# A Geometrically Characterized Reference Frame For the Study of Cartan Hypersurfaces In N-Dimensional Projective Space[1]

W. DALE MANESS[2]

1. *Introduction.* In the projective differential geometry of ordinary space a problem of fundamental importance is that of obtaining a covariantly determined reference frame for the definition of local point coordinates and associated power series developments for the equations of curves and surfaces. Much of the celebrated memoir [1] of G. M. Green was devoted to this problem for a surface or 2-dimensional Cartan variety.[3] However, the complete geometric characterization of the reference frame used by Green was not completed until sixteen years later by Bell [2].

In this paper, an extension of Green's "relation R" is given for a linear $(n-2)$-space $L$ and a line $l$. This extended relation will also be known as the *relation* $R$. The power series expansion in local non-homogeneous coordinates for the equation of an $(n-1)$-dimensional variety $V_0$ in $n$ space is obtained using the method introduced by Bell [3]. This power series is simplified by choosing a suitable reference frame. The complete geometric characterization of this reference frame is given and a generalization of the edges of Green for $n$-dimensional space is obtained by use of the relation $R$. The power series obtained and also the reference frame chosen are shown to be generalizations of those obtained by Green [1] using Wilczynski's normal coordinates.

2. *Preliminary Remarks.* Let $x_0$ denote the generating point of an $(n-1)$-dimensional analytic variety $V_0$ in an $n$-dimensional linear space $S_n$. Let the vertices of a local reference frame be denoted by $x_0, x_1, \ldots, x_n$ and the vertices $x_1, x_2, \ldots, x_{n-1}$ be taken on the respective tangents to the $u^1, u^2, \ldots, u^{n-1}$ parametric curves of $V_0$ at $x_0$. The general coordinates of the vertices satisfy a system of partial differential equations of the form

$$\frac{\partial x_i}{\partial u^\alpha} - \Gamma^k{}_{i\alpha}\, x_k = 0 \quad {}^4 \qquad (2.1)$$

$$(i, k = 0, 1, \cdots, n; \quad \alpha = 1, 2, \cdots, n-1). \quad {}^5$$

[1]The author takes this opportunity to acknowledge his indebtedness to Dr. P. O. Bell for his inspiration, encouragement and help during the preparation of this paper.

[2]Department of Mathematics, Howard Payne College.

[3]A variety which sustains asymptotic parametric curves is known as a Cartan variety.

[4]Throughout this paper unless otherwise indicated a repeated upper and lower index in a term indicates summation.

[5]Unless otherwise indicated a Greek index will have a range of $1, 2, \ldots, n-1$ and a Roman index range of $0, 1, \ldots, n$.

The left-hand side is called the intrinsic derivative of $x_i$ with respect to $u^\alpha$ and is denoted by $x_{i,\alpha}$. By a proper choice for the proportionality factors for $x_1, x_2, \cdots, x_{n-1}$ the coefficients may be made to satisfy the relations

$$\Gamma^k{}_{0\alpha} = \delta^k{}_\alpha \text{ and } \Gamma^0{}_{0\alpha} \neq 0 \tag{2.2}$$

where $\delta^k{}_\alpha$ represents the Kronecker delta.

The intrinsic derivatives of the local coordinates $x^i$ are defined by

$$x^i{}_{,\alpha} = x^i{}_\alpha + x^k \Gamma^i{}_{k\alpha} \tag{2.3}$$

in which $x^i{}_\alpha$ denotes the partial derivative of $x^i$ with respect to $u^\alpha$ [6]. It is easily verified that intrinsic derivatives of products obey the usual rules for derivatives. Moreover, the intrinsic derivative of an inner product $x^i x_i$ with respect to $u^\alpha$ is identical with the partial derivative with respect to $u^\alpha$.

The integrability conditions that the coefficients $\Gamma^k{}_{i\alpha}$ of the equations (2.1) satisfy may be written as

$$\Gamma^k{}_{i\alpha,\beta} = \Gamma^k{}_{i\beta,\alpha} \tag{2.4}$$

wherein the intrinsic differentiation is with respect to the index $k$.

The conditions that a point $X$ will remain fixed while $u^\epsilon$ varies are given in terms of non-homogeneous local coordinates $z^j$ in the form

$$z^j{}_\epsilon = z^j z^k \Gamma^0{}_{k\epsilon} - z^k \Gamma^j{}_{k\epsilon} \tag{2.5}$$

where

$$z^j = \frac{x^j}{x^0}$$

Let

$$f \equiv a_i x^i = 0,$$

in which the $a$'s are arbitrary functions of the $u$'s, be the equation of a hyperplane in $S_n$ and let a curve $C$ on $V_0$ be represented by

$$x^i = x^i(u^1, u^2, \cdots, u^{n-1}), \quad u^i = u^i(t).$$

If $a_i x^i = 0$ is the tangent hyperplane to $V_0$ at $x_0$ and the direction of $C$ at $x_0$ is a direction in which the tangent hyperplane has second order contact with $V_0$ at $x_0$, then $f = 0, f' = 0, f'' = 0$ at $(1, 0, 0, \cdots, 0)$. Solving these equations simultaneously gives

$$a_n \Gamma^n{}_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt} = 0. \tag{14}$$

[6]The convention of denoting a partial derivative of a function with respect to $u^\alpha$ by affixing a subscript $\alpha$ to the symbol which denotes the function will be used.

Directions satisfying this condition are known as asymptotic directions. Hence, the directions of the asymptotic curves of $V_0$ are given by an equation of the form

$$\varphi_n \Gamma^n{}_{\beta\alpha} du^\alpha du^\beta = 0, \tag{2.6}$$

where $\varphi_n$ is an arbitrary function. The equation (2.6) is the defining equation of a net of quadric cones of vertex $x_0$ which is called the asymptotic net of $V_0$ at $x_0$ and generalizes the asymptotic net of a surface in $S_3$. The tangents to the parametric curves of $V_0$ at $x_0$ are generators of the asymptotic net of quadric cones of $V_0$ if the coefficients of (2.6) satisfy the relations

$$\Gamma^n{}_{\beta\alpha} \neq 0 \ (\alpha \neq \beta) \text{ and } \Gamma^n{}_{\alpha\alpha} = 0. \tag{2.7}$$

An analytical hypersurface for which the relations (2.7) are true is a Cartan hypersurface.

Making use of (2.2) and (2.7) in the integrability conditions (2.4) the following relations between the coefficients of (2.1) can be established:

$$\Gamma^p{}_{\alpha\beta} = \Gamma^p{}_{\beta\alpha}. \tag{2.8}$$

$$\Gamma^\epsilon{}_{\beta\alpha} = \Gamma^\epsilon{}_{\alpha\beta} + \delta^\epsilon{}_\beta \Gamma^0{}_{0\alpha} - \delta^\epsilon{}_\alpha \Gamma^0{}_{0\beta}. \tag{2.9}$$

$$\Gamma^p{}_{\alpha\beta_\alpha} = \Gamma^h{}_{\alpha\alpha} \Gamma^p{}_{h\beta} - \Gamma^h{}_{\alpha\beta} \Gamma^p{}_{h\alpha}. \tag{2.10}$$

$$\Gamma^p{}_{\alpha\beta_\alpha} = \Gamma^\epsilon{}_{\alpha\alpha} \Gamma^p{}_{\epsilon\beta} - \Gamma^\epsilon{}_{\alpha\beta} \Gamma^p{}_{\epsilon\alpha} - \Gamma^r{}_{\alpha\beta} \Gamma^p{}_{r\alpha}. \tag{2.11}$$

3. *The Relation R.* As $x_0$, the generating point of an analytic hypersurface $V_0$ in $n$-space, moves along a fixed parametric curve $u^\beta$, $x_\alpha$, a point on the tangent to the $u^\alpha$ parametric curve, moves along a corresponding $\bar{u}^\beta$ curve and $x_{\alpha\beta}$ is a point on the tangent to this $\bar{u}^\beta$ curve at $x_\alpha$ defined as follows

$$x_{\alpha\beta} = (x_\alpha)_\beta = \Gamma^h{}_{\alpha\beta} x_h = \Gamma^0{}_{\alpha\beta} x_0 + \Gamma^\alpha{}_{\alpha\beta} x_\alpha + \Gamma^\epsilon{}_{\alpha\beta} x_\epsilon + \Gamma^n{}_{\alpha\beta} x_n \tag{3.1}$$

(3.2) $(\epsilon \neq \alpha, \ \alpha \neq \beta)$ where by use of equations (2.1) and (2.2)

$$x_\alpha = x_{0_\alpha} - \Gamma^0{}_{0\alpha} x_0.$$

Now $x_0$, $x_\alpha$ and the $(n\text{-}2)$ points $x_{\alpha\beta}$ $(\beta = 1, 2, \ldots, \alpha\text{-}1, \alpha + 1, \ldots, n\text{-}1)$ determine a linear space usually of dimension $(n\text{-}1)$ whose determinant equation is

$$
(3.3)\qquad
\begin{vmatrix}
x^0 & x^1 & x^2 & \dots x^\alpha & \dots x^n \\
1 & 0 & 0 & \dots 0 & \dots 0 \\
0 & 0 & 0 & \dots 1 & \dots 0 \\
\Gamma^0{}_{\alpha 1} & \Gamma^1{}_{\alpha 1} & \Gamma^2{}_{\alpha 1} & \dots \Gamma^\alpha{}_{\alpha 1} & \dots \Gamma^n{}_{\alpha 1} \\
\Gamma^0{}_{\alpha 2} & \Gamma^1{}_{\alpha 2} & \Gamma^2{}_{\alpha 2} & \dots \Gamma^\alpha{}_{\alpha 2} & \dots \Gamma^n{}_{\alpha 2} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\Gamma^0{}_{\alpha\alpha-1} & \Gamma^1{}_{\alpha\alpha-1} & \Gamma^2{}_{\alpha\alpha-1} & \dots \Gamma^\alpha{}_{\alpha\alpha-1} & \dots \Gamma^n{}_{\alpha\alpha-1} \\
\Gamma^0{}_{\alpha\alpha+1} & \Gamma^1{}_{\alpha\alpha+1} & \Gamma^2{}_{\alpha\alpha+1} & \dots \Gamma^\alpha{}_{\alpha\alpha+1} & \dots \Gamma^n{}_{\alpha\alpha+1} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\Gamma^0{}_{\alpha n-1} & \Gamma^1{}_{\alpha n-1} & \Gamma^2{}_{\alpha n-1} & \dots \Gamma^\alpha{}_{\alpha n-1} & \dots \Gamma^n{}_{\alpha n-1}
\end{vmatrix} = 0.
$$

As $\alpha$ assumes all possible values $1, 2, \dots, n-1$ in (3.3) there is a system of $n-1$ linear $(n-1)$-spaces (hyperplanes) determined each of which passes through $x_0$. This system of hyperplanes is represented by the system of equations

$$
(3.4)\qquad \sum_{i=1}^{n} u_{\alpha i}\, x^i = 0 \quad (\alpha \neq i)
$$

and the coefficient $u_{\alpha i}$ is the determinant

$$
|\Gamma^k{}_{\alpha\beta}|;\ \alpha = 1, 2, \dots, n-1;
$$
$$
\beta = 1, 2, \dots, \alpha-1, \alpha+1, \dots, n-1;
$$
$$
k = 1, 2, \dots, i-1, i+1, \dots, \alpha-1, \alpha+1, \dots, n.
$$

If the rank of this system of $(n-1)$ equations in $n$ unknowns is $n-1$, the dimension of the linear space of intersection is $n-(n-1) = 1$. This is equivalent to the statement that the intersection of these $(n-1)$ hyperplanes is a line if, and only if, they are linearly independent. If the rank of the system is less than $n$–1, the dimension of the linear space of intersection is greater than one and the $(n-1)$ hyperplanes are linearly dependent.

The system of $(n-1)$ hyperplanes given by the equations (3.4) are taken to be linearly independent. It is evident from the equation (3.3) that $x_n$ is in the hyperplane corresponding to a fixed $\alpha$ if, and only if,

$$
(3.5)\qquad |\Gamma^u{}_{\alpha\beta}| = 0,\ (\beta, u = 1, 2, \dots, \alpha-1, \alpha+1, \dots, n-1).
$$

If equation (3.5) holds for each $\alpha$, then $x_n$ is in each hyperplane and hence $x_0x_n$ is the line of intersection of the $(n-1)$ linearly independent hyperplanes given by the equations (3.4).

Now to each $x_\alpha$ [7] there corresponds a hyperplane, (3.3), passing through the $u^\alpha$-tangent and the line $l \equiv x_0x_n$. Hence, to the linear $(n-2)$-

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[7]This point will be uniquely determined with the determination of $\Gamma^0{}_{0\alpha}$ in Section 5.

space $L$ in the tangent hyperplane determined by the $(n-1)$ linearly independent points $x_\alpha$ there corresponds the line $l$ and conversely. The linear $(n-2)$-space $L$ and the line $l$ which are related to each other in this manner are said to be in *the relation R.* The geometric relation that has been used here to define $l$ and $L$ is an extension of the known "relation $R$" of Green [1:pp. 86-87].

4. *Calculation Of The Power Series Of $V_0$.* If an arbitrary local reference frame is chosen, the power series development in non-homogeneous coordinates of $V_0$, an $(n-1)$-dimensional analytic variety in $n$-space, will contain all terms, even the constant term. The constant term disappears when $x_0$, the generating point of $V_0$, is taken as a vertex of the local reference frame. The $z^\alpha$ terms vanish if the tangent hyperplane to $V_0$ at $x_0$ is taken as a face of the reference frame. If the asymptotic tangents ($u^\alpha$-tangents) are taken as edges of the reference frame, the $(z^\alpha)^2$ terms vanish.

If the above choices for the local reference frame are made and the method of Bell [3] is used, the power series development of $V_0$ expressed in terms of non-homogeneous coordinates assumes the form

$$z^n = a_{\alpha\beta}z^\alpha z^\beta + a_{\alpha\beta\gamma}z^\alpha z^\beta z^\gamma + a_{\alpha\beta\gamma\delta}z^\alpha z^\beta z^\gamma z^\delta + \dots, \tag{4.1}$$

where the coefficients are functions of $u^\epsilon$.

To determine the coefficients of the power series, differentiate both sides of (4.1) with respect to $u^\epsilon$ and substitute for $z^i{}_\epsilon$ the values given by the fixed point conditions (2.5) and for $z^n$ the value given by (4.1). The two sides are then identical power series by the uniqueness theorem, and by equating corresponding terms the coefficients are determined. The asymptotic curves are taken to be parametric, so upon making use of (2.2) and (2.6), the coefficients of the terms of (4.1) out to and including those of degree four are found to be

$$2!\ a_{\alpha\beta} = \Gamma^n{}_{\alpha\beta},$$

$$3!\ a_{\alpha\beta\gamma} = \Gamma^n{}_{\alpha\beta_\gamma} + \Gamma^n{}_{\alpha\beta}\Gamma^0{}_{0\gamma} + \Gamma^n{}_{n\gamma}\Gamma^n{}_{\alpha\beta} - \Gamma^n{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\beta\gamma} - \Gamma^n{}_{\beta\epsilon}\Gamma^\epsilon{}_{\alpha\gamma} = \Gamma^n{}_{\alpha\beta,\gamma} + \Gamma^n{}_{\alpha\beta}\Gamma^0{}_{0\gamma},$$

$$4!\ a_{\alpha\beta\gamma\delta} = (\Gamma^n{}_{\alpha\beta,\gamma} + \Gamma^n{}_{\alpha\beta}\Gamma^0{}_{0\gamma})_{,\delta} + 2\Gamma^0{}_{0\delta}(\Gamma^n{}_{\alpha\beta,\gamma} + \Gamma^n{}_{\alpha\beta}\Gamma^0{}_{0\gamma}) + 3\Gamma^0{}_{\gamma\delta}\Gamma^n{}_{\alpha\beta} - \frac{3\Gamma^n{}_{\alpha\epsilon}\Gamma^\epsilon{}_{n\delta}\Gamma^n{}_{\beta\gamma}}{2} - \frac{3\Gamma^n{}_{\beta\epsilon}\Gamma^\epsilon{}_{n\delta}\Gamma^n{}_{\alpha\gamma}}{2},$$

where $\Gamma^n{}_{\alpha\beta,\gamma}$ is the restricted convariant derivative [3:p.210] of $\Gamma^n{}_{\alpha\beta}$ with respect to $u^\gamma$ defined by the relations

$$\Gamma^n{}_{\alpha\beta,\gamma} = \Gamma^n{}_{\alpha\beta_\gamma} + \Gamma^n{}_{n\gamma}\Gamma^n{}_{\alpha\beta} - \Gamma^n{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\beta\gamma} - \Gamma^n{}_{\beta\epsilon}\Gamma^\epsilon{}_{\alpha\gamma}. \tag{4.2}$$

Hence, (4.1) may be written to as many terms as will be needed in the subsequent discussion as

$$
(4.3)\qquad
\begin{aligned}
z^n = {} & \frac{\Gamma^n{}_{\alpha\beta}\, z^\alpha z^\beta}{2} - \frac{\Gamma^n{}_{\alpha\epsilon}\,\Gamma^\epsilon{}_{\alpha\alpha}\,(z^\alpha)^3}{3} \\
& + \frac{[\Gamma^n{}_{\alpha\beta,\alpha} + \Gamma^n{}_{\alpha\beta}\,\Gamma^0{}_{0\alpha} - \Gamma^n{}_{\alpha\epsilon}\,\Gamma^\epsilon{}_{\alpha\beta}]\,(z^\alpha)^2 z^\beta}{3} \\
& + \frac{[\Gamma^n{}_{\alpha\beta,\gamma} + \Gamma^n{}_{\alpha\beta}\,\Gamma^0{}_{0\gamma}]z^\alpha z^\beta z^\gamma}{6} \\
& + \frac{[(\Gamma^n{}_{\alpha\alpha,\alpha} + \Gamma^n{}_{\alpha\alpha}\,\Gamma^0{}_{0\alpha})_{,\alpha} + 2\Gamma^0{}_{0\alpha}\,\Gamma^n{}_{\alpha\alpha,\alpha}]\,(z^\alpha)^4}{24} + \ldots\ldots
\end{aligned}
$$

Let the $(n-1)$ points $x^\alpha$ (3.2) on the $u^\alpha$-tangents be taken as the vertices on those edges of the local reference frame and the $(n-1)$ hyperplanes (3.4) passing through those edges will be taken as faces of the reference frame. Since $x_0x_n$ (the line $l$) lies in the intersection of these faces, it will be taken as an edge of the reference frame.

In three-space, Green [1:pp. 116-117] has shown that if the pair of lines in relation $R$ which are now known as the edges of Green are the lines $x_0x_3$ and $x_1x_2$ of the local reference frame, then the $x^2y$ and $xy^2$ terms in the power series expansion for $z$ in terms of $x$ and $y$ vanish. The converse is also true. If the $x^2y$ and $xy^2$ terms vanish, $x_0x_3$ and $x_1x_2$ of the local reference frame are the edges of Green (i.e., if the asymptotic net is parametric).

In the preceding section, it was shown that (3.5) were the necessary and sufficient conditions for the line $l$ to be in relation $R$ to the $(n-2)$-space $L$ in the tangent hyperplane. These are also the necessary and sufficient conditions for the vanishing of the coefficients of $(z^\alpha)^2z^\beta$ in the power series expansion for $z^n$ in terms of $z^\epsilon$ in (4.3). That is, the following theorem can now be proved.

*Theorem.* The coefficients of $(z^\alpha)^2z^\beta$ vanish in the power series expansion for $z^n$ in terms of $z^\epsilon$ (4.3) if, and only if,

$$|\Gamma^u{}_{\alpha\beta}| = 0.$$

From the power series (4.3), the coefficient of $(z^\alpha)^2z^\beta$ *is*

$$(4.4)\qquad 1/3\,(\Gamma^n{}_{\alpha\beta,\alpha} + \Gamma^n{}_{\alpha\beta}\,\Gamma^0{}_{0\alpha} - \Gamma^n{}_{\alpha\epsilon}\,\Gamma^\epsilon{}_{\alpha\beta}).$$

If $\Gamma^n{}_{\alpha\beta,\alpha}$ is replaced by use of (4.2) and $(\Gamma^n{}_{\alpha\beta_\alpha}$ by use of (2.11), (4.4) becomes

$$
\begin{aligned}
1/3\,[\Gamma^\epsilon{}_{\alpha\alpha}\,\Gamma^n{}_{\epsilon\beta} - \Gamma^\epsilon{}_{\alpha\beta}\,\Gamma^n{}_{\epsilon\alpha} - \Gamma^n{}_{\alpha\beta}\,\Gamma^n{}_{n\alpha} + \Gamma^n{}_{\alpha\beta}\,\Gamma^0{}_{0\alpha} + {}^n{}_{\alpha\beta}\,\Gamma^n{}_{n\alpha} \\
- \Gamma^n{}_{\epsilon\alpha}\,\Gamma^\epsilon{}_{\alpha\beta} - \Gamma^n{}_{\epsilon\alpha}\,\Gamma^\epsilon{}_{\beta\alpha} - \Gamma^n{}_{\epsilon\beta}\,\Gamma^\epsilon{}_{\alpha\alpha}],
\end{aligned}
$$

which upon collecting terms assumes the form

$$-1/3\ (2\Gamma^{\epsilon}{}_{\alpha\beta}\,\Gamma^{n}{}_{\epsilon\alpha}-\Gamma^{\epsilon}{}_{\beta\alpha}\,\Gamma^{n}{}_{\epsilon\alpha}+\Gamma^{n}{}_{\alpha\beta}\,\Gamma^{0}{}_{0\alpha}).$$

Upon substituting from (2.9) for $\Gamma^{\epsilon}{}_{\beta\alpha}$ and writing $\Gamma^{n}{}_{\alpha\beta}$ as $\delta^{\epsilon}{}_{\beta}\,\Gamma^{n}{}_{\epsilon\alpha}$ (4.4) is then

$$-1/3\ [2\Gamma^{\epsilon}{}_{\alpha\beta}\,\Gamma^{n}{}_{\epsilon\alpha}-\Gamma^{\epsilon}{}_{\alpha\beta}\,\Gamma^{n}{}_{\epsilon\alpha}-\delta^{\epsilon}{}_{\beta}\,\Gamma^{0}{}_{0\alpha}\,\Gamma^{n}{}_{\epsilon\alpha}+\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}\,\Gamma^{n}{}_{\epsilon\alpha}+\delta^{\epsilon}{}_{\beta}\,\Gamma^{0}{}_{0\alpha}\,\Gamma^{n}{}_{\epsilon\alpha}]$$

or

$$-\Gamma^{n}{}_{\epsilon\alpha}\left(\Gamma^{\epsilon}{}_{\alpha\beta}-\frac{\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}}{3}\right),\quad (\alpha\neq\beta).$$

If (4.4) vanishes then

$$(4.5)\qquad -\Gamma^{n}{}_{\epsilon\alpha}\left(\Gamma^{\epsilon}{}_{\alpha\beta}-\frac{\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}}{3}\right)=0.$$

As $\alpha$ assumes all possible values $1, 2, \ldots, n-1$, (4.5) represents $(n-2)$ equations in the $(n-2)$ unknowns $\Gamma^{n}{}_{\epsilon\alpha}$.
Since

$$\Gamma^{n}{}_{\epsilon\alpha}\neq 0 \text{ when } \epsilon\neq\alpha \text{ and } \Gamma^{n}{}_{\alpha\alpha}=0,$$

then

$$\left|\Gamma^{\epsilon}{}_{\alpha\beta}-\frac{\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}}{3}\right|=0.\quad \text{But } \epsilon\neq\alpha,$$

But $\epsilon\neq\alpha$; therefore,

$$|\,\Gamma^{\epsilon}{}_{\alpha\beta}\,|=0.$$

Conversely, if

$$|\,\Gamma^{\epsilon}{}_{\alpha\beta}\,|=0\quad \alpha\neq\beta,$$

then

$$\left|\Gamma^{\epsilon}{}_{\alpha\beta}-\frac{\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}}{3}\right|=0,\quad \alpha\neq\epsilon,$$

and since

$$\Gamma^{n}{}_{\epsilon\alpha}\neq 0,$$

then

$$\Gamma^{n}{}_{\epsilon\alpha}\left(\Gamma^{\epsilon}{}_{\alpha\beta}-\frac{\delta^{\epsilon}{}_{\alpha}\,\Gamma^{0}{}_{0\beta}}{3}\right)=0$$

or (4.4) vanishes.

Much of the preceding discussion is dependent upon a suitable choice of the $\Gamma^{0}{}_{0\alpha}$. In fact, as soon as $\Gamma^{0}{}_{0\alpha}$ is chosen, the vertices $x_{\alpha}$ of the local reference frame are determined, the edge $x_0x_n$ is determined by the relation $R$ and as will be seen by a suitable choice of the $\Gamma^{0}{}_{0\alpha}$, the coefficients of the $(z^{\alpha})^{4}$ terms in the power series (4.3) will vanish.

If the coefficients of $(z^{\alpha})^{4}$ in (4.3) are to vanish, then

$$(\Gamma^{n}{}_{\alpha\alpha,\alpha}+\Gamma^{n}{}_{\alpha\alpha}\,\Gamma^{0}{}_{0\alpha})_{,\alpha}+2\Gamma^{0}{}_{0\alpha}\,\Gamma^{n}{}_{\alpha\alpha,\alpha}=0,$$

which upon expansion and collection of terms becomes

$$-2\Gamma^n_{\alpha\epsilon_\alpha}\Gamma^\epsilon_{\alpha\alpha}-\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha_\alpha}-2\Gamma^n_{n\alpha}\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}+\Gamma^n_{\alpha\pi}\Gamma^\pi_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}$$
$$+\Gamma^n_{\alpha\pi}\Gamma^\pi_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha}+\Gamma^n_{\epsilon\pi}\Gamma^\pi_{\alpha\alpha}\Gamma^\epsilon_{\alpha\alpha}-3\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}\Gamma^0_{0\alpha}=0.$$

Substituting from (2.11) for $\Gamma^n_{\alpha\epsilon_\alpha}$ and again collecting terms gives

$$-\Gamma^n_{\epsilon\pi}\Gamma^\pi_{\alpha\alpha}\Gamma^\epsilon_{\alpha\alpha}+3\Gamma^n_{\pi\alpha}\Gamma^\pi_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha}+\Gamma^n_{\alpha\pi}\Gamma^\pi_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}$$
$$-3\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}\Gamma^0_{0\alpha}-\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha_\alpha}=0.$$

When $\Gamma^\pi_{\epsilon\alpha}$ is replaced by (2.9), the equation becomes

$$-3\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}\Gamma^0_{0\alpha}-\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha_\alpha}-\Gamma^n_{\epsilon\pi}\Gamma^\pi_{\alpha\alpha}\Gamma^\epsilon_{\alpha\alpha}$$
$$+\Gamma^n_{\alpha\pi}\Gamma^\epsilon_{\alpha\alpha}[\Gamma^\pi_{\epsilon\alpha}+3(\Gamma^\pi_{\epsilon\alpha}+\delta^\pi_\alpha\Gamma^0_{0\epsilon}-\delta^\pi_\epsilon\Gamma^0_{0\alpha})]=0.$$

But $\pi\neq\alpha$, hence

$$-3\Gamma^0_{0\alpha}(\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}+\delta^\pi_\epsilon\Gamma^n_{\alpha\pi}\Gamma^\epsilon_{\alpha\alpha})+4\Gamma^n_{\alpha\pi}\Gamma^\epsilon_{\alpha\alpha}\Gamma^\pi_{\epsilon\alpha}$$
$$-\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha_\alpha}-\Gamma^n_{\epsilon\pi}\Gamma^\pi_{\alpha\alpha}\Gamma^\epsilon_{\alpha\alpha}=0.$$

Then if the function $\Gamma^0_{0\alpha}$ is defined by

$$\Gamma^0_{0\alpha}=\frac{4\Gamma^n_{\alpha\pi}\Gamma^\pi_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}-\Gamma^n_{\epsilon\pi}\Gamma^\pi_{\alpha\alpha}\Gamma^\epsilon_{\alpha\alpha}-\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha_\alpha}}{3(\Gamma^n_{\epsilon\alpha}\Gamma^\epsilon_{\alpha\alpha}+\delta^\pi_\epsilon\Gamma^n_{\alpha\pi}\Gamma^\epsilon_{\alpha\alpha})},\tag{4.6}$$

the coefficient of $(z^\alpha)^4$ vanishes, each vertex $x_\alpha$ and the edge $x_0x_n$ of the local reference frame are determined. The linear $(n-2)$-space $L$ determined by these points $x_\alpha$ and the line $x_0x_n$ in relation $R$ to $L$ are generalizations for $n$-dimensional space of the edges of Green.

The power series expansion (4.3) has now become

$$z^n=\frac{\Gamma^n_{\alpha\beta}z^\alpha z^\beta}{2}-\frac{\Gamma^n_{\alpha\epsilon}\Gamma^\epsilon_{\alpha\alpha}(z^\alpha)^3}{3}+\frac{(\Gamma^n_{\alpha\beta,\gamma}+\Gamma^n_{\alpha\beta}\Gamma^0_{0\gamma})z^\alpha z^\beta z^\gamma}{6}+\ldots,\tag{4.7}$$

where the value of $\Gamma^0_{0\gamma}$ is given by (4.6) and the local reference frame has been chosen as follows: the generating point $x_0$ as a vertex, the tangent hyperplane at $x_0$ as a face, the asymptotic tangents as edges, each point $x_\alpha$ on the asymptotic $u^\alpha$-tangent as a vertex, the line $l$ in relation $R$ to the linear $(n-2)$-space $L$ as an edge.

5. *The Geometric Characterization of the $(n+1)^{st}$ Vertex.* The reference frame will be completely characterized if the vertex $x_n$ is characterized. The point $x_n$ is characterized by use of the invariant defined by Bell [4: pp. 34-35].

Let $X$ denote a point on $x_0x_n$ different from $x_0$. Let $\pi$ denote the tangent hyperplane to $V_0$ at $x_0$. Let $\pi_1$ be the tangent hyperplane to $V_0$ at a point $y$ whose general coordinates are given by

$$y=x_0(u^1+du^1,u^2+du^2,\ldots,u^{n-1}+du^{n-1}).$$

Let $X_1$ denote a point whose general homogeneous coordinates are given by

$$X_1 = X\,(u^1+du^1, u^2+du^2, \ldots, u^{n-1}+du^{n-1}).$$

The line joining $X$ and $X_1$ intersects $\pi$ in a point $Y$ and $\pi_1$ in a point $Y_1$. Bell [4] has shown that the principal part, $\Omega$, of the cross ratio $(X, Y, X_1, Y_1)$ is given by

$$\Omega = a_{\alpha\beta}\, du^\alpha\, du^\beta, \tag{5.8}$$

where

$$a_{\alpha\beta} = \frac{(x^i\,\xi_j\,\xi_{i,\alpha}\;x^j{}_{,\beta} - x^i\xi_i\,x^j{}_{,\alpha}\,\xi_{j,\beta})}{(x^i\,\xi_i)^2},$$

in which $\xi_i$ are the local plane coordinates of $\pi$ and $x^i$ are the local homogeneous point coordinates of $X$.

Let the general coordinates of $X$ be given by the equation

$$X = x_n - \eta x_0.$$

The local homogeneous coordinates of $x$ are

$$x^i = -\eta\delta^i{}_0 + \delta^i{}_n$$

and of $\pi$ are

$$\xi_i = \delta^n{}_i.$$

Substituting these into the expression for $a_{\alpha\beta}$ and simplifying gives

$$a_{\alpha\beta} = \Gamma^\epsilon{}_{n\alpha}\,\Gamma^n{}_{\epsilon\beta} - \eta\Gamma^n{}_{\alpha\beta}.$$

Hence

$$\Omega = (\Gamma^\epsilon{}_{n\alpha}\,\Gamma^n{}_{\epsilon\beta} - \eta\Gamma^n{}_{\alpha\beta})\, du^\alpha\, du^\beta. \tag{5.9}$$

Now if there is a point $P_\gamma$ on the line $x_0x_n$ such that $P_\gamma$ generates a curve whose tangent passes through the linear space determined by the tangent lines at $x_0$ to the $u^\alpha$-curves $\alpha \neq \gamma$, then the general homogeneous coordinates of the point $P_\gamma$ will be given by the equation

$$P_\gamma = x_n + \eta_\gamma x_0.$$

Hence,

$$\begin{aligned} \frac{\partial P_\gamma}{\partial u^\gamma} &= \frac{\partial x_n}{\partial u^\gamma} + \frac{\partial \eta_\gamma}{\partial u^\gamma} x_0 + \eta_\gamma \frac{\partial x_0}{\partial u^\gamma} \\ &= \Gamma^k{}_{n\gamma}\, x_k + \eta_\gamma\, x_\gamma + \eta_\gamma\, \Gamma^0{}_{0\gamma}\, x_0 + \frac{\partial \eta_\gamma}{\partial u^\gamma} x^0. \end{aligned} \tag{5.10}$$

The necessary and sufficient condition that there is such a point $P_\gamma$ is that the coefficient of $x_\gamma$ vanishes in the right hand side of (5.10) or that

$$\Gamma^\gamma{}_{n\gamma} + \eta_\gamma = 0 \quad (\gamma \text{ free index}).$$

Hence the coordinates of $P_\gamma$ are given by

$$P_\gamma = x_n - \Gamma^\gamma{}_{n\gamma}\, x_0 \quad (\gamma \text{ free index}).$$

Then at $P_\gamma$ the invariant $\Omega$ given by (5.9) is

$$\Omega_\gamma = (\Gamma^\epsilon{}_{n\alpha}\Gamma^n{}_{\epsilon\beta} - \Gamma^\gamma{}_{n\gamma}\Gamma^n{}_{\alpha\beta})\, du^\alpha du^\beta \quad (\gamma \text{ free index}).$$

Now as $\gamma$ assumes all possible values, the sum of the $\Omega_\gamma$'s is given by the equation

$$\sum_{\gamma=1}^{n-1} \Omega_\gamma = [(n-1)\,\Gamma^\epsilon{}_{n\alpha}\Gamma^n{}_{\epsilon\beta} - \Gamma^\gamma{}_{n\gamma}\Gamma^n{}_{\alpha\beta}]\, du^\alpha du^\beta.$$

Therefore the average value is

$$(5.11) \qquad \frac{\sum_{\gamma=1}^{n-1} \Omega_\gamma}{n-1} = \left[\Gamma^\epsilon{}_{n\alpha}\Gamma^n{}_{\epsilon\beta} - \frac{\Gamma^\gamma{}_{n\gamma}\Gamma^n{}_{\alpha\beta}}{n-1}\right] du^\alpha du^\beta.$$

Let $\bar{P}$ denote a point on $x_0x_n$ whose general homogeneous coordinates are given by the equation

$$\bar{P} = x_n + \bar{\eta}\, x_0$$

and let $\bar{\eta}$ be such that the value of $\Omega$ at $\bar{P}$ is given by (5.11). Then by (5.9) and (5.11)

$$\bar{\eta} = \frac{\Gamma^\gamma{}_{n\gamma}}{n-1}$$

and the coordinates of $\bar{P}$ are given by the equation

$$\bar{P} = x_n + \frac{\Gamma^\gamma{}_{n\gamma}\, x_0}{n-1}.$$

Now the point $\bar{P}$ will coincide with the point $x_n$ if

$$\frac{\Gamma^\gamma{}_{n\gamma}}{n-1} = 0, \quad \text{i.e., } \Gamma^\gamma{}_{n\gamma} = 0.$$

6. *Darboux's Canonical Expansion of $V_0$.* Darboux's canonical expansion in non-homogeneous coordinates for the equation of a surface $V_0$ in ordinary projective space as given by Green [1: pp. 115-118] can now be obtained by a specialization of (4.7).

The local reference tetrahedron has the vertices $y_0, y_1, y_2, y_3$ defined by

$$y_0 = y, \; y_1 = y_u - \beta y, \; y_2 = y_v - \alpha y,$$
$$y_3 = y_{uv} - \lambda y_u - \eta y_v - \gamma y$$

in which $y$ denotes Wilczynski's normal coordinates of $y_0$ and $u, v$ are asymptotic parameters. The following relations exist among the coefficients $\Gamma^n{}_{i\alpha}$ of equations (1.1) and the coefficients of Wilczynski's canonical form,

$$y_{uu} + 2b\, y_v + fy = 0$$
$$y_{vv} + 2a' y_u + gy = 0:$$

$$\Gamma^0{}_{01} = \beta, \Gamma^0{}_{02} = \alpha, \Gamma^h{}_{0\epsilon} = \delta^h{}_\epsilon \ (\epsilon = 1, 2;\ h = 1, 2, 3),$$
$$\Gamma^0{}_{11} = -(\beta^2 + \beta_u + 2b\alpha + f), \Gamma^1{}_{11} = -\beta, \Gamma^2{}_{11} = -2b,$$
$$\Gamma^3{}_{11} = 0,\ \Gamma^0{}_{21} = \eta\alpha + \beta\lambda + \gamma - \alpha\beta - \alpha_u,\ \Gamma^1{}_{21} = \lambda - \alpha,$$
$$\Gamma^2{}_{21} = \eta,\ \Gamma^3{}_{21} = 1,\ \Gamma^0{}_{12} = \eta\alpha + \beta\lambda + \gamma - \alpha\beta - \beta_v,$$
$$\Gamma^1{}_{12} = \lambda,\ \Gamma^2{}_{12} = \eta - \beta,\ \Gamma^3{}_{12} = 1,$$
$$\Gamma^0{}_{22} = -(\alpha^2 + \alpha_v + 2a'\beta + g),\ \Gamma^1{}_{22} = -2a',\ \Gamma^2{}_{22} = -\alpha,$$
$$\Gamma^3{}_{22} = 0, \Gamma^0{}_{31} = 4\beta a'b + 2bg - 2\beta b_v - \beta f - f_v - \beta\lambda_u$$
$$+ \lambda f + 2a\lambda b - \alpha\eta_u - \alpha\eta^2 - \beta\lambda\eta - \eta\gamma - \gamma_u - \beta\gamma,$$
$$\Gamma^1{}_{31} = 4a'b - \lambda_u - \eta\lambda - \gamma, \Gamma^2{}_{31} = 2\lambda b - 2b_v - f - \eta_u - \eta^2,$$
$$\Gamma^3{}_{31} = -\eta,\ \Gamma^0{}_{32} = 4\alpha a'b + 2a'f - 2\beta a'_u - \beta g - g_u - \alpha\lambda\eta$$
$$- \beta\lambda^2 - \lambda\gamma - \beta\lambda_v - \alpha\eta_v + 2\beta\eta a' + \eta g - \gamma_v - \alpha\gamma,$$
$$\Gamma^1{}_{32} = 2\eta a' - 2a'_u - g - \lambda^2 - \lambda_v,\ \Gamma^2{}_{32} = 4a'b - \lambda\eta - \eta_v - \gamma,$$
$$\Gamma^3{}_{32} = -\lambda.$$

The local reference tetrahedron is characterized as follows: the generating point of the surface $y_0$ is a vertex, the tangent plane at $y_0$ is a face, the asymptotic tangents are edges, the points $y_1$, $y_2$ on the asymptotic tangents are vertices, the reciprocal line $y_0y_3$ of the line $y_1y_2$ in the tangent plane is an edge. The position of the vertex $y_3$ is usually not determined except that it is not in the tangent plane. Then as in section 4, the coefficients of

$$z^1,\ z^2,\ (z^1)^2\, z^2,\ z^1(z^2)^2,\ (z^1)^4 \text{ and } (z^2)^4$$

all vanish. Hence the power series expansion is

$$z^3 = z^1 z^2 + \frac{2b\,(z^1)^3}{3} + \frac{2a'\,(z^2)^3}{3} + \frac{2\,(b_v - 2b\alpha)\,(z^1)^3 z^2}{3}$$
$$+ \frac{2\,(a'_u - 2a'\beta)\, z^1\,(z^2)^3}{3} + (\gamma + \alpha\beta)\ (z^1)^2 (z^2)^2 + \ldots.$$

In Green's expansion, $\gamma + \alpha\beta = 0$ from another consideration [1: p. 118].

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