequency distribution functions of the outcomes of individual investments can lead to serious misestimations of the frequency distribution function of the overall portfolio. Thus, the investor always runs the risk that he has misestimated the frequency distribution function of the portfolio he chose, and that another portfolio would have been a better a priori choice.[3] The investor risks being in a situation similar to that of a pollster whose sampling procedure is biased; increasing the sample size may not make the pollster's predictions perceptibly more accurate, just as increased "diversification" cannot correct for systematic bias in the investor's valuation estimation procedures. Further, investments which have to be "managed" may turn out to be more variable in unskilled hands or may be costly to undertake because of the expense necessary to hire an expert manager.

Regardless of what portfolio selection criteria the investor adopts, unless he is extremely conservative,[4] Monte Carlo simulations will likely show considerable divergence between expected value of portfolio assets at the end of some reasonably long time period and their actual value. Markowitz gives an interesting illustration of this phenomenon.[5] The probability distribution of return on a hypothetical portfolio (which does not include cash) is

Return Rate	−.2	−.1	0	+.1	+.2
Probability	.1	.2	.3	.3	.1

The expected value of return rate is +.01; yet at the end of ten periods in the particular simulation, the investor who had invested in this particular portfolio had only 81 percent of what he started with. At the end of 600 periods, he had 2,690 percent; yet by the end of 650 periods, this had dropped to 571 percent! In other words, if the dispersion of possible outcomes is at all large, the investor may be subject to some violent whipsaws over time, even though he is following some "optimal" policy.

While there is agreement that investors dislike variability of returns —especially variability which would result in a return lower than expected—there has been no agreement over an appropriate objective measure which can be used to classify investment portfolios according to degree of "riskiness." Nonetheless, the word "risk" is in such widespread use that it is difficult not to use it in the following chapters. It will be used to denote the sort of variability in expected returns that investors do not like.

II

Uncertainty and Optimal Bidding Strategy for Offshore Petroleum Leases

1. Risk in the Petroleum Exploration Industry

DISCOVERING commercial petroleum deposits is a notoriously risky[1] business. Excellent discussions of the character of these uncertainties may be found in McDonald[2] and Kaufman.[3] However, these presentations tend to center around the difficulties in obtaining and interpreting geophysical and other information about areas of interest, the expenses of leasing and drilling which must be incurred in order to discover a suspected deposit, and the uncertainties connected with development and production. In other words, these discussions tend to describe a frequency distribution function for the outcomes of investments in an area. Usually there is a large chance of a small loss—that geophysical and other preliminary work will be done but that prospects will be deemed so poor that further work is at least temporarily abandoned. There also may be a moderate chance for a large loss. Such losses will occur if preliminary indications are so favorable that leasing and drilling are undertaken but no commercial deposit is found. Finally, there is some chance for a gain. The gain which will result if a really valuable hit is made, of course, provides the incentive for the whole exploration process.

The fact that there is a large dispersion in the frequency distribution functions of typical individual investments, however, is not sufficient evidence of a "high risk" industry. If adequate opportunity for meaningful diversification is available, then the dispersion of the outcome of a portfolio of investments, each with an individually high dispersion, may be relatively low. If risk-reducing diversification is possible, then it is no longer rational to consider each investment as a separate entity; the ultimate desirability of a potential investment will depend largely upon how its frequency distribution function of possible outcomes is related to the frequency distribution function of other investments in the portfolio.[4] To the extent that each investment will tend to behave in the same manner as others in the portfolio, this diversification will not reduce risk, and each investment may be considered independently of others. Therefore, the important question to be considered in forming hypotheses about the behavior of the firms bidding for Louisiana offshore petroleum lands is the extent to which the expected return from the different tracts can be considered to be independent, negatively dependent, or positively dependent.

Commercial petroleum deposits are classified as pools, fields, and provinces. Terms such as "pool," "field," "province," and "subprovince" are useful in describing and locating the various oil and gas accumulations and occurrences. They combine both geographical and geological factors that are commonly understood by the geologists, geophysicists, and engineers of the petroleum industry. But these terms, like many others in geology, grade into one another, which makes it difficult, at times, to define their exact meaning. Local usage generally prevails eventually, even though it may not reflect the best or most accurate scientific classification and terminology.

Pool. The simplest unit of commercial occurrence is the pool. It is defined as the body of oil or gas occurring in a separate reservoir and under a single pressure system. A pool may be small, underlying only a few acres, or it may extend over many square miles. Its content may be entirely gas, or it may be entirely or mainly oil. The term major pool is arbitrarily taken to mean a pool that will ultimately produce 50 million barrels or more of oil.

Field. When several pools are related to a single geologic feature, either structural or stratigraphic, the group of pools is termed a field. The individual pools comprised in a field may occur at various depths, one above another, or they may be distributed laterally throughout the geologic feature. Geologic features that are likely to form fields are salt plugs, anticlinally folded multiple sands, and complex combinations of faulting, folding, and stratigraphic variation. . . . The amount of oil that a pool or field will produce is not a distinguishing characteristic. In the East Texas pool and in many of the Middle East pools, the oil is obtained from a single reservoir; yet the ultimate production of each of these pools will be greater than that of many fields or even provinces. Since a field may contain several closely related pools, the terms "pool" and "field" are often confused, especially during the early stages of development.

Province. A petroleum province is a region in which a number of oil and gas pools and fields occur in a similar or related geological environment. Since the term is loosely used to indicate the larger producing regions of the world, the boundaries of a so-called province are often indistinct. The Mid-Continent province of the south-central United States, for example, has definite regional characteristics of stratigraphy, structure, and oil and gas occurrence. Consequently, the term has a specific meaning for geologists and the petroleum industry. Subprovinces may occur within provinces; with the Mid-Continent province, for example, we find the Cherokee sand subprovince of southeastern Kansas and northeastern Oklahoma, the Anadarko Basin subprovince of western Oklahoma and northwestern Texas, the Reef subprovince of westcentral Texas, the Panhandle subprovince of northwestern Texas, and many others.[5]

Within a province, it is very difficult to imagine how there might be a negative dependency between the outcomes of investments in tracts. This would mean that knowledge of success in one tract would increase the a priori probability of failure on another, or vice versa. On the other hand, because a province "is a region in which a number of oil and gas pools and fields occur in a similar or related geological environment,"[6] a positive dependency is likely. If success is obtained from a particular structure type at one place in the province, then it becomes more likely that drilling a similar structure elsewhere in the province would also be successful. The degree of positive dependency will tend to vary inversely with the distance between the structures and directly with their geological similarity. Also there may be reasons, perhaps connected more with the origin of the petroleum than with the trap in which it is contained, for the existence of general positive tendencies within an area. In fact, Kaufman gives considerable evidence for believing that the amounts of oil contained in the various fields in a basin[7] tend to be distributed according to a definite frequency distribution function.[8] If enough fields have already been discovered in the basin, the parameters of the distribution can be estimated. This distribution can then be a valuable aid in estimating the amount of oil which may be found in an undiscovered field in the basin. In other words, the probability distribution for the amount of oil which may be found if the well is successful is, in part, a function of the amounts of oil contained in all known fields in the basin. Similarly, if oil is not found on one tract, it becomes less likely that oil will be found in similar tracts in the province.

Biases in value estimation procedures may also cause positive dependency. Such biases may be most prevalent in the interpretation of geological and geophysical data. Suppose that a firm consistently overestimates the expected value of certain type structures. Whatever method the company adopts for selecting a desired optimal portfolio, it is probable that more tracts containing the overvalued structures will be bid high, and thus more such tracts will probably be won than if there had been no such bias. But these tracts will on the average turn out more poorly than expected. Thus the diversification is not as effective as the company thought, and the expected value of the portfolio is lower.

Such biases in estimation procedures are likely for several reasons. A considerable time may elapse after a decision is made to bid for a lease before even one well is completed or the lease is abandoned. Even if the first well turns out to be a promising producer, the full extent of the deposit and a good estimate of the potential productive capability may not be known for several years. Also, new or modified exploration techniques are occasionally introduced; before these techniques are properly evaluated there may be divergent opinions concerning their effectiveness. And the typical type of formation being examined changes with the introduction of the new techniques and the relative completion of evaluation of structures and formations recently popular. For all these reasons a firm can never be sure that the predictions it makes concerning the expected value and shape of frequency distribution functions are not biased, at least for large classes of tracts.

Also, diversification within or among provinces within the United States cannot help protect against crude or gas price changes or widespread allowable production changes. These factors will tend to affect many producing tracts in a somewhat similar manner, even though the tracts be widely scattered and producing from assorted structures. Too, diversification is costly in terms of the extra information which must be collected and evaluated for each investment possibility considered. Thus, even though further diversification might be expected to increase the portfolio's desirability somewhat, a point will eventually be reached when the expected gain from more diversification is offset by the loss caused by money spent for information about possible portfolio additions. When such a state is reached, there will, at least temporarily, be no more incentive toward diversification.

Most of the firms that have submitted bids for federally owned offshore tracts are quite large, subsidiaries of large firms, or members of a group of firms that is large in the aggregate. For instance, in the October 13, 1954, sale the following companies submitted more than five bids—Shell, Gulf, Standard of California, Humble, Standard of Indiana, Placid, Magnolia, Phillips, and Forest. Humble was controlled by Standard of New Jersey and Magnolia by Socony Mobil. Of these nine companies or parent companies, only two did not have assets reported in excess of one billion dollars in the July, 1955, *Fortune* list of the five hundred largest industrial corporations in the United States (neither Placid nor Forest was listed either in the *Fortune* tabulation or in *Moody's Industrials*[9]). In addition, two groups of companies or combines made in excess of five bids. Continental, Atlantic Refining, Tidewater, and Cities Service made up the most active combine with twenty-six bids submitted; the Pure, Standard of Ohio, and Sun group submitted fourteen bids. The aggregate assets of each of these combines were also quite respectable. Of course, many other assets than those directly related to production are included in total asset figures; however, these companies and combines apparently do have the means to diversify and to obtain information for intelligent diversification.

In summary, each company or group seemingly has the resources to invest in enough different production opportunities so that it can realize the full benefits possible from available diversification.[10] However, the available diversification is limited because of the high cost of information and evaluation, and the benefits of diversification are limited because of positive interdependencies of returns from tracts within provinces, and even among provinces.

2. The Allocation of Money to a Province

The "Gulf Coast of the United States" is listed by Levorsen as a petroleum province.[11] The continental shelf area off the coast of Louisiana is, of course, only a part of this province. Suppose a new wildcat sale is announced by the Bureau of Land Management. How should a firm go about preparing a set of optimal bids?

Of course, a firm may not wish to bid at all. Actual bidding will be the result of a multistage process which can be terminated short of bidding. At the end of any step, the next step must appear desirable or the process will be terminated. For instance, a firm may decide that it is not interested in the offshore Louisiana area; that is, the company believes, a priori, that it has better investment possibilities elsewhere. To make a further investigation into the possibilities of the area is felt to be an unwarranted expense. Even though a company is initially interested, it may still decide, after gathering information, not to bid. In this case the firm felt the expense of gathering and interpreting information to be justified by the prospects which might have been uncovered, but sufficiently attractive prospects did not appear. Also, a firm may not wish to investigate all the tracts offered. This may occur because of budgetary or time constraints, or because the company feels that only a subset of the tracts are worthy of investigation.

Suppose, however, that a firm has investigated at least some of the tracts and found a number with positive expected values. What then? First, it would seem very pertinent for the company to evaluate the potentialities of the tracts with positive expected values in terms of how the acquisition of some of them would tend to affect the firm's overall investment position with respect to its desired goals. Particularly to be considered are the company's current activities or lack of activities in the area; are there potential economies of operation available on the one hand, or is there danger of overconcentration on the other? What other alternative uses are available for the money which might be spent? How badly is the potential production needed? And, especially, how are the expected returns from this area related to those from other areas? These and similar questions are apt to sound vague, but they must be dealt with if the company is to proceed on a unified (and thus risk-reducing) plan rather than on a piecemeal basis. At this stage the ultimate decisions will depend on the experience and goals of the management. The task of the corporate staff is to supply the active management with views of the alternative investment combinations which are as clear as possible, so that the managers can make well-informed decisions.[12]

If a firm decides to bid at all, it may wish to win some minimum acreage or number of tracts. There may be some economies of scale available from supplying and operating several drilling platforms in the same general area. Also, most integrated oil companies seem to have a strong desire to produce as large a fraction as is economically feasible of their refinery crude requirements. While a firm may be able to trade crude produced far from one of its refineries for crude produced nearby, the number of provinces in which a given company may operate is not unlimited and goals may be assigned to each area in an attempt to reach self-sufficiency. To reach such goals, a certain amount of acreage will need to be tested. In addition, the wish to keep competitors from obtaining too many desirable tracts may be important. Considerations like these will also help determine the amount of money which should be allocated to an area for the purchase of leases at a sale.

3. The View-Span of the Investor

A basic problem of many discussions of investment under uncertainty, including that of the previous chapter, is that the term "available investment" is never properly defined. An idea of the view-span of the investor is necessary in order to think about the smallest investment or item of investment that he will ordinarily consider. The president of a multi-department corporation may consider the individual departments as his individual investment opportunities; at the most, he will probably not concern himself regularly with anything less than the more important projects at the department level. The department manager, in turn, must be concerned with all his projects, but will not wish to get involved in the detailed administration of each of them; his view-span is mostly limited to the individual projects. The project manager is concerned with how best to allocate funds among alternative ways for proceeding with the project, the foreman with how best to perform the particular task he has been assigned, and so on.

Thus, a very real problem in a large company is how far down the overall company investment planning procedure should go. Regardless of the level chosen, there will generally be those below this level with the authority to make some kinds of investment decisions. Presumably, the best advice that can be given to these people is to do the best they can with what they have, i.e., to try to maximize the profit from the project they are working on from their viewpoint of that project. Down to what level should the corporate staff of a large petroleum company plan investment activities in detail? Perhaps only down to the basin or subprovince level. If so, then within such a region as the Louisiana offshore area, constrained expected profit maximization might well be an optimal policy.

4. An Areal Bidding Strategy

There are at least three kinds of arguments for proposing expected profit maximization as a goal within an area. The first sort of argument is that the opportunities for risk-reducing diversification within an area may be limited. This contention is discussed in section 1 of this chapter. Second, as explained in the immediately preceding section, the view-span of the corporate staff limits the area down to which concern about possible inter-areal relationships of outcomes of separate investments may go. Finally, the profit maximization model developed in chapter 6 does in some sense take risk into account. For a given expected value for a tract, this model requires the optimal bid to be lower, the larger the expected dispersion of competitors' bids. A major source of the expected variation in competitors' bids appears to be differences in opinion about the expected value of the tract, such differences reflecting a general uncertainty about the tract value. Thus, if the model developed is a useful one, the uncertainty connected with the evaluation of the expected value of a tract is, in a sense, taken into account and the effect of the uncertainty is to cause a lower bid to be offered. This lowering of the bid is not a risk discount, but has a similar effect.

Therefore, the following company strategy for purchasing leases is proposed: (1) on the basis of the economically available information, the firm should allocate money to all areas in which it is interested (such as the Louisiana offshore region) in such a manner as is expected to produce that available combination of expected return and risk which is most satisfactory to the top management; (2) within each area, the goal of profit maximization subject to applicable constraints should be pursued. One such constraint, the desire to win a specified minimum acreage, has already been mentioned in section 2 of this chapter. Other possible constraints will be discussed in the following chapter, in which the elementary mathematics of constrained profit maximization for bidding at a lease sale will be presented.

III

The Development of a Bidding Strategy

1. An Abstract Bidding Strategy

SUPPOSE THAT the goal of a company is strict profit maximization. The firm is preparing to submit a bid for a valuable object at a closed bid auction. There exist no constraints on the amount which may be bid or on the size of the return expected. Let R be the firm's estimate of the value of the object, X an amount which might be bid for the object, and P(X) the probability the bid of amount X will be the winning bid. Define the profit to the company if it wins as R — X, the difference between the estimated value and the cost of the object. The expected profit from a bid of amount X is thus (R — X) P(X) and the goal of the firm is to maximize this product.[1]

Let E(X) equal this expected profit from the bid of amount X. The first order conditions for maximizing E(X) require that

$$\frac{dE(X)}{dX} = \frac{d[(R - X)\ P(X)]}{dX} = 0. \tag{3.1}$$

If P′(X) is used to stand for $\frac{dP(X)}{dX}$, then (3.1) can be written

$$R\ P'(X) - X\ P'(X) - P(X) = 0 \tag{3.2}$$

or

$$R = X + \frac{P(X)}{P'(X)}. \tag{3.3}$$

It is not possible to solve (3.3) directly for X unless P(X) can be written as an explicit function of X. It may be possible, however, to approximate the optimum X if P(X) can be estimated for a sufficiently wide range of values for X.

If bids for a number of objects must be submitted simultaneously and there are no constraints on the total amount which may be bid, then the optimum bid for each object may be determined through the use of equation (3.3). If constraints exist, a solution may be obtained from the application of Lagrangian multipliers. This method sometimes yields results which are easy for an economist to interpret. Suppose the only constraint requires the sum of all bids to equal a given amount C. Let E_T equal the sum of the expected profits from the bids which are placed on each of the objects up for sale. Then the constraint may be written,

$$C - X_1 - X_2 - \ldots - X_n = 0$$

and the function

$$V = E_1 + E_2 + \ldots + E_n + \lambda(C - X_1 - X_2 - \ldots - X_n)$$

formed, where λ is an undetermined Lagrange multiplier. The first order condition for maximizing E_T may now be obtained by partially differentiating V with respect to $X_1, X_2, \ldots, X_n$ and λ and setting each partial differential equal to zero. Inspection of the resultant system of equations shows that

$$\frac{\partial E_i}{\partial X_i} = \frac{\partial E_j}{\partial X_j} = \lambda. \tag{3.4}$$

This requirement is the very familiar equalization of marginal returns. In this context, the total expected profit will be maximized when the marginal expected profits from the last money unit bid on each object are all equalized, provided the required second order conditions are satisfied; λ is the expected marginal profit from the marginal money unit bid.

If company constraints also require a minimum expected profit from every dollar bid, the Lagrangian multiplier method is still capable of yielding a solution, since it is possible to use more than one constraint. However, it is quite possible that all constraints cannot be satisfied simultaneously. In such a case, the constraints can be varied until either an acceptable solution is found or it becomes apparent that the firm should make no bids at all on this collection of objects. Linear programming may be useful in complex problems.

Other kinds of constraints, such as that which requires the expected amount actually spent to equal some specified amount,

$$\sum_n P(X_n)\, X_n = C',$$

may also be treated by this method, but the interpretation may not be so straightforward.

2. An Analogy to Monopsony Theory

An interesting analogy to equation (3.3) can be found in the theory of monopsony. Consider a very simple case in which a firm produces a product Y using only one input, Z; the production function is linear and Y and Z are measured in such units that $dY/dZ = 1$. The price P_z of Z is a function of the quantity of Z which the firm purchases per time period. The product selling price P_y does not depend on the amount of Y produced and sold. Under these assumptions the firm's profit π per time period may be written

$$\pi = P_y Y - P_z Z. \tag{3.5}$$

The firm wishes to employ that quantity of Z which will enable it to maximize its profit. The first order conditions for maximizing π require that

$$\frac{d\pi}{dZ} = P_y - P_z - Z\frac{dP}{dZ} = 0, \text{ since } \frac{dY}{dZ} = 1.$$

Rearranging,

$$P_y = P_z + \frac{Z}{dZ/dP} \cdot \tag{3.6}$$

Now X is the amount offered in the bidding model; it corresponds to P_z, the price of the input in the monopsony model. Similarly, R corresponds to P_y. P(X) may be interpreted as a quantity. Of course, if the probability of winning is, say, one-third, this does not mean that the bid will surely purchase one-third of the object. But if one can imagine the auction being repeated a large number of times under identical circumstances, then one would expect to win one-third of the time. The expected number of objects purchased per auction is thus one-third; this is the sense in which P(X) may be thought of as a quantity. Since the object itself undergoes no transformation simply by being auctioned off, P(X) corresponds both to the quantity of input bought and the output sold; this is the reason for assuming that $dY/dZ = 1$ in the monopsony model. Thus, the analogy between (3.3) and (3.6) is shown. In either case, the value of a unit of output (R or P_y) equals the price offered for a unit of input (X or P_z) plus the quantity of the input (P(X) or Z) divided by the rate of change of the quantity of the input with respect to the input price (dP(X)/dX or dZ/dP). The sum of the terms to the right of the equals sign in (3.6) has been called the marginal resource cost;[2] it is the rate of change of total cost with respect to changes in the quantity of input purchased. Thus the firm's profit is maximized when the marginal resource cost is equal to the value of a unit of output, provided, of course, that the second order conditions are satisfied.

This analogy is introduced in order to show something about the relationship between the competitive bidding model developed here and conventional nonprobabilistic price theory. A profit maximizing firm preparing to bid at a sealed bid auction has much the same motivation as a firm purchasing an input in a monopsonistic market. A bidder can increase his probability of winning (i.e., the expected quantity purchased) only by increasing the amount bid (and thus paid if the bid is accepted); the monopsonist can purchase greater quantities of the input only if he is willing to pay a higher price for all units of the input.

3. Competitive Bidding Useful Only in Markets in Which Price Is Uncertain

There must be some uncertainty about the value of the offered object in order to provide incentive for an auction. If there exists complete agreement among potential bidders about the value of an object, then the seller need not go to the trouble of holding an auction; he can realize as much by selling to an arbitrarily selected buyer at the commonly agreed price.

There are at least two reasons why the seller may be uncertain about the price which can be charged for an object. First, the offered object may be a "pig in a poke" in the sense that its physical characteristics are only imperfectly known at the time the bids are submitted. This lack of knowledge may be due to the reluctance of the party offering the object to having it thoroughly examined, or it may result from a combination of the high cost of accurate information and the difficulty of keeping discovered information secret. If important information about the object is costly but hard to keep secret, the potential buyer may prefer to delay detailed evaluation until after he has purchased the object. Otherwise, he may find himself in the awkward position of supplying costly information free to his competitors; he may have spent money that has not purchased commensurate competitive advantage.

Second, the value of durable objects is dependent upon conditions which will prevail in the uncertain future. Thus, value estimates for durable objects are dependent upon assumptions about the future. Even if two parties agree in every detail about the present physical characteristics of the object, their estimates of its value may be quite different because of divergent expectations.

For these reasons, the seller may not know the highest price at which he could sell the object. This uncertainty may make some sort of auction sale attractive.

If the seller decides to dispose of the object by means of a sealed bid auction, an additional source of uncertainty enters—the bidding strategies of those who might be interested in purchasing the object. A bidding strategy may be defined as a procedure for arriving at a decision about the optimum amount to bid which takes into account not only the estimated value of the object, but also some assumptions about the expected behavior of the other bidders.

The fact that all bids are not expected to be identical means that over a range of bids it will not be possible to predict with virtual certainty whether or not a particular bid will win. At best, only a probability that a given bid will win may be associated with that bid. This probability of winning has been denoted as P(X).

4. Assumptions About Competitor Behavior

The probability that a given bid will win is obviously a function of the expected behavior of each potential competitor. Upon what does the expected behavior of each competitor depend? Upon the information he believes he possesses about the object, his expectations about the future, and his beliefs about the probable actions of *his* competitors.

In order to discuss competitor behavior it is often necessary to consider simultaneously the decision problems of several bidders; therefore, it will be convenient to adopt the convention that firm O is the "client" firm on which the analysis is directly focused. The actions of other bidders will be important because of their influence on the optimal course of action for firm O; when necessary for clarity, symbols may be subscripted with appropriate firm numbers.[3]

There are two extreme types of hypotheses about competitor behavior which the client might adopt. On the one hand, firm O may consider that its opponents will generate their bids by means of processes which are independent of the strategies of other bidders. An example of such a bid-generating process is described in the following quotation:

Some land men representing large oil companies contend that one should bid an amount which will yield a normal long-run profit adjusted for competitive realities and then not be too concerned about either losing to a higher bid, or winning and leaving a substantial sum of money on the table.[4] These are two hazards which must be accepted as part of the sealed bidding procedure.[5]

In other words, bids should be determined strictly on the basis of what the tract in question is expected to be worth to the bidder; the behavior of competitors is not to be considered. Firm O might think that all other bidders act in this way. Presumably, the reason they would so act is that they feel they do not have significant information for predicting possible behavior of opponents.

On the other hand, the client firm may believe that *each* participant in the bidding expressly considers his opponents' probable actions as part of his own analysis of the problem. This is the assumption which is commonly made in game theory analyses. Christenson devotes chapters of his book to the study of the implications of both hypotheses.[6] A strict game theory approach to the problem of bidding for offshore petroleum does not appear promising to this writer and will not be further pursued in this monograph. But strategies which do not explicitly consider the reactive behavior of competitors in certain circumstances suffer from a grave defect. It is necessary to understand why this problem may exist in order that it may be avoided.

If firm O adopts the first hypothesis about competitor behavior—and thus believes that it is the only bidder who is explicitly considering the probable actions of its competitors—it will be very advantageous for it to keep the fact that it is using a bidding strategy secret. If the client firm does not keep its strategy secret, its opponents may now find that they do have some information which will help in determining the probable action of firm O; if so, they would be foolish if they did not use this information in preparing their own bids. If they do use this information, it will change the distribution of possible competitor actions which firm O expects; thus the bid which would have been optimal for the client is now no longer optimal, and the client's advantage from the use of the strategy will be lessened if not negated. These considerations may help explain why petroleum firms tend to be very secretive not only about tract geological information but also about the procedure by which their bids are derived from the geological and other information available.

However carefully a firm may keep this information secret, if it consistently follows a strategy alert competition should soon be able to discern the general strategic pattern:

> With the large dollar amounts involved, every potential competitor must be in possession of certain geological and geophysical information regarding the tracts of interest. Most of this information is obtained by the same geological and geophysical procedures, and it will be a safe assumption, therefore, that most of the competition is looking at the same basic data.[7]

The firm's competitors are likely to know at least some of the data on which its decisions are based; therefore clever competitors should be able to find out something about the process which converts the raw data into actual bids. In sum, a firm with keen and intelligent competitors cannot hope that they will stand by and watch it reap the benefits of a good bidding strategy without reacting themselves. Therefore, a bidding strategy for a firm which repeatedly bids against nearly the same set of competitors should take this reaction into account.

Specifically, if the competitors have similar assets and goals, there may exist an equilibrium strategy which it is optimal for *each* competitor to follow. This equilibrium would presumably be arrived at after a "learning" period forced by competitive pressures and based on the reactions to actions of the participants.

Of course, there may exist periods in which one or more firms have a temporary advantage which can be exploited. For instance, one firm may discover a method for gaining better information from geophysical exploration records. One firm might develop a better bid formulation procedure while others are still using techniques which do not so efficiently use the available information. Or, one or more firms may be temporarily operating with either unusually high or low amounts of available capital. If a firm does have special information about the likely strategies of its competitors, this information should certainly be taken into account in the optimum bid generating procedure.

Yet expecting a competitor's strategy to be markedly different from one's own can be a dangerous procedure. Assessing the degree of skill which an opponent will play has always been an important problem in game theory. A quite useful assumption has been that the opponent pursues consistently a strategy which is optimum for him. To expect otherwise is a reflection on the opponent's intelligence and competitive ability. But if all who bid are "equals" in assets and goals, then what is optimal for the opponent will also be an optimal strategy for the client firm! Though sometimes useful, assumed strategy differences should be used with caution. The model developed in chapter 6 is based on the assumption of equal competitors; the optimal strategies for all the bidders are identical, though the offered bids are not necessarily the same because each firm probably estimates the uncertain value of the tract as a different amount.

5. A Difficulty Inherent in a Previous Model

Of course, a basic difficulty in trying to implement the strategy outlined in the first section of this chapter lies in estimating P(X). Two widely referenced articles about bidding strategies unfortunately contain some potentially misleading statements about how P(X) may be estimated when there will probably be more than one competing bidder.[8]

There are, of course, two common kinds of sealed bidding situations. The first type may be called purchaser bidding; prospective purchasers submit bids for a valuable object. The highest bid wins. The second type may be called seller bidding. In this case, someone who desires to purchase a specified collection of goods or services asks for bids from potential suppliers. The qualified supplier who bids the lowest amount wins the purchase agreement.

The theory discussed in this monograph is all developed in the context of purchaser bidding. It is obvious, however, that the same basic equation,

$$E(X) = (R - X)\,P(X), \tag{3.7}$$

can serve both cases by interchanging R and X in the parenthesis and redefining R. In purchaser bidding (R − X) represents the expected profit if the bid of amount X wins; in seller bidding (X − R) represents expected profit conditional on X being the winning bid, if R is redefined to be the cost of producing the required collection of goods or services. Thus, the basic theory for optimal seller bidding will formally be little different than that for purchaser bidding. There is thus no necessity for exhaustively discussing both types.

In his article in *Operations Research* Friedman develops his theoretical formulation in the context of seller bidding.[9] Hanssmann and Rivett, who base their development on that of Friedman, use the context of purchaser bidding.[10] The following discussion of the method for estimating $P(X)$ espoused by these authors is made for the familiar purchaser bidding situation.

Suppose, these authors argue, that the client firm knew that there would be only one other firm bidding against it, and that further it could somehow estimate the probability density function of its opponent's bid, $f(X_1)$. In order for firm O to win, its bid X_0 must be higher than that of its competitor. The probability that X_0 will be greater than X_1 is

$$P(X_0) = \int_0^{X_0} f(X_1)\ dX. \tag{3.8}$$

In other words, the probability that X_0 will win is equal to the fraction of the area of the probability density function for X_1 which lies below X_0. This result follows clearly and properly from the assumptions.

However, suppose the client firm knows there will be n competitors bidding against it, but can estimate density functions $[f(X_1), f(X_2), \ldots, f(X_n)]$ for each of the competing bidders. Friedman concludes that the probability of winning with a bid of X_0, "when the competitors are known, is simply the product of the probabilities of defeating each of the known competitors."[11] In formal terms, this statement asserts that[12]

$$\text{Prob}\ (X_0 > X_1, X_0 > X_2, \ldots, X_0 > X_n) = [\text{Prob}\ (X_0 > X_1)]\ [\text{Prob}\ (X_0 > X_2)] \ldots [\text{Prob}\ (X_0 > X_n)]. \tag{3.9}$$

This result is valid only if for all $0 < i \leqslant n$, $0 < j \leqslant n$, and $i \neq j$[13]

$$\text{Prob}\ (X_0 > X_i \mid X_0 > X_j) = \text{Prob}\ (X_0 > X_i). \tag{3.10}$$

Equation (3.10) will not in general be valid; if it is known that $X_0 > X_i$, it will typically be more likely that $X_0 > X_j$ than if no information about the relative magnitudes of X_0 and X_i is known. Friedman's statement is true only if the frequency distribution functions $[f(X_1), f(X_2), \ldots, f(X_n)]$ are completely independent of each other; the more usual situation will be that one or more of the parameters of each frequency distribution function are dependent upon the observable characteristics of the collection of goods or services on which the bid is being made and the current "state of the world." In such a case, X_i and X_j will not be unconditionally independent of each other and equations (3.9) and (3.10) will not be valid, though X_i and X_j may well be mutually independent *given their joint depen-*

dence on the characteristics of the object of the bidding and current environmental conditions.[14] The possibility of dependence between X_i and X_j should be explicitly taken into account in formulating the bidding strategy; otherwise, paradoxical results may be obtained.

This may perhaps be made clearer by an analogy. Imagine a handicapped horse race with three entrants—horses A, B, and C. Suppose that the handicapping had been done with such skill that the owner of each horse thought that his horse had a probability of exactly ½ of defeating either of the other horses. Using the symbolism $A > B$ to denote horse A crossing the finish line sooner than horse B, each owner thus would agree on the following probabilities:

$$\text{Prob}\ (A > B) = \tfrac{1}{2}, \qquad \text{Prob}\ (A > C) = \tfrac{1}{2},$$
$$\text{Prob}\ (B > A) = \tfrac{1}{2}, \qquad \text{Prob}\ (B > C) = \tfrac{1}{2},$$
$$\text{Prob}\ (C > A) = \tfrac{1}{2}, \qquad \text{Prob}\ (C > B) = \tfrac{1}{2}.$$

Suppose now that each owner wishes to calculate the probability that his horse will win the race. The owner of A, if he followed Friedman's method, would calculate the probability of his horse winning the race as

$$\begin{aligned}&\text{Prob}\ (A > B,\ A > C)\\ &= [\text{Prob}(A > B)]\ [\text{Prob}\ (A > C)]\\ &= (\tfrac{1}{2})\ (\tfrac{1}{2}) = \tfrac{1}{4}.\end{aligned}$$

Similarly, the owner of B would calculate the probability of his horse winning as ¼, as would C. But clearly this method cannot be correct, for the sum of the individual probabilities of winning ought to be 1. Of course, a correct expression for the probability of horse A winning is

$$\begin{aligned}&\text{Prob}\ (A > B,\ A > C)\\ &= [\text{Prob}\ (A > B)]\ [\text{Prob}\ (A > C \mid A > B)].\end{aligned} \tag{3.11}$$

$\text{Prob}\ (A > C \mid A > B)$ may be evaluated by examining the list of possible finish orders, all of which are equally likely according to the assumptions: (1) $A > B > C$, (2) $A > C > B$, (3) $B > C > A$, (4) $B > A > C$, (5) $C > A > B$, and (6) $C > B > A$. If it becomes known that $A > B$, then finish orders (3), (4), and (6) are not possible. In two of the remaining possible finish orders $A > C$; therefore, $\text{Prob}\ (A > C \mid A > B)$ equals ⅔. Then, using (3.11),

$$\text{Prob}\ (A > B,\ A > C) = (\tfrac{1}{2})\ (\tfrac{2}{3}) = \tfrac{1}{3}.$$

In a perfectly handicapped race between three horses, the probability of a given horse winning is ⅓.

It is also important to note that (3.11) is not the only formula by which Prob $(A > B, A > C)$ may be calculated. The following formula is equally valid and will give the same numerical result:

$$\begin{aligned}&\text{Prob } (A > B, A > C)\\ &= [\text{Prob } (A > C)]\,[\text{Prob } (A > B \mid A > C)].\end{aligned}$$

This analogy is introduced to help make clear why equation (3.9) might not be correct and to show a situation in which Friedman's prescription might be misleading. Clearly, in this example the probabilities of defeating each competitor individually cannot just be multiplied in order to get the probability of defeating both.

Returning to the bidding context, one correct expression for the probability of the client firm winning with the bid X_0 over the bids of n competitors is

$$\begin{aligned}&\text{Prob } (X_0 > X_1, X_0 > X_2, \ldots, X_0 > X_n)\\ &= [\text{Prob } (X_0 > X_1)]\,[\text{Prob } (X_0 > X_2 \mid X_0 > X_1)] \ldots\\ &[\text{Prob } (X_0 > X_n \mid X_0 > X_1, X_0 > X_2, \ldots, X_0 > X_{n-1})]. \qquad (3.12)\end{aligned}$$

The right hand side of (3.12) can no doubt be simplified somewhat for many practical problems.

6. Estimating Probability Density Functions of Competitors' Bids

In order to complete the development of a bidding strategy, it is necessary to find some way to estimate the probability distributions of competitors' bids, since these distributions are necessary for estimating $P(X_0)$, the probability that a bid of amount X made by the client firm will win. Fortunately, an extensive body of data exists for a particular bidding situation. These data make it possible to test statistically hypotheses about bidding behavior and to infer the a priori probability distributions of competitors' bids. These data are the records of the bids for petroleum leases for tracts on the federally owned outer continental shelf off the Louisiana coast. Before describing the hypotheses and the results of the statistical tests, it is necessary to discuss the setting in which these bids were generated.

IV

Petroleum Leasing on the Louisiana Outer Continental Shelf

THE CONTINENTAL SHELF is a submarine plain which borders nearly every continent. Off the Louisiana coast, the downward slope of this plain is relatively gentle. For instance, the ocean depth 90 miles south of Cameron, Louisiana is only about 120 feet; 20 miles south of Grand Isle, the ocean depth is about 130 feet. There is no sharp geological discontinuity marking the Louisiana shoreline. Therefore, when oil and gas discoveries were made near the coast, there was every reason to expect that similar discoveries might be made offshore. However, technological difficulties connected with drilling, production, and transportation were so severe that only a very few wells were drilled off the coast prior to World War II, and these were in shallow water. At this time the state of Louisiana claimed ownership of all its offshore lands. Just prior to the close of the war, farsighted members of the Louisiana Mineral Board and its staff saw that technological improvements would likely be made which would allow drilling and production in ever deeper water at reasonable costs. They therefore set about establishing a "Modern Leasing Program" to systematize leasing procedures. The offshore lands with a water depth of roughly 120 feet or less were divided into twelve areas, each of which was further subdivided into 5,000-acre blocks.[1] Such blocks were the largest tracts which could be covered by a single lease. Each 5,000-acre block was further subdivided into 64 equal areas.[2] Such a 78.1-acre tract was the smallest area which could be leased.

Between August, 1945, when the "Modern Leasing Program" was inaugurated, and October, 1948, approximately seven hundred leases were granted. In November, 1948, all offshore leasing under this program was suspended after the Department of the Interior claimed that all lands more than three miles from shore belonged to the federal government. Louisiana immediately made a counterclaim of ownership out at least three leagues (10.35 miles) from the coast. Eventually the federal courts ruled in favor of the Department of the Interior. Immediately, however, a new dispute arose, this time about the location of the shore. In many places there is no definite seacoast; there is only a brackish marsh in which the average water depth increases slowly as one moves toward the sea. The Department of the Interior claims one coastline; the state of Louisiana claims another, which is in some places as much as twenty miles from the federally defined shore. The dispute has not yet been finally settled. The Louisiana outer continental shelf lands are thus divided into four zones. Zone 4 lands are conceded by both parties to be under federal control. The area in Zone 3 is also under effective federal control, but would come under state control if Louisiana's claim for control of a three-league strip offshore should be revived. Zone 2 lands are claimed by both parties because of the dispute in regard to the shoreline.

In 1954 the state resumed granting leases on Zone 1 land in accordance with the "Modern Leasing Program," and the Bureau of Land Management, a division of the Department of the Interior charged with the administration of public lands, began its own leasing program. At present, the Bureau of Land Management has jurisdiction over Zones 2, 3, and 4, though all proceeds from Zone 2 tracts are being held in escrow pending settlement of the ownership dispute.

The Outer Continental Shelf Lands Act[3] established the leasing procedures to be used by the Bureau of Land Management. The lease blocks designated by Louisiana were taken over intact and new, similar ones added to extend the coverage out to a depth of about five hundred feet. A lease conveys the right to drill wells on the property and to produce petroleum in conformity with regulations issued by the Bureau of Land Management and the Louisiana Department of Conservation. The lease continues for five years and as long thereafter as oil or gas is produced in commercial quantities or approved drilling or well-reworking operations are being conducted.

Permits for geophysical exploration of unleased tracts are freely granted. When a firm discovers a tract on which it might like to bid, it may nominate this tract to the Bureau of Land Management for inclusion in the next sale. Sales of offshore lands have occurred at irregular intervals. The Bureau of Land Management may also nominate tracts which are adjacent to those on which production is being obtained; otherwise petroleum lying beneath these tracts may be drained away without additional lease bonus payments to the Department of the Interior.

Lease payments are of three distinct types. The lease bonus is a sum of money which is paid at the time the contract is consummated. Rentals are annual payments which must be paid until the lease expires or is surrendered back to the leasing authority, or until commercial production is obtained. This system of rental payment is designed to provide an incentive to the lessor to test the lease as soon as possible and either to get into production or to surrender the tract back to the state quickly, while leaving the lessor some flexibility of operation. Royalty payments are calculated as percentages of the total well-head value of the oil and gas produced. The Outer Continental Shelf Lands Act specified that the bid variable might be either bonus or royalty, but not both, with the royalty in no case less than 12½ percent. For all sales to date, the secretary of the interior has prescribed lease bonus as the sale variable and set the royalty demanded at 16⅔ percent. The secretary of the interior, on the advice of the Bureau of Land Management, has the right to reject all bids for a tract if none is deemed sufficiently high. Out of the 786 tracts on which offers have been received, all bids have been rejected on 40 tracts. The Outer Continental Shelf Lands Act does not specify an exact procedure for such rejections. Most appear to have occurred on tracts adjacent to producing tracts.

The staff of the Bureau of Land Management has a policy of not recommending acceptance by the secretary of the interior of any lease bonus bids of less than $15 per acre. It is clear that the participants in the bidding understand this policy, as no bids less than $15 per acre have been offered. This minimum is designed to make outright speculation expensive. If there were no minimum, leasing tracts for which current prospects seem very poor might appear attractive through the hope that improved exploratory techniques might make some of these apparently worthless tracts valuable.

The Outer Continental Shelf Lands Act also provides for annual rentals to be set by the secretary of the interior at the time of the lease offering. Rentals demanded have ranged up to $3 per acre per year, but the average has been lower than this amount. When compared with average lease bonuses and drilling costs, these rentals appear small, and it seems unlikely that they really provide much incentive for early drilling.

The New Orleans office of the Bureau of Land Management also handles the lease sales of federally controlled lands off Texas, Mississippi, Alabama, and Florida coasts, and has published summaries of all sales to date. No bids have been received for tracts off Mississippi or Alabama. Only one sale has been held for Florida offshore lands. At this sale, held May 26, 1959, twenty-three tracts were bid on, but in no case was there more than one bid per tract. Thus, these data cannot be used for testing hypotheses about the distribution of bids on a tract. Considerable interest has been shown in Texas outer continental shelf lands, and tracts in this area have been offered at several different sales. However, these bids were not included in the data used to test hypotheses about bidding behavior and the functional form of $P(X)$ because the Texas offshore area has not proved nearly so productive as that of Louisiana and the average per acre winning lease bonus bid changed over time from \$347 at the sale of November 9, 1954, to \$19 at the March 16, 1962, sale. A similar decline did not occur for the Louisiana offerings; therefore it seemed best not to mix the two sets of data.

Only bids received for Louisiana outer continental shelf lands from October, 1954, through April, 1964, have been included in the data for statistical analysis. Table 1 provides a summary of the various sales included. In all cases but one, tracts on which all bids were rejected were included in the data on the hypothesis that these bids were as legitimately offered as any others. Of course, these bids generated no receipts to be included in the total bonus figure. The one exception is the sale of August 11, 1959. Here, there was only one bid on each of the nine tracts for which the bids were rejected by the secretary of the interior; these nine bids have been deleted from the analyzed data. After the October 13, 1954, sale, officials at Kerr-McGee found that they had misinterpreted their maps and placed winning bids on seven tracts they did not want. After an investigation the Bureau of Land Management allowed Kerr-McGee to retract these bids, which have also been omitted from the tested data. Finally, in the February 24, 1960, sale record one bid is reported for tract 779 and two bids for tract number 780, but both tracts have identical location identifiers. It seems reasonable to suppose that one of the location identifiers is in error; however, the safest course seemed to be to delete these three bids also. Table 1 thus shows eight sales at which 2,350 bids were made on 777 tracts. The average tract size was 4,130 acres, and the average per acre lease bonus paid was \$361. In all, 3,078,000 acres were leased for a total bonus of \$1,109,000,000.

TABLE 1
SUMMARY OF LOUISIANA OUTER CONTINENTAL SHELF SALES

Date	No. Tracts Offered	No. Tracts on Which Bids Were Made	No. Tracts with More Than One Bid	No. Bids Received	Acreage of Leased Tracts (Thousands)	Total Bonus from Leased Lands (Millions of Dollars)	Ave. Leased Tract Size (Acres)	Ave. Per Acre Bonus (Dollars)	Remarks (See Text for Explanation)
10/13/54	199	90	59	327	395	116	4,390	294	Seven Erroneous Kerr-McGee Bids Omitted
7/12/55	171	94	65	351	253	100	2,690	396	
8/11/59	38	19	13	47	39	88	2,050	2,270	Nine Rejected Bids Not Included
2/24/60	288	125	71	336	464	247	3,710	532	Three Bids on OCS Tracts Nos. 799 and 780 Omitted
3/13/62	401	212	121	538	951	177	4,490	186	
3/16/62	380	200	133	656	927	268	4,640	288	
10/ 9/62	19	14	6	26	16	44	1,140	2,710	
4/28/64	28	23	19	69	33	60	1,430	1,850	
Total	1,524	777	487	2,350	3,078	1,109	4,130	361	

SOURCE: U.S., Department of the Interior, Bureau of Land Management, *Bid Recap of OCS Sales* (New Orleans, mimeographed, undated).

The sales of August 11, 1959, October 29, 1962, and April 28, 1964, are of a somewhat different character from the others. They are distinguished by a smaller number of tracts offered and bid on, and a much higher per acre bonus paid for the tracts leased. These are called "drainage" sales; the tracts offered are adjacent to currently producing tracts. This proximity accounts for the higher per acre bonus paid. All the other sales are called "wildcat" sales. Here, the tracts offered are not usually contiguous to current production. At drainage sales, the owner of the adjacent producing lease has access to information gathered during the drilling and operation of the producing wells which will not be known to others, who must rely almost exclusively on geophysical data. This extra information may include an estimate of the thickness of the producing strata, the reservoir pressure, and usually hints about the total volume of oil and gas contained in the deposit and the directions in which the deposit probably extends. Thus, bidding by others than the owner of the adjacent producing property is especially risky, for it means bidding against another who probably has a more accurate estimate of the value of the tract. The drainage sales were included in the data subjected to statistical analysis to find out whether the bids followed the same pattern as those in wildcat sales.

The Department of the Interior has chosen to use sealed rather than open bidding for the Louisiana outer continental shelf lands. At open bidding, strategy often dictates that participants initiate the bidding at levels considerably below the maximum they are willing to offer. As the bidding progresses, each party may actively participate in the bidding until a competitor's offer exceeds the maximum he is willing to pay; then he drops out. As the bids continue to increase, eventually a bid will be made which no one is willing to raise. This bid wins. How high, then, has the price been bid up? To the maximum that any participant is willing to pay? Not necessarily. The winner may have been prepared to offer much more than he actually paid, but all that was required to win was an offer slightly in excess of the maximum anyone else was willing to make. Thus, in an open bidding situation, the amount realized is just in excess of the *second highest* amount anyone is willing to offer.[4]

In a closed bidding situation, the amount realized will be that of the highest bid. There is no assurance, however, that the bid a firm will make at a closed auction will be identical with the maximum bid it would make at an open auction for the same object. At a closed auction, it may well be more attractive to enter a bid below the maximum one would be willing to pay at an open sale in hopes of winning at a lower price. In fact, the profit maximizing model developed in chapter 6 is built around this idea. If one knows the value of the object to himself, can regard the bids of others for the object as random drawings from a known frequency distribution function, and can estimate the number of competing bids which will be entered, then one can determine a bid which will maximize the expected profit. The resulting optimal bid does not necessarily equal the maximum amount one would offer in an open bidding situation; it may be considerably lower.

Also, if more than one object is offered at a sale another important distinction arises. At closed bid sales, all bids on all offered objects are usually collected before the results of the bidding on any of the objects are announced. At open bidding, however, the objects are usually offered serially; that is, the public bidding for one object is completed before the bidding on the next is begun. In this case, optimal bidding strategy seems likely to become dynamic in the sense that the outcome of previous sales affects planned bids on yet unsold objects.

After an extensive study, Mead has concluded that it is not possible to recommend uniformly either open or sealed bidding as the most appropriate method for selling government-owned natural resources.[5] The structure of the industry which uses the resource, the degree of competition in the market for the resource, and the probability of tacit collusion among buyers are important determining factors in his analysis.

There appear to be two major reasons why the secretary of the interior has chosen to use sealed rather than open bidding for the Louisiana outer continental shelf lands. The first is an honest belief on the part of many concerned public officials that the aggregate expected revenue to the government from these sales will be higher under the closed bidding system.[6] These officials seem to think that, in general, the highest bid offered at a closed bid auction will be in excess of the second highest amount anyone would be willing to bid at an open auction. Second, closed bidding is typically a somewhat more orderly procedure, taking less time to conduct and being easier to control and administer.

V

Statistics of Louisiana Outer Continental Shelf Lease Bids

1. The Purpose of the Statistical Tests

THE PROFIT MAXIMIZING STRATEGY discussed in chapter 3 requires that $P(X_0)$, the probability that a bid of amount X_0 will win, be estimable over the range of feasible bids for each tract. In order to estimate $P(X_0)$ for a given X_0, the client must possess some beliefs about the probable bids of his competitors. If the bids of the competitors can be assumed to be random drawings from a population which is distributed according to some theoretical frequency function whose parameters are known, and the number of competitors who will bid is also known, then it may be possible to express $P(X_0)$ in a fairly simple form. If one or more of the parameters of the frequency distribution function of the number of bidding competitors is not known, but must be estimated, the determination of $P(X_0)$ becomes more complicated, but may still be possible. Therefore, in this chapter three major questions will be treated: (1) Can an acceptable frequency distribution function for the bids on a tract be found? (2) If so, how can the parameters of this frequency function be estimated? (3) How can the number of competing bids best be estimated?

2. The Distribution of Bids

The most striking feature apparent from a cursory examination of the bid data is the wide variation of bids on many of the tracts for which there was more than one bidder. Table 2 shows all the bids received for the first four tracts listed in the Bureau of Land Management summary of the sale of October 13, 1954, for which there were multiple bids.

Hypotheses about the form of the distribution of bids can be developed in two stages. First, one may speculate about the likely

TABLE 2
SAMPLE OF PER ACRE BIDS FROM SALE OF OCTOBER 13, 1954

Tract[a] Number 481		Tract Number 405		Tract Number 409		Tract Number 408	
Company	Amt. Bid	Company	Amt. Bid	Company	Amt. Bid	Company	Amt. Bid
CATC[b]	\$451.20	Shell	\$220.00	Shell	\$520.00	Gulf	\$800.60
Shell	223.00	Gulf	100.30	CATC	200.80	Shell	511.00
Gulf	100.30	CATC	40.40	Gulf	200.40	CATC	451.20
POS[c]	100.03			Cal Std	85.10	POS	350.03
Cal Std[d]	85.20					Cal Std	272.10

SOURCE: U.S. Department of the Interior, Bureau of Land Management, *Bid Recap of OCS Sales*, p. 1.
[a]Each tract had an area of 5,000 acres.
[b]CATC is the abbreviation for a consortium formed by Continental Oil, Atlantic Refining, Tidewater Oil, and Cities Service.
[c]POS is the abbreviation for a similar combination of Pure Oil, Standard Oil (Ohio), and Sun Oil.
[d]Standard Oil Company of California.

distribution of value estimates on a tract. Second, bidding strategy may be considered as a transformation of value estimates into actual bids. This transformation may either preserve the form of the distribution or alter it, depending upon the type of transformation and the original distribution of value estimates. One might, for example, assume the value estimates for a tract to be random drawings from a normal distribution, each company's estimate differing from the "true value" by some error term which tends to be normally distributed. The error term, in this context, may be considered to have components arising from three sources: (1) differences in interpretation of available geophysical information; (2) different expected contributions of the tract to each firm's current investment objectives; and (3) differences in expectations about the future. If one also assumes a uniform bidding strategy in which each company bids the same fraction of its value estimate, then the bids on each tract should appear to be random drawings from a normally distributed population, since the linear transformation of a normal distribution is still a normal distribution. After a hypothesis about the frequency distribution of the bids on individual tracts has been formed, the next step obviously is to test it with the data available. The hypothesis that the bids on a tract are normally distributed will not survive even an inspection for symmetry. The normal distribution is symmetrical; therefore, the expected number of observations greater than the arithmetic mean equals the expected number of observations less than the arithmetic mean. The distributions of bids on tracts are typically not symmetrical in this sense. Table 2 is illustrative of this asymmetry. The arithmetic mean per acre bid on tract 481 is $191.95; there are two bids above the mean and three below. For tract 405 the mean bid is $120.23; there is one bid greater than the mean and two less. There is only one bid above the arithmetic mean bid of $251.27 for tract 409 (and three below the mean); for tract 408 there are two bids above the mean of $476.99 and three below. For all the tracts in the sale of October 13, 1954, on which there were more than two bids,[1] there were 110 bids greater than the respective arithmetic mean tract bids and 166 bids less.

J. J. Arps has advanced the hypothesis that the bids on a tract tend to be distributed lognormally.[3] "The lognormal distribution may be defined as the distribution of a variate whose logarithm obeys the normal law of probability."[4] In his paper Arps advances no theoretical explanation why the bids might tend to be distributed lognormally; he offers lognormality as an empirical observation. However, a rationalization for this phenomenon can be fairly easily developed.

> It [the lognormal distribution] arises from a theory of elementary errors combined by a multiplicative process, just as the normal distribution arises from a theory of elementary errors combined by addition.[5]

Thus, "the distribution of the product of N independent random variables tends to lognormality as $N \rightarrow \infty$, under very general conditions."[6] In the process of arriving at a bid many different factors are considered. Is it not possible that these factors are subjectively combined multiplicatively rather than additively in arriving at a bid? Such a multiplicative process could account for the lognormal distribution of bids on a tract.

The hypothesis of lognormality was tested by the Kolmogorov-Smirnov test.

> The Kolmogorov-Smirnov one-sample test is a test of goodness of fit. That is, it is concerned with the degree of agreement between the distribution of a set of sample values [observed] and some specified theoretical distribution. It determines whether the scores [observations] in the sample can reasonably be thought to have come from a population having the theoretical distribution.
>
> Briefly, the test involves specifying the *cumulative* frequency distribution which would occur under the theoretical distribution and comparing that with the observed cumulative frequency distribution. The theoretical distribution repre-

sents what would be expected under H_0 [the hypothesis to be tested]. The point at which these two distributions, theoretical and observed, show the greatest divergence is determined. Reference to the sampling distribution indicates whether such a large divergence is likely on the basis of chance. That is, the sampling distribution indicates whether a divergence of the observed magnitude would probably occur if the observations were really a random sample from the theoretical distribution.[7]

In order to use the Kolmogorov-Smirnov one-sample test for goodness of fit to a lognormal distribution on each tract, it is necessary to know the mean and the variance of the theoretical lognormal distribution with which the actual distribution is being compared. If these parameters must be estimated from the observed distributions themselves, the sensitivity of the test is reduced. It is thus desirable to avoid estimating one or both of the parameters (the mean and the variance) of each theoretical lognormal distribution directly from each set of tract bids if possible. Therefore, it seems reasonable to investigate the problem of estimating these parameters under the assumption that the bids are lognormally distributed, hoping to avoid estimating both of the parameters from the individual observed bid distributions. In this manner, the Kolmogorov-Smirnov test can be made as sensitive as possible. Then, if the hypothesis of lognormality of bid distributions is not rejected, some of the information necessary for estimating $P(X_o)$ will already have been developed.

Suppose one assumes that the bids on a tract are indeed distributed lognormally. If one looks at all the bids offered at a sale, there exist two possible hypotheses about this set of bids. The first is that the bids on the individual tracts are not statistically distinguishable from each other; that is, that all bids offered at the sale may be regarded as random drawings from the same distribution. The second hypothesis is that the sets of bids offered for the individual tracts are statistically distinguishable from each other. There are three ways in which the second hypothesis may be true: (1) the means of the individual tract distributions might differ, but the variances be the same; (2) the variances might differ but the means be the same; and (3) neither the variances nor the means might be the same.

The hypothesis of equality of means among the various tracts offered at a sale is not very attractive, since it seems far more likely that the geophysical and other information available would lead to different mean bids on tracts with different prospects. However, it might be the case that the tracts offered at a sale tended to be of such similar quality that the difference among the tract mean bids would not be statistically significant. Such a finding would be of great importance in determining $P(X_0)$, since it would mean that a firm would have only to estimate the mean bid for the entire sale rather than the mean bid for each individual tract.

The F-test may be used to test the hypothesis of equality of means among various samples. Assuming a lognormal distribution, the appropriate hypothesis to be tested is the equality of the arithmetic means of the logarithms of the bids on each tract within a sale. The procedure for applying the F-test is discussed in many references, among them Richmond.[8] Application of the F-test uniformly led to the rejection of the hypothesis of equal means. The results are summarized in table 3.

Thus, when applying the Kolmogorov-Smirnov test it will be necessary to estimate the mean bid for each tract from the observed bids on that tract.

The hypothesis of constant variance of bids on tracts within sales is interesting because of its implications about the bid generating process. It implies that large bids do not tend to be proportionately more or less precise than small ones.[9] If all firms bidding on a tract have similar needs for the tract and are following the same bidding strategy, then the differing bids may be viewed as arising from differ-

TABLE 3
RESULTS OF F-TEST FOR EQUALITY OF MEANS AMONG TRACTS, ALL SALES, 1954-1964, ASSUMING LOGNORMAL DISTRIBUTION

Sale Date	Degrees of Freedom: Between Tracts	Degrees of Freedom: Within Tracts	F[a]
10/13/54	58	237	2.42
7/12/55	64	255	5.30
8/11/59	12	26	4.81
2/24/60	70	213	1.67
3/13/62	120	324	1.86
3/16/62	132	456	2.53
10/ 9/62	5	12	2.80
4/28/64	18	45	2.32

SOURCE: The entries in this table were calculated from data contained in U.S., Department of the Interior, Bureau of Land Management, *Bid Recap of OCS Sales*.

[a]All F values are significant at the .01 level except those for the 4/28/62 sale which is significant at the .05 level and the 10/9/62 sale which is not significant at the .05 level.

ing interpretations of commonly known information about the geophysics of the tract and the course of future events which will affect the tract returns.

> With the large dollar amounts involved, every potential competitor must be in possession of certain geological and geophysical information regarding the tracts of interest. Most of this information is obtained by the same standard geological and geophysical procedures, and it will be a safe assumption, therefore, that most of the competition is looking at the same basic data. The interpretation of such data, however, and the evaluation of a tract's potential value may vary over a wide range.[10]

It is possible that the uncertainties in evaluation may be roughly proportional to the expected value of the tract. This would be an explanation of constancy of variance of logarithms of bids on tracts within sales.

Hoel has discussed testing for the equality of variance among samples using likelihood ratio methods.[11] In particular, he has defined a variable with an approximate chi-square distribution and $k - 1$ degrees of freedom, where k is the number of samples being compared.[12] This formulation is particularly applicable to the problem at hand, since it is said to be accurate even though the sample sizes tend to be small. The results of applying this test to the bids at each sale are displayed in table 4. In no case was the value of chi-square large enough to reject the hypothesis of equality of variances within a sale at a significance level of 0.05 or lower.

Therefore, in computing the theoretical cumulative distribution which was compared to the actual cumulative distribution of bids on each tract, the mean of the theoretical distribution was estimated as the natural logarithm of the geometric mean of the bids actually received on the tract, but the variance used was the mean variance for the respective entire sale.[13] Thus it was feasible to avoid estimating both parameters of the theoretical distribution for each tract from the observations on that tract, and the sensitivity of the Kolmogorov-Smirnov test was preserved as much as possible. The "standard" Kolmogorov-Smirnov tables for the critical value of the maximum divergence between the theoretical and observed distributions were calculated under the assumption that none of the parameters of the theoretical distribution were estimated from the observed sample. However, Lilliefors has calculated critical values for use in testing whether or not a set of observations was likely to have come from a normal population when either the mean[14] or both the mean and the variance[15] of the theoretical distribution were estimated from the observed sample. During the period over which bids were analyzed,[16] 251 tracts received four or more bids.[17] The maximum divergence between the theoretical and observed distributions was tested at two different confidence levels—20 percent and 5 percent. For forty-five (17.9 percent) of the tracts the maximum divergence exceeded the appropriate 20 percent confidence level critical value; for nine of these forty-five tracts (3.6 percent of the total population of 251) the maximum divergence exceeded the appropriate 5 percent confidence level critical value. Both of the observed percentage rejections are below the expected percentage rejections of 20 and 5 percent respectively. Therefore, the hypothesis of lognormal distribution of bids on a tract is not refuted and seems a reasonable one to adopt.[18]

TABLE 4

RESULTS OF CHI-SQUARE TESTS FOR EQUALITY OF VARIANCE AMONG TRACTS, ALL SALES, 1954-1964, ASSUMING LOGNORMAL DISTRIBUTION

Sale Date	Degrees of Freedom[a]	Chi-Square	Mean Variance of Lognormal Distribution[b]
10/13/54	58	56.0	1.31
7/12/55	64	65.3	1.10
8/11/59	12	12.0	0.72
2/24/60	70	69.9	1.75
3/13/62	120	141	0.88
3/16/62	132	146	1.10
10/ 9/62	5	2.5	1.45
4/28/64	18	24.6	1.11

SOURCE: The entries in this table were calculated from the data contained in U.S., Department of the Interior, Bureau of Land Management, *Bid Recap of OCS Sales.*

[a]The number of degrees of freedom is the number of tracts in the sale which received more than one bid, less one.

[b]The properly weighted mean of all the individual tract variances in the sale. See Hoel, equation (21), p. 199. All logarithms are natural logarithms.

3. THE PREDICTION OF THE VARIANCE AND THE MEAN

In order to estimate $P(X_0)$, the probability that a bid of amount X_0 will win the tract in question, it is necessary to predict the mean and the variance of the assumed lognormal distribution of bids. In the previous section it was noted that the variance of bids around their respective tract means could be considered constant within a sale. Unfortunately, this constancy of variance does not hold between sales.

Using the same method as was used to calculate the results displayed in table 4 gives a value of chi-square of 551 with 486 degrees of freedom. If the variance were constant between sales, a value of chi-square as large as 551 would be expected with a probability of only 0.022; thus the hypothesis does not seem likely. However, this apparent nonconstancy of variance between sales does not necessarily mean that a prediction of the variance at the present sale based on the observed variances at past sales is useless. Such a prediction may well be the best obtainable and quite valuable. The sensitivity of the expected results from the bidding model to misestimations in the variance is of paramount importance here. This sensitivity will be investigated after the bidding model has been developed.

Examination of the individual sale variances shown in table 4 shows no apparent time trend. If all the tracts in all the sales are lumped together, the "pooled" variance about the individual tract means is 1.17. This method of calculation gives equal weight to each bid, no matter when it occurred. If it seemed that the variance had a time trend, one might use regression analysis to predict for the present sale, or at least use a weighting for calculating the variance in which the weight given to observations declines as one moves farther into the past. Neither of these techniques appears useful here, however. In any case, if future sales can be assumed to behave as past ones have, it will be possible to predict a variance for the sale, though the prediction may not be as accurate as might be desired.

Since the mean bids on tracts do not tend to be constant even within sales, much less between sales, historical averages will probably not be of much use in estimating the mean of the assumed lognormal distribution of bids which will be made on a particular tract. However, since the hypothesis of uniform geometric mean bids on tracts was rejected, one would predict the natural logarithms of the bids of individual companies to be positively correlated with the mean natural logarithm of all *other* bids on the tract.[19] Such positive correlations are indeed to be found. Table 5 presents these correlation coefficients for six companies who were frequent bidders at the wildcat sales. The drainage sales were not included because of the small number of bids which were offered at these sales. In the absence of any better information, it would therefore seem possible and desirable to base an estimate of the mean logarithm of the competitors' bids on one's own bid. How this might be done is discussed in connection with the bidding model developed in chapter 6.

4. The Expected Number of Bids on a Tract

The purpose of this chapter is to investigate how an objective estimate for $P(X_o)$ may be formed. So far, evidence has been presented indicating that the bids on a tract tend to be distributed lognormally, and that it is possible for a company to estimate both the variance and the mean of the lognormal distribution of bids for any tract in which it is interested. However, there is one other variable necessary for estimating $P(X_o)$—the number of competing bidders.

TABLE 5

Correlation Coefficients between Logarithms of Company Bids and the Mean Logarithm of Other Bids on Each Tract, Wildcat Sales Only, Selected Companies

	Sale of 10/13/54		Sale of 7/12/55		Sale of 2/24/60		Sale of 3/13/62		Sale of 3/16/62	
Company Name	No. Bids	Corr. Coef.	No. Bids	Corr. Coef.	No. Bids	Corr. Coef.	No. Bids	Corr. Coef.	No. Bids	Corr. Coef.
CATC	26	.61	37	.67	20	—.35	39	.31	17	.63
Shell	33	.58	28	.66	31	.59	48	.25	79	.52
Gulf	28	.40	3	.73	----	----	20	.45	28	.68
Socal	44	.35	49	.79	35	.16	52	.38	73	.12
Phillips	22	.68	13	.72	21	.21	7	.78	10	.46
Humble	47	.59	3	.99	4	.63	25	.09	36	.31

Source: The entries in this table were calculated from data contained in U.S., Department of the Interior, Bureau of Land Management, *Bid Recap of OCS Sales.*

Tracts which are potentially more valuable might be expected to attract a larger number of bidders than those from which less is expected. This hypothesis may be supported through least squares regression analysis. If the number of bids on a tract is used as the dependent variable and the logarithm of the geometric mean of the bids received as the independent variable, then for each wildcat sale, the regression coefficient for the independent variable is positive and significantly different from zero at the .01 level. The multiple correlation coefficient varies from .63 for the sale of February 24, 1960, down to .33 for the sale of March 16, 1962. Thus, the tracts with higher mean bids do seem to attract more bidders.

However, the individual firm does not know what the geometric mean bid on the tract will be. It probably has only a very rough estimate of this mean. Therefore, to attempt to estimate the number of bidders by first estimating the mean bid from one's own bid and then using this projected mean to estimate the number of bidders would seem likely to lead to poor predictions. Since the expected values of the tracts to the bidding companies are unknown, there is no way to see how poor this estimation procedure might be. The closest one can come with the available data is to regress the logarithms of all bids received at a sale against the number of bidders on the respective tracts. The multiple correlation coefficients thus obtained are not high, the largest being .44 for the sale of February 24, 1960. However, in the absence of reliable "grapevine" information about the probable number of competitor bids, estimating the number of competing bids from one's own value estimate for the tract may be the best method available, though the estimates obtained are not likely to be very good. In the following chapter, the sensitivity of the expected profit to misestimation of the number of competing bidders will be carefully examined.

VI

A Bidding Model

1. Introduction

IN THE PREVIOUS CHAPTER, the following hypotheses about the lease bonus bids for Louisiana outer continental shelf lands were developed: (1) the frequency distribution of the bids on a tract tends to be lognormal; (2) it is possible to estimate the variance of this distribution from the records of previous sales; and (3) a rough estimate of the number of competitors who will bid can be made. However, these hypotheses are not sufficient to form the sole foundation for a bidding model; additional assumptions about the behavior of the client firm and its competitors are also necessary. In the next section a sufficient set of assumptions is stated and discussed, and the model is developed. The following section explores the sensitivity of expected profit to misestimates of the variance and the number of competing bidders. The fourth section discusses briefly some bidding models based on alternate firm goals; the final section indicates how the model may be extended to include bidding on more than one object subject to possible constraints.

2. The Formulation of the Model

The following assumptions are sufficient to form the basis for a bidding model:

1. There is only one tract for lease. This assumption is made for convenience in developing the model. It will be relaxed in section 5.

2. The lease bonus will be the auction sale variable. Offers will be made by sealed bids. When the bids are opened, the highest lease bonus offered will win the lease.

3. Each firm's goal is to maximize expected profit. Two alternative goals are discussed in section 4.

4. There are no constraints on the amount which any firm may bid if it so desires. Relaxing this assumption will also be discussed in section 5.

5. Each of the firms is able to make an estimate of the present value (the sum of the expected values of the differences of all future returns and costs, except lease bonus cost, discounted to the present) of a lease on the tract. The estimates of the firms are not necessarily identical. However, each firm feels that its estimates of the present value of the tract are unbiased in the statistical sense; that is, no firm knows of any reason why its present value can be expected to be either higher or lower than the geometric mean of the estimates of its competitors.

6. Each firm knows that n *other firms* will submit bids for the tract; the total number of bids will thus be n + 1.

7. All the firms believe that the submitted bids will be distributed as if they were random drawings from a lognormal population with a known variance but an unknown mean. This assumption implies either that the bidding firms do not know the identity of their competitors, or, if they do, that no special information is available about how the known competitors are likely to bid relative to the mean.

If R is the client firm's present value estimate for the tract, X_0 the amount bid, and $P(X_0)$ the probability that a bid of amount X_0 will win, then the expected profit from the bid may be written

$$E(X_0) = (R - X_0)\, P(X_0).$$

In order to find the value of X_0 which maximizes expected profit, it is necessary to write out the specific functional form of $P(X_0)$.

Assumption 7 implies that any dependence among competitors' bids can be expressed as a joint dependence on the value of the mean of the parent distribution. Given this mean, μ, competitors' bids are mutually independent, even though they are not independent unconditionally.[1] Suppose, temporarily, that there is only one competing bidder, firm 1. If the true value of the mean of the distribution from which X_1 can be regarded as a random drawing were known, the density function for X_1 could be written as[2]

$$f(X_1) = \frac{1}{X_1\, \sigma \sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X_1 - \mu)^2}{\sigma^2},$$

where σ is the population variance of the parent lognormal distribution. The probability that a bid of amount X_0 would win is then

$$P(X_o) = \text{Prob } (X_o > X_1) =$$
$$\int_0^{X_o} \frac{1}{X_1 \sigma \sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X_1 - \mu)^2}{\sigma^2} dX_1. \qquad (6.1)$$

But μ is not known; only an estimate $\hat{\mu}$ is available.[3] One cannot simply replace μ by $\hat{\mu}$ in (6.1) in order to calculate $P(X_o)$, since this probability depends on the true, not the estimated value of the mean. However, suppose that the client firm believes its estimate to be generated by an unbiased process such that $\hat{\mu}$ appears to be a random drawing from a normal distribution with a variance of $\sigma^2_{\hat{\mu}}$. The unbiasedness of the process implies that $\hat{\mu}$ is the best available estimate of the mean of this normal distribution. In this case, the expected value of the probability that the bid X_o will win is[4]

$$P(X_o) = \int_{-\infty}^{\infty} P(X_o \mid \mu) f(\mu)\, d\mu =$$
$$\int_{-\infty}^{\infty} \left[\frac{1}{\sigma_{\mu} \sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\mu - \hat{\mu})^2}{\sigma^2_{\mu}}\right]$$
$$\left[\int_0^{X_o} \frac{1}{X_1 \sigma \sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X_1 - \mu)^2}{\sigma^2} dX_1\right] d\mu. \qquad (6.2)$$

It has been hypothesized that, given μ, competitors' bids will be independent. Therefore, if more than one competing bidder is expected, the joint density function for all n competing bids may be factored so that[5]

$$f(X_1, X_2 \ldots, X_n \mid \mu) = f(X_1 \mid \mu)\, f(X_2 \mid \mu) \ldots f(X_n \mid \mu).$$

Further, since no special information is known about any of the X_i, that is, since all the $f(X_1 \mid \mu)$ are considered identical, the joint density function can be written

$$f(X_1\, X_2, \ldots, X_n \mid \mu) = [f(X_1 \mid \mu)]^n =$$
$$[f(X_2 \mid \mu)]^n = \ldots = [f(X_n \mid \mu)]^n. \qquad (6.3)$$

Therefore, it is usually not necessary to keep the subscripts denoting the individuality of the competitors; in most places it will be sufficient to use X_o as the symbol for the client firm's bid and an unsubscripted X to denote a generalized opponent's bid. Using this notation, the probability that a bid of X_o will win against n competing bids is

$$P(X_o) = \int_{-\infty}^{\infty} [\frac{1}{\sigma_\mu \sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\mu - \hat{\mu})^2}{\sigma_\mu^2}]$$

$$[\int_{o}^{X_o} \frac{1}{X \sigma\sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X - \mu)^2}{\sigma^2} dX]^n d\mu. \qquad (6.4)$$

Recalling the definition of expected profit as

$$E(X_o) = (R - X_o)\, P(X_o), \qquad (6.5)$$

it is apparent that for positive values of X_o, $\hat{\mu}$, σ and σ_μ, $P(X_o)$ as defined by (6.4) is a strictly increasing function of X_o; but $(R - X_o)$ is a strictly decreasing function of X_o. Therefore, the expected profit $E(X_o)$ as a function of X_o is unimodal with a unique maximum. Table 6 shows $E(X_o)$ calculated by numerical integration for various values of X_o, given specified $\hat{\mu}$, σ, and σ_μ. As X_o increases, the difference between R and X_o decreases, thus decreasing the profit expected if the tract is won. On the other hand, increasing X_o raises the proba-

TABLE 6

EXPECTED PROFITS FROM BIDS OF VARYING AMOUNTS, ASSUMING $P(X_o)$ CALCULATED ACCORDING TO EQUATION (6.4),

$\hat{\mu} = \ln 52.18$, $\sigma = \sigma_\mu = 2.0$, $n = 5$, AND $R = \$100$

X_o, the Amt. Bid	$R - X_o$	$P(X_o)$	$E(X_o)$
0.19	\$ 99.81	0.0004	\$ 0.04
1.29	98.71	.0054	0.54
5.44	95.56	.0269	2.54
14.22	85.78	.0644	5.52
37.14	62.86	.1332	8.37
49.14	50.86	.1604	8.16
50.13	49.87	.1625	8.10
51.14	48.86	.1646	8.04
52.18	47.82	.1666	7.97
53.23	46.77	.1688	7.89
54.30	45.70	.1709	7.81
55.40	44.60	.1731	7.72
87.76	12.24	.2272	2.78
229.20	−129.20	.3652	−27.18
598.61	−498.61	.5224[a]	−260.45[a]
1563.38	−1463.38	.6731[a]	−987.94[a]
10663.79	−10563.79	.8927[a]	−9430.13[a]

[a]These values are not exact because of slight numerical integration errors.

bility of winning. In this case, the maximum expected profit occurs for a bid of \$37.14. The expected profit over a range is not very sensitive to relatively large percentage changes in the amount bid. For example, increasing the amount bid per acre from \$37.14 to \$55.40, an increase of 49 percent, only decreases the expected return by 8 percent from \$8.37 to \$7.72. This lack of sensitivity of expected profit to variation in amount bid is due to the fact that, over this range of X_0, the greater probability of winning with the bid of \$55.40 almost offsets the smaller difference between R and X_0.

The conventional technique for maximizing expected profit—finding $dE(X_0)/dX_0$, setting the resultant derivative equal to zero, and solving for X_0—is awkward in this case because of the form of $P(X_0)$. However, if μ and σ_μ were known, the bid which would lead to maximum expected profit could be approximated as closely as desired through a search technique similar to that which generated table 6.

How might the client firm estimate μ and σ_μ? If each firm believes that its estimate of the expected value of the tract is unbiased and that it is following the same strategy as each of its competitors, then each company should regard its bid as the best estimate available for the mean of the distribution from which the other bids can be regarded as random drawings. This statement is implied by assumptions 5 and 7. To estimate the mean of the parent bid distribution as some other value than one's own bid implies either that at least some firms are not arriving at unbiased estimates of the present value of the tract or that all firms are not following the same bidding strategy. In chapter 3, section 4, it was argued that it is not likely such a disequilibrium situation could long continue; no firm with alert competitors can hope to enjoy a favored position for long because of a special strategy. A similar argument can be made with regard to evaluation techniques. Therefore the usual game theory assumption—that opponents are as skilled as the client—appears appropriate here and it seems reasonable to estimate μ as the client's own optimal bid.[6] This, of course, implies that the expected value for each of the X_i, the optimal bids of the client's competitors, also equals the client's optimal bid.

Continuing along this line of reasoning, σ_μ may be estimated as equal to the variance of the bid parent distribution σ. Why this can be done may be explained by reference to a hypothetical statistical problem. Suppose there exists a normally distributed population with a known variance but unknown mean. It is necessary to estimate the mean of the population from *one* random drawing from the population.

The value drawn will be the best estimate available for the true mean.[7] The variance of this estimated mean around the true mean will be the population variance. The bid which the firm plans to place is analogous to the single drawing allowed in the hypothetical example; it represents a random drawing from the distribution from which other bids can be regarded as drawn.

But how can the client firm decide on the optimum bid when this bid is a function of $\hat{\mu}$? One way is through the following iterative process:

1. Given values for R and σ, and some rough estimate of μ, use the search technique to find that value of X_o which maximizes the expected profit in (6.5).
2. Use the natural logarithm of the optimum X_o just calculated as a new estimate of μ and calculate a new value of X_o which now maximizes $E(X_o)$.
3. Repeat step 2 until the difference between two successive values of X_o is smaller than some preassigned value. The resultant X_o is the amount to be bid.

This procedure can be justified as a simulation of a learning process in which firms have, over time, learned to relate their opponents' bidding to their own value estimates and thus to their own bids. Table 7 shows the expected profit resulting from bids of varying amounts after convergence of the iterative procedure. The value of $\hat{\mu}$ for the initial step was $52.18 as shown in table 6. The values of σ, R, and n are identical for both tables. Four iterations were required to move from table 6 to table 7.

Denote the optimum bid generated by the above process $X_o{}^*$. If $\hat{\mu} = \ln X_o{}^*$ and $\sigma_\mu = \sigma$ are substituted into (6.4), then

$$P(X_o{}^*) =$$

$$\int_{-\infty}^{\infty} \left[\frac{1}{\sigma\sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\mu - \ln X_o{}^*)^2}{\sigma^2}\right]$$

$$\left[\int_{0}^{X_o{}^*} \frac{1}{X\sigma\sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X - \mu)^2}{\sigma} dX\right]^n d\mu. \qquad (6.6)$$

It can be shown that

$$P(X_o{}^*) = \frac{1}{n+1} \qquad (6.7)$$

regardless of the value of σ. This result is hardly surprising. If firm

TABLE 7

EXPECTED PROFIT FROM BIDS OF VARYING AMOUNTS AFTER CONVERGENCE OF ITERATION PROCESS

$\ln X_o^* = \hat{\mu} = \ln 38.65,\ \sigma = \sigma_\mu = 2,\ n = 5$ AND $R = \$100.00$

X_o, the Amt. Bid	$R - X_o$	$P(X_o)$	$E(X_o)$
0.14	\$ 99.86	0.0004	\$ 0.36
0.96	90.04	.0054	0.54
4.03	95.97	.0269	2.58
10.53	89.47	.0644	5.76
27.51	72.49	.1332	9.66
36.40	63.60	.1604	10.20
37.14	62.86	.1625	10.21
37.89	62.11	.1646	10.22
38.65[b]	61.35	.1666	10.22
39.43	60.57	.1688	10.22
40.23	59.77	.1709	10.22
41.04	58.96	.1731	10.20
65.01	34.99	.2272	7.95
69.80	−69.80	.3652	−25.49
443.46	−343.46	.5224[a]	−179.41[a]
1158.18	−1058.18	.6751[a]	−714.39[a]
7892.92	−7799.92	.8927[a]	−6962.87[a]

[a]These values not exact because of slight numerical integration error.
[b]This is the value of X_o^*.

0 has n equally shrewd bidding competitors, then the a priori probability that any one of the $n + 1$ bidders will win ought to be $1/(n+1)$.

While the iterative procedure may be used to arrive at X_o^*, the expected profit maximizing bid, it is both interesting and convenient to develop a formula for calculating X_o^* given R, σ, and n. Rewrite equation (3.3) as

$$R = X_o^* + \frac{P(X_o^*)}{P'(X_o^*)}. \tag{6.8}$$

$P(X_o^*)$ has just been said to be equal to $1/(n + 1)$ in equation (6.7) above. The only remaining unknown other than X_o^* in (6.8) is thus $P'(X_o^*)$. Differentiating (6.4) with respect to X_o and evaluating the result at $X_o = X_o^*$ and $\sigma_\mu = \sigma$ gives[8]

$$P'(X_o^*) = \frac{n}{X_o^* \sigma} \int_{-\infty}^{\infty} \left[\frac{1}{\sigma \sqrt{2\pi}} \exp - \frac{1}{2} \frac{(\mu - \hat{\mu})^2}{\sigma^2} \right]$$

$$\left[\frac{1}{\sqrt{2\pi}} \exp - \frac{1}{2} \frac{(\ln X_o^* - \mu)^2}{\sigma^2} \right]$$

$$\left[\int_0^{X_o^*} \frac{1}{X \sigma \sqrt{2\pi}} \exp - \frac{1}{2} \frac{(\ln X - \mu)^2}{\sigma^2} dX \right]^{n-1} d\mu = \frac{n}{X_o^* \sigma} G(n), \tag{6.9}$$

where G(n) denotes the value of the integral over μ. It can be shown that this integral is a function only of n and not of σ. Table 8 displays the values of G(n) for n from one through sixteen. Note that $\hat{\mu}$ may not be regarded as identically equal to $\ln X_o^*$ in the process of obtaining $P'(X_o^*)$; *after* convergence of the iterative process $P(X_o^*)$ is not a function of X_o^*, but is equal to $1/(n + 1)$—see equation (6.7). $P'(X_o^*)$ as defined by (6.9) may be described as the rate of change of the expected probability of winning with respect to a change in amount bid in the vicinity of X_o^* with $\hat{\mu}$ fixed at $\ln X_o^*$.

TABLE 8
G(n) Evaluated for n from One through Sixteen

n	G(n)
1	0.28208
2	.14104
3	.08578
4	.05815
5	.04224
6	.03219
7	.02542
8	.02062
9	.01710
10	.01442
11	.01234
12	.01069
13	.00936
14	.00827
15	.00736
16	.00660

Through the use of table 8, X_o^* may be calculated directly without going through the iterative procedure. Substituting $P(X_o^*)$ from (6.7) and $P'(X_o^*)$ from (6.9) into (6.5) one obtains

$$R = X_o^* \left[1 + \frac{\sigma}{n(n+1)\,G(n)}\right]$$

$$X_o^* = \frac{R}{1 + \dfrac{\sigma}{n(n+1)\,G(n)}} \qquad (6.10)$$

Substituting $n = 5$ and $\sigma = 2$ into (6.10) gives $X_o^* = \$38.76$. This compares with the value of X_o^* shown in table 7 after the convergence of the iterative process of $38.65. This slight difference arises from the fact that the computer program which generated the table evaluated $E(X_o)$ at intervals of $(\ln X_o - \mu)/\sigma = 0.01$. Had finer calculation intervals been used, the value shown in table 7 would have been closer to $38.76.

For given n and R, X_o^* decreases as σ increases. In other words, the smaller the variance of the distribution from which the bids appear drawn, the higher will be the expected profit maximizing bid. This result is intuitively satisfying, for σ is a measure of the expected dispersion of the bids. But the dispersion of the bids has been postulated as caused by the variations in estimating the present value of the tract. If there is no collusion and the dispersion of tract estimates is generally thought to be low, no one will have much incentive to bid considerably below his value estimate, for if he does, it is very unlikely that he will win. On the other hand, a high σ indicates the probability of wide variations in value estimates and increases the possibility that a bid considerably below one's own value estimate might win, thus providing an incentive to bid low. The usual argument states that profits should be higher in risky industries in order to attract capital from risk-averse investors. The implication of this discussion is that uncertainty about the outcome of individual investments can create a situation in which higher profits may be expected, given some sort of barrier to entry into the industry.[9]

A barrier to entry is still necessary, however, for above-normal profits to exist according to this model since X_o^* increases as n increases for fixed R and σ. This may be demonstrated from equation (6.10) with the aid of table 8. As n increases, $n\,(n+1)$ increases faster than $G(n)$ decreases; therefore, as n gets larger R is divided by smaller numbers, so that X_o^* increases. In fact,

$$\lim_{n \to \infty} (X_o^*) = R.$$

In other words, if there exists a "large" number of competitors (i.e., a "pure" competition situation), a bid below one's value estimate has a vanishingly small probability of winning, and there is no incentive to so bid.

In summary, the above model has been developed specifically for a bidding situation in which all the competitors are presumed to be equally shrewd and equally well informed on the average. They have learned to anticipate each other's reactions and have converged to a stable strategy pattern. The strategy is said to be stable because once all competitors use it, there is no incentive for anyone to depart from it, *unless he finds himself with better information about the expected value of the tract than others*; otherwise, the client firm's expected profit will be maximized if it bids the X_o^* given by (6.10). Further, the model predicts that among a total of n + 1 "equal" bidders, the probability that a specific bidder will win is 1/(n + 1). Finally, it predicts that the greater the uncertainty about the value of the object, the smaller percentage of its expected value will be the optimal bid for it. On the other hand, the larger the number of competing bids expected the higher the percentage of the expected value of the object should be bid.

3. The Sensitivity of Expected Profit to Misestimates of the Variance and the Number of Bidders

The just-developed model takes into account the fact that $\hat{\mu}$, the mean of the lognormal distribution from which the bids of competitors can be regarded as drawn, is only an estimate with an associated probability distribution. Should not similar modifications be introduced because n and σ are estimates also? Unfortunately, no satisfactory frequency distribution function has been found for either n or σ; thus a simple analytical treatment like that for $\hat{\mu}$ is impossible. Therefore, it is interesting to investigate the sensitivity of expected profit to misestimates in the values of n and σ in order to see how important it might be to take into account the frequency distribution functions of these parameters. Table 9 shows the average percentage loss in expected profit from misestimating the number of competing bidders for two different values of σ. The entries in this table were calculated as follows:

1. Assume some arbitrary value for R.
2. Let $\hat{n}$ be the estimated number of other bidders, n denote the actual number of competing bidders, $\hat{X}_o^*$ be the profit maximizing

value of X_o calculated from (6.10) using $\hat{n}$, and X_o^* be the profit maximizing value of X_o calculated using n.

3. $E(X_o^*)$, the expected profit from a bid of X_o if n other bids are placed, is calculated equal to

$$(R - X_o^*)\ P(X_o^*) = (R - X_o^*)\left(\frac{1}{n+1}\right).$$

4. $\hat{E}(X_o^*)$, the expected profit if $\hat{X}_o^*$ is bid because $\hat{n}$ bids were expected when actually n bids occurred, is calculated from equations (6.4) and (6.5) using $\hat{\mu} = \ln X_o^*$.

5. Finally, the expected percentage loss in expected profit from the misestimation of the number of bidders is calculated as

$$\text{Percent loss} = \frac{[E(X_o^*) - \hat{E}(X_o^*)]}{E(X_o^*)} \times 100\%.$$

Examination of table 9 reveals that the expected percentage loss in profit from misestimating the number of competing bidders is sur-

TABLE 9

EXPECTED PERCENTAGE PROFIT LOSS FROM MISESTIMATING NUMBER OF BIDDERS FOR A TRACT

Estimated Number of Other Bidders, $\hat{n}$	Expected Percent Profit Loss — Actual Number of Other Bidders, n: 1	3	5	7	9	11	13
	Standard Deviation of Lognormal Distribution of Bids $= \sqrt{0.50}$						
1	0.0	14.2	26.1	34.0	39.5	43.6	47.0
3	10.1	0.0	1.8	4.4	6.9	9.1	11.0
5	17.1	1.6	0.0	0.6	1.6	2.7	3.8
7	21.4	3.7	0.5	0.0	0.3	0.8	1.4
9	24.4	5.5	1.4	0.3	0.0	0.1	0.4
11	26.6	7.1	2.4	0.7	0.1	0.0	0.1
13	28.4	8.4	3.2	1.3	0.4	0.1	0.0
	Standard Deviation of Lognormal Distribution of Bids $= \sqrt{2.0}$						
1	0.0	8.8	17.0	22.8	27.1	30.4	33.2
3	6.5	0.0	1.2	3.1	4.8	6.4	7.8
5	11.5	1.1	0.0	0.4	1.1	2.0	2.7
7	14.8	2.6	0.4	0.0	0.2	0.6	1.0
9	17.1	4.0	1.0	0.2	0.0	0.1	0.3
11	18.9	5.1	1.7	0.5	0.1	0.0	0.1
13	20.4	6.1	2.4	0.9	0.3	0.1	0.0

prisingly small in many cases. Therefore, in the absence of a known frequency distribution function for n, it seems permissible to neglect the fact that n is an estimate and not an observed parameter. However, if a satisfactory discrete frequency distribution function for n could be obtained, then the expected profit from a bid of X_0 could be calculated as

$$\hat{E}(X_0) = \sum_{n=1}^{k} [P(n)]\ [R - X_0]\ [P(X_0, n)]$$

where P(n) is the estimated probability that exactly n competitors will bid, k the largest number of competitor bids possible, and $P(X_0, n)$ the probability that a bid of amount X_0 by the client firm will win over n competing bids. In this case, the computations become more cumbersome, but some gain in efficiency is possible.

TABLE 10

EXPECTED PERCENTAGE LOSS FROM MISESTIMATING STANDARD DEVIATION OF BIDS FOR A TRACT

Estimated Standard Deviation $\hat{\sigma}$	Expected Percent Loss Actual Standard Deviation, σ						
	0.707	0.866	1.000	1.118	1.247	1.323	1.414
	Number of Competing Bidders = 1						
0.707	0.0	1.1	2.9	4.6	6.1	7.4	8.6
0.866	1.3	0.0	0.5	1.4	2.5	3.5	4.4
1.000	3.9	0.6	0.0	0.3	0.9	1.5	2.2
1.118	6.9	1.8	0.3	0.0	0.2	0.6	1.0
1.247	9.8	3.3	1.0	0.2	0.0	0.1	0.4
1.323	12.7	5.0	1.9	0.6	0.1	0.0	0.1
1.414	15.4	6.6	2.9	1.2	0.4	0.1	0.0
	Number of Competing Bidders = 12						
0.707	0.0	1.5	4.0	6.5	8.8	10.9	12.8
0.866	1.8	0.0	0.7	2.1	3.7	5.3	6.8
1.000	5.2	0.8	0.0	0.4	1.3	2.4	3.5
1.118	9.1	2.6	0.5	0.0	0.3	0.9	1.7
1.247	13.1	4.8	1.5	0.3	0.0	0.2	0.6
1.323	16.9	7.1	2.9	1.0	0.2	0.0	0.1
1.414	20.5	9.6	4.5	1.9	0.7	0.2	0.0

Table 10 shows the average percentage loss in expected profit from misestimating the standard deviation of the lognormal distribution of the opponents' bids for two values of n. The entries in this table are calculated as follows:

1. Assume some arbitrary value for R.

2. Let $\hat{\sigma}$ be the estimated standard deviation, σ denote the actual standard deviation, $\hat{X}_o^*$ be the profit maximizing value of X_o calculated from (6.10) using $\hat{\sigma}$, and X_o^* be the profit maximizing value of X_o calculated using σ.

3. $E(X_o^*)$, the expected profit from a bid of X_o^* if the actual standard deviation turns out to be σ, is calculated equal to

$$(R - X_o^*)\ P(X_o^*) = (R - X_o^*)\ (\frac{1}{n+1}).$$

4. $\hat{E}(X_o^*)$, the expected profit if $\hat{X}_o^*$ is bid because $\hat{\sigma}$ was the expected standard deviation when actually the standard deviation was equal to σ, is calculated from equations (6.4) and (6.5) using $\hat{\mu} = \ln X_o^*$.

5. Finally, the expected percentage loss in expected profit from misestimation of the standard deviation is calculated as

$$\text{Percent loss} = \frac{E(X_o^*) - \hat{E}(X_o^*)}{E(X_o^*)} \times 100\%.$$

The largest mean sale variance observed was 1.75 for the sale of February 24, 1960;[10] the corresponding standard deviation is 1.32. The smallest mean sale variance and standard deviation observed were 0.72 and 0.85, respectively, for the sale of August 11, 1959.[11] Within this range of observed standard deviations, misestimates of σ would appear to lead to very small percentage reductions of expected profits. For example, if the standard deviation were estimated to be 1.323 but actually turned out to be 0.866, the expected loss in profit due to misestimation of σ amounted to 5.0 percent if there was only one competing bidder and to 7.1 percent if there were 12 competitors bidding for the tract. The mean standard deviation over all sales is 1.08; if this value had been used as the predicted σ at all sales to date, the expected percentage profit loss from misestimating the standard deviation would, at most, have been less than two percent (see table 10). Therefore, in this case, it seems feasible to neglect the fact that σ is an estimate and to treat it as a known parameter.

4. Model Stability and the Goal of the Firm

The model developed in section 2 is based on the assumption that each firm wishes to maximize the profit expected from bidding on the object for sale. This model is stable in the sense that an iterative process in which one uses the just-prior estimate of the optimum bid as an estimate of the mean of the distribution from which other bids are regarded as drawn in order to make a new estimate of the optimum bid always converges.[12] Other possible assumptions about company goals do not always lead to stable models. For example, suppose that the goal of the firm is to maximize the ratio of expected profit to the expected amount spent. If $P(X_o)$ is the probability that a bid of amount X_o will win, then $X_oP(X_o)$ is the expected amount spent, and the ratio of the expected profit to this amount is

$$r = \frac{E(X_o)}{X_oP(X_o)} = \frac{(R - X_o)P(X_o)}{X_oP(X_o)} = \frac{R - X_o}{X_o} = \frac{R}{X_o} - 1,$$

provided $P(X_o) \neq 0$. However,

$$\lim_{X_o \to 0} r = \lim_{X_o \to 0} \left(\frac{R}{X_o} - 1\right) = \infty.$$

Therefore, without additional assumptions, it is not possible to maximize this ratio, much less develop a stable model.

Yet this formulation is interesting for two reasons. First, it may give some insight into why minimum acceptable bids are often stipulated by sellers when auctions are announced. The reason usually given for setting minimum bids is that of protecting the seller against possible collusion among potential bidders.[13] However, in a world where capital rationing is often apparently important, the seller may need to set a refusal price as a protection against those who would enter low bids without expecting to win many objects, but hoping for large net returns (relative to the amounts paid to the seller) on those few objects purchased.

Second, in any practical situation, some cost C will be associated with entering a bid. If this cost is included, then the ratio to be maximized will be

$$r = \frac{(R - X_o)P(X_o) - C}{X_oP(X_o)} = \frac{R}{X_o} - 1 - \frac{C}{X_oP(X_o)}. \qquad (6.11)$$

Whether or not a bidding model based on the above equation will be stable depends on the form of $P(X_o)$. It is possible to differentiate r as defined in (6.11) with respect to X_o, set it equal to zero, and thus find the first order conditions for maximizing r. However, the algebra is tedious and the result expressed in the form of equation (6.10) is so complicated that interpreting it is difficult and not particularly enlightening.

It is important to note that including the cost of bidding as in (6.11) is really not sufficient for developing a complete model for offshore bidding. The difficulty lies in the fact that there are other outlays than lease bonus–drilling and producing costs if oil or gas is found–which will be necessary if the lease is won. The firm would presumably be interested in maximizing the ratio of expected profit to expected total present value of expenses, rather than the ratio of expected profit just to expected lease bonus.

Of course there exist a number of "plausible" assumptions about firm goals. One possible firm goal is maximizing the ratio of expected profit to amount spent if the bid wins. This goal is interesting because if it replaces assumption 3 in the list at the beginning of section 2 a model results which is stable for some values of σ and n, but not for others. Call the ratio of expected profit to amount bid r. Then,

$$r = \frac{E(X_o)}{X_o} = \frac{(R - X_o)P(X_o)}{X_o} = P(X_o)\left[\frac{R}{X_o} - 1\right].$$

The first order condition for maximizing this ratio may be found by differentiating r with respect to X_o and setting the derivative equal to zero.

$$\frac{dr}{dX_o} = R\left[\frac{X_o P'(X_o) - P(X_o)}{X_o^2}\right] - P'(X_o) = 0.$$

Thus, the first order condition is satisfied when

$$X_o^2 P'(X_o) = R[X_o P'(X_o) - P(X_o)]. \tag{6.12}$$

Call the value of X_o which satisfies (6.12) $\check{X}_o$. Now, equations (6.6), (6.7), and (6.9) define $P(X_o^*)$ and $P'(X_o^*)$ in terms of the distribution of the bids expected; the goals of the bidders do not enter into their development so long as the goals and strategies of all the bidders are the same. Therefore, these equations can be used to derive $P(\check{X}_o)$ as equal to $1/(n + 1)$ and $P'(\check{X}_o)$ as equal to $(n/\check{X}_o\sigma)G(n)$. Substituting these values into (6.12) one obtains

$$\check{X}_o^2 \frac{n}{\check{X}_o\sigma} G(n) = R\left[\check{X}_o \frac{n}{\check{X}_o\sigma} G(n) - \frac{1}{n+1}\right],$$

$$\frac{\check{X}_o n}{\sigma} G(n) = R[\frac{n}{\sigma} G(n) - \frac{1}{n+1}], \text{ and}$$

$$\check{X}_o = R[1 - \frac{\sigma}{n(n+1)G(n)}]. \qquad (6.13)$$

If $\sigma > n(n+1)G(n)$ then (6.13) gives a negative value for $\check{X}_o$, an obviously irrational prediction;[14] in this case the model is not stable and the iterative procedure will not converge. If $\sigma < n(n+1)G(n)$ the model is stable, the iterative procedure converges, and equation (6.13) accurately predicts the convergent value of $\check{X}_o$.

Neither of the above goals is presented as a very likely actual company objective; the two examples were given to illustrate the sensitivity of the stability of the model to assumptions about the goal of the bidding firms.

5. Multiplicity of Tracts and Bidding Constraints

The assumption of only one tract up for lease is unrealistic in connection with the petroleum land bids under consideration in this study; in all sales so far, at least nineteen tracts were offered. A multiplicity of available tracts will not affect the optimum bid calculated according to the model unless there are constraints on the amount which may be bid or spent or on the rate of return expected.

It seems quite probable that a firm might have a limit on the amount which may be spent for leases at a particular sale. Most firms budget an amount each year for exploration expenditures. The size of this exploration budget is a function of many factors, including the cash flow position of the company, its ability to borrow, its beliefs about future demand for and price of its products, and its expectation of return from the exploration money expended. At any rate, the capital available for exploration seems not to be unlimited; such capital rationing appears to be almost universal not only in the petroleum industry but also in many other industries as well.[15] Buying leases is obviously not the only use for exploration funds; competitive uses are geophysical exploration, purchasing interests in already leased but untested property, etc. Further, money allocated to any one area is often limited because of the desire for areal diversification.

However, since it is usually not possible to predict with certainty which tracts among those bid on will be won, it is also usually not possible to predict exactly how much will actually be spent at a sale.

In such circumstances, a constraint on the expected amount spent might be used. If $P_i(X_{io}^*)$ is the probability of winning the i'th tract, then $\Sigma_i X_{io}^* P_i(X_{io}^*)$ is the expected amount which will be spent. This sort of constraint can be handled by the Lagrange multiplier method described in section 1 of chapter 3. It is conceivable that some sort of safety factor might be desired. In such a case, if C represents the maximum it is desirable to spend and a is some safety factor larger than 1, the relevant constraint can be formulated as

$$a \sum_i X_{io}^* P_i(X_{io}^*) = C.$$

Of course, in no case should a be so large that

$$a \sum_i X_{io}^* P_i(X_{io}^*) > \sum_i X_{io}^*,$$

since it is not possible to spend more than the sum of all amounts bid.

A constraint expressing a minimum acceptable rate of return for each tract presents no major problem. The Lagrange multiplier technique is capable of handling several constraints simultaneously; standard linear programming techniques may also be used. Other constraints, such as the requirement for a specified minimum or maximum expected number of tracts to be won, may also be handled. Of course, when more than one constraint is used, it may turn out to be impossible to satisfy all simultaneously. If this situation occurs, one or more constraints may be changed until it is possible to satisfy all at once, or the problem may be restructured.

A firm may wish to neglect the effect of constraints on the bidding of its competitors. Or, if a firm wishes to assume that the constraints of others will affect their bids in a manner similar to the effect on one's own bids, then a further iteration can be used to arrive at the optimum bids. The procedure may be outlined as follows:

1. For each tract, R, n, and σ are estimated; then X_o^* is calculated from (6.10) and $\hat{\mu}$ is evaluated as $\ln X_o^*$. Now $P(X_o)$ can be evaluated for any bid on any tract.

2. The Lagrange multiplier technique including relevant constraints can now be applied and the solution obtained for that set of bids which maximizes the aggregate expected profit subject to the constraints applied.

3. On the assumption that all other bidders are subject to similar constraints, the logarithms of the resultant bids can be respectively substituted for the previous estimates of the μ.

4. Steps 2 and 3 can be repeated in sequence until successive bid values converge.

If a firm feels that the constraints affecting competitors are not identical with its own, then special procedures will probably be necessary depending upon the particular assumptions involved.

VII

Summary and Conclusions

THIS MONOGRAPH is focused on a specific type of sealed bidding. Such bidding necessarily takes place in an atmosphere of uncertainty. Current discussions about decision-making under uncertainty often fall into one of two categories. To the first category belong attempts to relate the psychology of investment allocation to subjective assessments of characteristics of possible investments. This study contributes little to this area. Particularly acute are problems concerning the length of the time period over which the investor's expected utility is to be maximized, the interdependencies of the outcomes of possible concurrent investments, and the indefinite character of an investment unit and the view-span of investors and managers. All these things make it presently impossible to view a few specific acts by a group of companies—say, bids for petroleum leases on tracts in certain areas—and to attempt to infer comparative measures of each firm's attitude toward uncertainty.

To the second category of discussions about decision-making under uncertainty belong attempts to derive optimum procedures for attaining goals in particular situations. The main thrust of this study lies in this area. Hypotheses about bidding behavior were tested against the records of the Louisiana offshore sales. The hypotheses not rejected were combined with other plausible assumptions in order to construct a model for maximizing expected profit. This model is interesting in at least three ways. First, it may provide a convenient conceptual framework for a firm's thinking about submitting a bid at a current sale, though in many situations a firm will probably not wish to consider its competitors a homogeneous group, for information about the probable action of individual competitors may be available. Even if the formulation of the model is not regarded as practical in an actual situation, it may still serve to focus attention on some variables which almost certainly will be important no matter what conceptual framework is adopted for thinking about the problem. These variables include the estimated present value of the tract, the likely number of competing bidders, and the probability distributions of competitor bids.

The second way in which the developed model is interesting lies in its implications about behavior in imperfect markets under conditions of uncertainty. If the number of potential competitors is limited, and selling and buying opportunities periodically occur, then the model shows how a stable industry situation might result with each firm earning economic profits on the average. The greater the uncertainty, *ceteris paribus*, the greater the expected economic profit from each transaction, for the presence of uncertainty about the behavior of competitors in effect provides incentive to bid below the expected value when purchasing or to offer above cost when selling.

Finally, the relative insensitivity of the expected profit to the amount bid over a considerable range of possible bids is somewhat surprising until one examines the cause—that the difference between the expected value of the object and the amount bid and the probability of winning move in opposite directions with proposed changes in the amount to be bid. The insensitivity means that it is not so important to place precisely *the* right bid as might otherwise be the case.

It was not possible to test how well this model described the actual bidding behavior of oil companies for the Louisiana offshore lands because no data about company estimates of the expected values of tracts were available. Thus, this model stands in need of empirical verification. If such verification were forthcoming, this would tend to substantiate the assumptions made in the development of the model which could not be tested directly against the bid data. If the present model does not describe actual behavior adequately, then it might well be possible to modify it to increase its descriptive power.

Several areas for possible further research suggest themselves. An obvious question is whether or not the bids in other situations tend to be distributed lognormally with a predictable variance. Investigating optimal bidding strategy when one firm has better information about the present value of the offered object than the others might yield interesting results.[1] Mead's research about conditions under which sealed bidding is more advantageous to the seller than open bidding, and vice versa,[2] may have opened a fertile area for further work. Certainly there is much to be learned about the psychology of competitor reactions. In a larger framework, there are many areas where the traditional deterministic models of mathematical microeconomics need to be modified to introduce properly the uncertainties under which actual firms must act.[3]

As Friedman has pointed out, the more uncertainty there is in a business problem, the more likely a formal analysis is to generate a solution which leads to significantly more profitable results than those obtained by informal methods.[4] While it has not been demonstrated that the bidding model developed in this monograph is the best one that a firm engaged in bidding for Louisiana offshore leases might use, the development of the model is illustrative of the type of formal analysis of a specific uncertain situation which may lead to better decision-making under uncertainty by the firms which use it.

Notes

CHAPTER I

1. For an interesting discussion of this assertion see Robert Schlaifer, *Probability and Statistics for Business Decisions* (New York: McGraw-Hill, 1959), pp. 2-49.

2. Although diversification has long been practiced, Markowitz apparently deserves the credit for the first systematic theoretical analysis of "efficient" diversification. See Harry Markowitz, *Portfolio Selection: Efficient Diversification of Investments* (New York: John Wiley & Sons, 1959), pt. 3.

3. Of course, some other portfolio choice would almost always have been a better a posteriori choice.

4. Even the most conservative investor runs the risk of inflation if he has invested in fixed income securities or decline in value of precious metals if he has invested in gold or silver, etc., or of changes in the tax structure.

5. Markowitz, *Portfolio Selection*, pp. 116-21.

CHAPTER II

1. No attempt has been made in this discussion to distinguish between risk and uncertainty. The assumption adopted here is that a businessman, prior to making an investment decision, at least subjectively assigns probabilities to the various possible outcomes and evaluates the proposed investment with these probabilities in mind.

2. Stephen M. McDonald, *Federal Tax Treatment of Income From Oil and Gas* (Washington: Brookings Institution, 1963), pp. 32-45.

3. Gordon M. Kaufman, *Statistical Decision and Related Techniques in Oil and Gas Exploration* (Englewood Cliffs, N.J.: Prentice-Hall, 1963), pp. 7-44.

4. See above, p. 4.

5. A. I. Levorsen, *Geology of Petroleum* (San Francisco: W. H. Freeman & Co., 1954), pp. 29-30.

6. Ibid., p. 30.

7. A basin is roughly equivalent to a subprovince in Levorsen's terminology.

8. Kaufman, *Statistical Decision*, pp. 103-54.

9. Apparently neither is publicly held.

10. Smaller firms can also reap available benefits of diversification through trading agreements with other firms. See C. Jackson Grayson, Jr., *Decisions Under Uncertainty: Drilling Decisions by Oil and Gas Operators* (Boston: Division of Research, Harvard Business School, 1960), pp. 189-96.

11. Levorsen, *Geology of Petroleum*, p. 31.

12. An interesting account of how this has been attempted at Tidewater Oil Company may be found in James C. Wilson, "A Comprehensive Approach to Investment Planning"

(Report presented to the fall meeting of the Cost Study Committee, Independent Petroleum Association of America, Houston, Texas, October 24, 1964).

CHAPTER III

1. Lawrence Friedman, "A Competitive Bidding Strategy," *Operations Research* 4 (February, 1956): 105-6; F. Hanssmann and B. H. P. Rivett, "Competitive Bidding," *Operational Research Quarterly* 10 (March, 1959): 45-50.

2. Richard H. Leftwich, *The Price System and Resource Allocation* (New York: Holt, Rinehart, & Winston, 1966), pp. 279-81.

3. This notational system is developed in Charles Christenson, *Strategic Aspects of Competitive Bidding for Corporate Securities* (Boston: Division of Research, Harvard University Graduate School of Business Administration, 1965), p. 36.

4. The difference between the highest and second-highest bid at a closed auction is sometimes referred to as "money left on the table."

5. Walter J. Mead, "Natural Resource Disposal Policy—Oral Auction Versus Sealed Bids," *Natural Resources Journal* 7 (April, 1967): 211-12.

6. Christenson, *Competitive Bidding*, pp. 52-89. See also Lawrence Friedman, "Competitive Bidding Strategies" (Ph.D. diss., Case Institute of Technology, 1957), pp. 52-74.

7. John J. Arps, "A Strategy for Sealed Bidding," *Journal of Petroleum Technology* 17 (September, 1965): 1035.

8. See below, notes 9 and 10.

9. Friedman, "Competitive Bidding Strategy," p. 105.

10. Hanssmann and Rivett, "Competitive Bidding," p. 49.

11. Friedman, "Competitive Bidding Strategy," p. 107.

12. Prob $[X_0 > X_1, X_0 > X_2]$ is a shorthand notation for "the probability that X_0 is greater than X_1 and X_0 is also greater than X_2."

13. Prob $[X_0 > X_1 \mid X_0 > X_2]$ is the notation for "the probability that X_0 is greater than X_1 given that X_0 has been determined to be greater than X_2."

14. Christenson, *Competitive Bidding*, p. 77.

CHAPTER IV

1. A few blocks on the boundaries of the designated areas were less than 5,000 acres because of boundary irregularities.

2. Blocks smaller than 5,000 acres were divided into 32 or fewer equal parts.

3. U.S., Congress, *Outer Continental Shelf Lands Act*, Public Law 212, 83rd Cong., 2d sess., 1953.

4. For a discussion of simple open bidding see Lawrence Friedman, "Competitive Bidding Strategies," Ph.D. diss., Case Institute of Technology, pp. 36-51.

5. Walter J. Mead, "Natural Resource Disposal Policy—Oral Auction Versus Sealed Bids," *Natural Resources Journal* 7 (April, 1967): 194-224.

6. Interview with C. J. Bonnecarrere, Secretary to the Louisiana State Mineral Board, September 13, 1965.

CHAPTER V

1. If there are only two nonidentical bids on a tract, obviously one is greater than the arithmetic mean and one is less.

2. This approximation will be very good because of the large number of observations.

3. John J. Arps, "A Strategy for Sealed Bidding," *Journal of Petroleum Technology* 17 (September, 1965): 1035.

4. J. Aitchison and J. A. C. Brown, *The Lognormal Distribution* (Cambridge: Cambridge University Press, 1957), p. 1.

5. Ibid., pp. 1-2.

6. Arthur S. Goldberger, *Econometric Theory* (New York: John Wiley & Sons, 1964), p. 216.

7. Sidney Siegel, *Nonparametric Statistics for the Behavioral Sciences* (New York: McGraw-Hill, 1956), p. 47.

8. Samuel B. Richmond, *Statistical Analysis* (New York: Ronald Press Co., 1964), pp. 304-17. The F-test is, of course, an application of the analysis of variance method.

9. See Goldberger, *Econometric Theory*, pp. 215-16.

10. Arps, "Strategy for Sealed Bidding," p. 1035.

11. Paul G. Hoel, *Introduction to Mathematical Statistics* (New York: John Wiley & Sons, 1954), pp. 193-96.

12. Ibid., p. 195, equation (15).

13. See table 4, footnote b.

14. H. W. Lilliefors, personal communication, Sept. 15, 1967. Necessary interpolations were performed by this author.

15. H. W. Lilliefors, "On the Kolmogorov-Smirnov Test for Normality With Mean and Variance Unknown," *Journal of the American Statistical Association* 62 (June, 1967): 399-402.

16. See above, table 1.

17. Lilliefors did not show critical values for the Kolmogorov-Smirnov statistic for sample sizes of two or three.

18. The bids on each tract, strictly speaking, constitute a truncated distribution because of the $15 per acre minimum on bids accepted by the Bureau of Land Management. It was thought that the estimated mean and variance of the distributions of the bids on tracts should be corrected for this truncation by the method described in Aitchison and Brown, *Lognormal Distribution*, pp. 87-91, and more fully in A. Hald, "Maximum Likelihood Estimation of the Parameters of a Normal Distribution Which Is Truncated at a Known Point," *Skandinavisk Aktuarietidskrift* 32 (1949): 119-34. However, sample checks showed these corrections to be negligible except for a few tracts on which there were several bids between $15 and $25. Further, the method of calculation and the functions tabulated would not have been easy to generate on the computer. For these reasons, corrections for truncation were omitted in this study. Such corrections might be more important, however, in other bidding situations.

19. The firm's own bid must be eliminated from the calculation of the mean logarithm, for in samples as small as these, inclusion of the firm's own bid would so strongly bias the mean as to introduce strong spurious correlation.

One should also expect positive correlation coefficients for the logarithms of pairs of companies who each bid on some of the tracts of a sale. These correlation coefficients were calculated for many of the company pairs at several of the sales and are almost always positive when more than four or five observations are involved. However, they vary greatly in magnitude among various company pairs in a sale and among different sales for the same company pair. It was not possible to draw any useful conclusions from these correlation coefficients; therefore, tables showing them are not presented here.

CHAPTER VI

1. See Charles Christenson, *Strategic Aspects of Competitive Bidding for Corporate Securities* (Boston: Division of Research, Harvard University School of Business Administration, 1965), p. 77.

2. D. V. Lindley, *Introduction to Probability and Statistics* (Cambridge: Cambridge University Press, 1965), 1:143.

3. In order to avoid confusion it is important to remember that μ is the logarithm of the geometric mean and $\hat{\mu}$ is the estimate of this logarithm.

4. Arthur S. Goldberger, *Econometric Theory* (New York: John Wiley & Sons, 1964), p. 82.

5. Ibid., p. 56.

6. Obviously, there may exist situations in which one or more firms have "the jump" on the others. In such cases, the bidding model would have to be developed in the context of the specific situation.

7. Paul G. Hoel, *Introduction to Mathematical Statistics* (New York: John Wiley & Sons, 1954), p. 202.

8. Letting

$$g(\mu) = \frac{1}{\sigma_u\sqrt{2\pi}} \exp -\tfrac{1}{2} \frac{(\mu - \mu)^2}{\sigma_u^2}$$

and

$$h(X, \mu) = \frac{1}{X\sigma\sqrt{2\pi}} \exp - \tfrac{1}{2} \frac{(\ln X - \mu)^2}{\sigma^2},$$

equation (6.4) may be written as

$$P(X_0) = \int_{-\infty}^{\infty} [\int_0^{X_0} h(X, \mu)dX]^n \; g(\mu)d\mu.$$

Thus,

$$\frac{dP(X_0)}{dX_0} = \int_{-\infty}^{\infty} n[\int_0^{X_0} h(X, \mu)dX]^{n-1} [\frac{d}{dX_0}\int_0^{X_0} h(X, \mu)dx] \; g(\mu)d\mu$$

$$= \int_{-\infty}^{\infty} n[\int_0^{X_0} h(X, \mu)dX]^{n-1} [h(X_0, \mu)\frac{dX}{dX_0}] \; g(\mu)d\mu.$$

Since the expected value of any of the other bids is equal to X_0^*, dX/dX_0 evaluated at X_0^* equals 1. Also $n/X_0^*\sigma$ may be brought out in front of the integral since it is not a function of μ. Thus, equation (6.9) is obtained.

9. The large capital requirements for leasing, drilling, and production are probably a very effective barrier to participation in the offshore bidding.

10. Above, table 4.

11. Ibid.

12. Above, p. 48.

13. Walter J. Mead, "Natural Resources Disposal Policy—Oral Auction Versus Sealed Bids," *Natural Resources Journal* 7 (April, 1967): 195-97.

14. It implies that the owner of an object with a positive expected value should be willing to pay to get rid of it.

15. Vernon L. Smith, *Investment and Production* (Cambridge, Mass.: Harvard University Press, 1961), pp. 185-87.

CHAPTER VII

1. See Robert B. Wilson, "Competitive Bidding with Asymmetric Information," *Management Science* 13 (July, 1967): 816-20.

2. Walter J. Mead, "Natural Resources Disposal Policy—Oral Auction Versus Sealed Bids," *Natural Resources Journal* 7 (April, 1967): 194-224.

3. For a very interesting effort in this direction see S. Sankar Sengupta, *Operations Research in Sellers' Competition: A Stochastic Microtheory* (New York: John Wiley & Sons, 1967).

4. Lawrence Friedman, "Competitive Bidding Strategies" (Ph.D. diss., Case Institute of Technology, 1957), p. 114.

Bibliography

BOOKS

AITCHISON, J., and BROWN, J. A. C. *The Lognormal Distribution.* Cambridge: Cambridge University Press, 1957.

CHRISTENSON, CHARLES. *Strategic Aspects of Competitive Bidding for Corporate Securities.* Boston: Graduate School of Business Administration, Harvard University, 1965.

FARRAR, DONALD E. *The Investment Decision Under Uncertainty.* Englewood Cliffs, N.J.: Prentice-Hall, 1962.

GOLDBERGER, ARTHUR S. *Econometric Theory.* New York: John Wiley & Sons, 1964.

GRAYSON, C. JACKSON. *Decisions Under Uncertainty: Drilling Decisions by Oil and Gas Operators.* Boston: Graduate School of Business Administration, Harvard University, 1960.

HENDERSON, JAMES M., and QUANDT, RICHARD E. *Microeconomic Theory.* New York: McGraw-Hill, 1958.

HOEL, PAUL G. *Introduction to Mathematical Statistics.* New York: John Wiley & Sons, 1954.

KAUFMAN, GORDON M. *Statistical Decision and Related Techniques in Oil and Gas Exploration.* Englewood Cliffs, N.J.: Prentice-Hall, 1963.

KEYNES, JOHN M. *The General Theory of Employment, Interest, and Money.* London: Macmillan, 1961.

LEFTWICH, RICHARD H. *The Price System and Resource Allocation.* New York: Holt, Rinehart, & Winston, 1966.

LEVORSEN, A. I. *Geology of Petroleum.* San Francisco: W. H. Freeman & Co., 1954.

LINDLEY, D. V. *Introduction to Probability and Statistics.* Cambridge: Cambridge University Press, 1965.

McDonald, Stephen M. *Federal Tax Treatment of Income From Oil and Gas.* Washington: Brookings Institution, 1963.

Markowitz, Harry M. *Portfolio Selection: Efficient Diversification of Investments.* New York: John Wiley & Sons, 1959.

Musgrave, Richard A. *The Theory of Public Finance.* New York: McGraw-Hill, 1959.

Richmond, Samuel B. *Statistical Analysis.* New York: Ronald Press, 1964.

Schlaifer, Robert. *Probability and Statistics for Business Decisions.* New York: McGraw-Hill, 1959.

Sengupta, S. Sankar. *Operations Research in Sellers' Competition: A Stochastic Microtheory.* New York: John Wiley & Sons, 1967.

Siegel, Sidney. *Nonparametric Statistics for the Behavioral Sciences.* New York: McGraw-Hill, 1956.

ARTICLES AND PERIODICALS

Arps, John J. "A Strategy for Sealed Bidding," *Journal of Petroleum Technology* 17 (September 1965): 1033-38.

Friedman, Lawrence. "A Competitive Bidding Strategy," *Operations Research* 4 (February 1956): 104-12.

Hald, A. "Maximum Likelihood Estimation of the Parameters of a Normal Distribution Which Is Truncated at a Known Point," *Skandinavisk Actuarietidskrift* 32 (1949): 119-34.

Hanssmann, F., and Rivett, B. H. P. "Competitive Bidding," *Operational Research Quarterly* 10 (March 1959): 49-55.

Latané, Henry. "Criteria for Choice Among Risky Ventures," *Journal of Political Economy* 67 (April 1959): 144-55.

____________. "Criteria for Choice Among Risky Ventures." In *Frontiers of Investment Analysis*, edited by E. Bruce Fredrickson, pp. 57-66. Scranton, Pa: International Textbook Co., 1965.

Lilliefors, Hubert W. "On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown," *Journal of the American Statistical Association* 62 (June 1967): 399-402.

Lintner, John. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics* 42 (February 1965): 13-37.

Massey, Frank J. "The Kolmogorov-Smirnov Test," *Journal of the American Statistical Association* 46 (March 1951): 68-78.

MEAD, WALTER J. "Natural Resource Disposal Policy—Oral Auction Versus Sealed Bids," *Natural Resources Journal* 7 (April 1967): 194-224.

PRATT, JOHN W. "Risk Aversion in the Small and in the Large," *Econometrica* 32 (January-April 1964): 122-36.

RYAN, J.M., and GARDER, A. O. "The Measurement of Dynamic Risk," *Southern Economic Journal* 31 (January 1965): 207-14.

SHUBIK, M. "Towards a Study of Bidding Processes," *Naval Research Logistics Quarterly* 10 (1963): 11-21, 151-73, 199-217.

WILSON, ROBERT B. "Competitive Bidding with Asymmetric Information," *Management Science* 13 (July 1967): 816-20.

OTHER SOURCES

DEAN, BURTON V. *Contract Award and Bidding Strategies.* Case Institute of Technology Tech. Memo 20, 1964.

FRIEDMAN, LAWRENCE. "Competitive Bidding Strategies." Ph.D. dissertation, Case Institute of Technology, 1957.

NOREK, BERNARD J. "Some Models of Competitive Bidding." M.S. thesis, Case Institute of Technology, 1964.

U.S. Congress. *Outer Continental Shelf Lands Act,* Public Law 212, 83rd Congress, 2d session, 1953.

U.S. Department of Interior, Bureau of Land Management. "Bid Recap of OCS Sales," n.d. Mimeographed.

WILSON, JAMES C. *A Comprehensive Approach to Investment Planning.* Report presented to the Cost Study Committee, Independent Petroleum Association of America, Fall Meeting, Houston, Texas, October, 1964.