1-1-1985

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Working Paper 85-121*

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Abstract

Combining the contributions of Cox, Ingersoll and Ross (1979, 1981) in stochastic duration of bonds and in equilibrium pricing of futures contracts, this paper develops stochastic duration as a dynamic risk measure of financial futures. The stochastic duration in this paper takes the general form and thus it is applicable for not only futures but also underlying securities. Some properties of the stochastic duration of the futures are examined and simulation results are provided for illustrative purpose.

Stochastic Duration and Dynamic Measure Of

Risk in Financial Futures

1. Introduction

Options, forward contracts, and futures contracts are the derivative assets, and they are very useful hedge instruments in portfolio management. In his recent work, Garman (1985) has applied the concept of duration to analyze the interest sensitivity of a portfolio of options that encompasses forward contracts. The purpose of this paper is to combine the contributions of Cox, Ingersoll and Ross (1979, 1981) (hereafter CIR) in the stochastic duration of bonds and in the equilibrium pricing of futures contracts to develop stochastic duration as a dynamic risk measure for financial futures and to examine some of its useful properties.

The introduction of futures contracts on several financial instruments into the exchanges in the recent years has generated a great deal of interest among financial economists as well as bond portfolio managers. Most of the studies on financial futures have focused either on the empirical investigation of hedging effectiveness of financial futures or on deriving optimal hedge-ratios in immunization strategies with financial futures using Macaulay's duration as a risk measure.[1] To the best of our knowledge, no study on financial futures to date has explicitly examined the validity of the traditional duration or proposed any alternative risk measure in the immunization strategies with financial futures.[2]

As Leibowitz (1981) has demonstrated, there are two basic kinds of yield-curve movements--parallel market shifts and yield-curve reshapings--and they lead to fundamentally different types of volatility behavior in the prices of financial futures. In particular, the prices of financial futures have been shown to be extremely sensitive to the yield-curve reshapings even when the cash security's yield remains unchanged. The risk embedded in a financial futures contract is not the same as that of a cash security. Therefore, determining a proper risk measure in financial futures is of significant importance if we attempt to devise effective hedging strategies with financial futures in the management of bond portfolios.

The traditional measures of duration, developed by Macaulay (1938) and Hicks (1939), have been used as measures of basis risk of bonds and as means to devise immunization strategies for bond portfolio management.[3] The concept of the traditional measures of duration has also been extended to assess the risk of other financial assets such as common stocks and financial futures.[4] However, as Cooper (1977), CIR (1979) and Ingersoll, Skelton and Weil (1978) have pointed out, the traditional duration is a valid risk measure only for parallel shifts in the entire yield-curve (i.e., preserving yield-curve shapings). Therefore, applying the traditional measures of duration to financial futures for immunization strategies might lead to improper results.[5]

It seems difficult to interpret the duration of futures contracts in a conventional way since they do not require initial investment. Thus, a "quasi-duration" is developed in this paper to measure the riskiness of financial futures contracts.[6] It is our hope that this paper will increase the understanding of the risk measure of financial futures and thus provide some insights for a better use of financial futures in devising immunization strategies in the management of bond portfolio. Section 2 reviews the literature on duration of bonds. Section 3 develops the stochastic quasi-duration of financial futures and its useful properties, and shows some simulation results. Section 4 contains a brief summary.

2. Brief Review of Duration of Bonds

Duration of a bond, originally developed by Macaulay and Hicks, is defined as follows:

$$D = \Sigma\ tC(t)P(t)/\Sigma\ C(t)P(t) \tag{1}$$

where C(t) is the stream of cash flows (coupons and principal repayment) and P(t) is the present value of $1.00 to be received at time t. Duration in (1) can also be expressed in the form of an elasticity:

$$-D = [(dB/B)/(dy/y)]/y = [dB/B]\cdot[1/dy] \tag{2}$$

where $B = \Sigma\ C(t)e^{-yt}$ and y is the continuously compounded yield-to-maturity on the bond.

CIR (1979) has demonstrated that measuring the risk of a bond by the elasticity given in (2), which is common in the bond market, is faulty since the result in (2) cannot, in general, be used to make cross-sectional comparisons of the riskiness of bonds (p. 52). In addition, Ingersoll, Skelton and Weil (1978) has proved that the duration in (1) can be a valid risk measure only when the entire yield curve is described by proportional shape-preservation under interest rate changes (see also CIR (1979) and Bierwag, Kaufman and Toevs (1983)). Thus it would be misleading if we apply the concept of the traditional duration directly to the financial futures contract since, as Leibowitz (1981) has shown, the futures price is more sensitive to yield curve reshapings than to parallel shifts.

As an alternative to the traditional duration, CIR (1979) has proposed stochastic duration as a dynamic measure of risk of bonds. This concept of duration allows the yield curve changes in shape as well as location. To derive the stochastic duration, CIR assumes that the instantaneous compounding risk-free interest rate, r, follows the first-order auto-regressive process as

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}\,dz \tag{3}$$

where μ is steady-state mean, κ is the parameter for the speed of adjustment toward μ, σ is the standard deviation and dz is a basic Wiener process.

Based upon a general process for interest rate in (3), CIR derives the stochastic duration as a proxy for basis risk of coupon bonds with units of time as follows:

$$\begin{aligned} D_B \equiv G^{-1}[-B_r/B] &= G^{-1}[-\Sigma\, C(t)P_r(t)/\Sigma\, C(t)P(t)] \qquad (4) \\ &= G^{-1}[\Sigma\, C(t)P(t)G(t)/\Sigma\, C(t)P(t)] \end{aligned}$$

where $P(\tau)$ = the price of a unit discount bond with time to maturity τ

$= A(\tau)\exp[-rG(\tau)]$

$$A(\tau) = \left\{ \frac{2\gamma \exp[(\gamma + \kappa + \lambda)\tau/2]}{(\gamma + \kappa + \lambda)[\exp(\gamma\tau) - 1] + 2\gamma} \right\}^{2\kappa\mu/\sigma^2}$$

$$G(\tau) = 2/[\kappa + \lambda + \gamma \operatorname{Coth}(\gamma\tau/2)]$$

$$\gamma = [(\kappa + \lambda)^2 + 2\sigma^2]^{1/2}$$

λ = the market risk parameter

$$G^{-1}(x) = \frac{2}{\gamma} \operatorname{Coth}^{-1}\left[\frac{2}{\gamma x} - \frac{\kappa + \lambda}{\gamma} \right]$$

CIR (1979) has compared the traditional duration in (1) with the stochastic duration in (4), and concluded that the traditional duration is not realistic.

3. Stochastic Duration of Financial Futures[7]

Taking into account the marking-to-market effect in futures contracts, CIR (1981) has derived the equilibrium pricing formula for the futures contract on a unit discount bond in a continuous time framework.

Let $F(t,\Delta)$ be the futures price as of time t for a contract with the maturity date s on a discount bond paying one dollar at time T ($t < s < T$), and let $\Delta = T - s$ and $\tau = T - t$ to be consistent with the notation in section 2. Then the equilibrium price of this futures contract is as follows:

$$F(t,\Delta) = A(\Delta)\left[\frac{\eta(s-t)}{G(\Delta) + \eta(s-t)}\right]^{2\kappa\mu/\sigma^2} \cdot \exp\left[-r \cdot \frac{\eta(s-t)G(\Delta)e^{-(\kappa+\lambda)(s-t)}}{G(\Delta) + \eta(s-t)}\right] \quad (5)$$

where $\eta(s-t) = \dfrac{2(\kappa + \lambda)}{\sigma^2(1 - e^{-(\kappa+\lambda)(s-t)})}$,

It is noteworthy to distinguish between the future price, $F(t,\Delta)$, and the market value of a futures contract. Because of the marking-to-market requirement, the futures contract is rewritten at the end of each day at the new futures price so that the market value of a futures contract is zero. Define the instantaneous changes in the futures prices as:

$$dF = (F\beta - \delta)dt + Fhdz \tag{6}$$

where β is the expected instantaneous percentage change in F, h is the instantaneous standard deviation of the percentage change in F, and δ is the payout received. CIR (1981) and others have shown that in a continuous-time the futures price must satisfy the following relationships:

$$\frac{1}{2} F_{rr} + (\beta - \lambda)F_r + F_t = 0 \tag{7a}$$

$$h = \sigma\sqrt{r}\, F_r/F \tag{7b}$$

where the subscripts on F denote partial derivatives. All other notations are as defined before. From equations (6) and (7b), one can see that the change in the futures price of a discount bond attributable to an unexpected shift in the spot interest rate is proportional to F_r/F and therefore, is an appropriate measure for the basis risk of a futures contract. Since the market value of a futures contract is zero, the term "stochastic quasi-duration" is used to distinguish from the conventional duration. It should be noted that the concept of the quasi-duration is similar to the one of the conventionally defined duration except that the denominator is replaced by the futures price.

From (4) and (5), the stochastic quasi-duration of the futures contract on a discount bond (D_F) can be derived as[8]

$$D_F \equiv G^{-1}[-F_r/F]$$

$$= G^{-1}\left[\frac{\eta(s-t)\cdot G(\Delta)e^{-(\kappa+\lambda)(s-t)}}{G(\Delta) + \eta(s-t)}\right] \tag{8}$$

Some of the properties of the stochastic quasi-duration of the futures contract on a discount bond can be stated as follows:

Lemma 1: The stochastic quasi-duration, D_F, as defined in (8) has the following properties:

(i) $\lim_{t \to s} D_F = T - s = \Delta$

(ii) $\frac{\partial D_F}{\partial (s-t)} < 0$

(iii) $\frac{\partial D_F}{\partial \Delta} > 0.$

Proof: (i) When $s \to t$, the futures price of a discount bond approximates the price of the discount bond. That is, when $s \to t$, $F \to B$. As a result, $D_F \to T - s = \Delta$.

(ii) Since $\mathrm{Coth}^{-1}(y)$ is meaningful only if $|y| > 1$, $G^{-1}(x)$ is meaningful only if $|\frac{2}{\gamma x} - \frac{\kappa+\lambda}{\gamma}| > 1$. For convenience, define $x = -F_r/F$. It can be shown that

$$\frac{\partial G^{-1}(x)}{\partial x} = \left(\frac{2}{\gamma x}\right)^2 \cdot \left[\left(\frac{2}{\gamma x} - \frac{\kappa+\lambda}{\gamma}\right)^2 - 1\right]^{-1} > 0.$$

$$\eta'(S-t) = -\frac{(\kappa+\lambda)e^{-(\kappa+\lambda)(s-t)}}{1 - e^{-(\kappa+\lambda)(s-t)}} \cdot \eta(s-t)$$

Then we can prove that

$$\frac{\partial x}{\partial (s-t)} = \frac{\partial(-F_r/F)}{\partial (s-t)}$$

$$= -\frac{(\kappa+\lambda)\cdot\eta(s-t)\cdot G^2(\Delta)\cdot e^{-(\kappa+\lambda)(s-t)}}{[G(\Delta)+\eta(s-t)]^2\cdot(1-e^{-(\kappa+\lambda)(s-t)})}$$

$$-\frac{(\kappa+\lambda)\cdot G(\Delta)\cdot\eta^2(s-t)\cdot e^{-(\kappa+\lambda)(s-t)}}{[G(\Delta)+\eta(s-t)]^2}$$

$$< 0, \text{ if } (\kappa+\lambda) > 0$$

Here we recognize the fact that if $(\kappa+\lambda) > 0$, then $\eta(s-t) > 0$ and $G(\Delta) > 0$. From the chain rule, we can obtain

$$\frac{\partial D_F}{\partial(s-t)} = \frac{\partial G^{-1}(-F_r/F)}{\partial(-F_r/F)}\cdot\frac{\partial(-F_r/F)}{\partial(s-t)} < 0.$$

(iii) Since

$$G'(\tau) = \frac{\gamma^2\cdot\mathrm{Sech}^2(\gamma\tau/2)}{[\kappa+\lambda+\gamma\coth(\gamma\tau/2)]^2} > 0,$$

it can be proved that

$$\frac{\partial(-F_r/F)}{\partial\Delta} = \frac{G'(\Delta)\eta^2(s-t)e^{-(\kappa+\lambda)(s-t)}}{[G(\Delta)+\eta(s-t)]^2} > 0.$$

As a result,

$$\frac{\partial D_F}{\partial\Delta} = \frac{\partial G^{-1}(-F_r/F)}{\partial(-F_r/F)}\cdot\frac{\partial(-F_r/F)}{\partial\Delta} > 0.$$ Q.E.D.

The property (i) in lemma 1 shows that D_B is a special case of D_F; (ii) indicates that the longer the time to expiration of a futures contract the less the stochastic quasi-duration, ceteris paribus; (iii) demonstrates that the longer the time to maturity of the underlying discount bond, the greater the stochastic quasi-duration and the risk.

We can also develop the pricing formula for a futures contract on a coupon bond since a coupon bond can be regarded as a portfolio of discount bonds. Consider a coupon bond which pays n constant coupons (C) with the equal time interval (δ) for the period $\Delta = T - s$ and principal of one dollar at time T, i.e., $\Delta/\delta = n$. This coupon bond can be thought of as a portfolio of n discount bonds ($i = 1,2,\ldots,n$). Let $F(t,i\delta)$ be the futures price on the ith discount bond. The futures price on the coupon bond as of time t ($f(t)$) can be written as

$$f(t) = C \sum_{i=1}^{n} F(t,i\delta) + F(t,n\delta) \tag{9}$$

$$= C \sum_{i=1}^{n} A(i\delta)\left[\frac{\eta(s-t)}{G(i\delta) + \eta(s-t)}\right]^{2\kappa\mu/\sigma^2} \cdot$$

$$\exp\left[-r\left\{\frac{\eta(s-t)G(i\delta)e^{-(\kappa+\lambda)(s-t)}}{G(i\delta) + \eta(s-t)}\right\}\right] + F(t,n\delta)$$

Following the same procedure, the stochastic quasi-duration of the futures contract on a coupon bond can be written as

$$D_f \equiv G^{-1}[-f_r/f]$$

where

$$-f_r = C\Sigma F(t,i\delta)\left\{\frac{\eta(s-t)G(i\delta)e^{-(\kappa+\lambda)(s-t)}}{G(i\delta) + \eta(s-t)}\right\} + F(t,n\delta)\left\{\frac{\eta(s-t)G(n\delta)e^{-(\kappa+\lambda)(s-t)}}{G(n\delta) + \eta(s-t)}\right\} \tag{10}$$

$$f = C\Sigma F(t,i\delta) + F(t,n\delta)$$

Equation (10) is the general form of stochastic duration for financial securities. For instance, if C is zero for discount bonds, then (10) reduces to (8) and we have $D_f = D_F$. In addition, when t is equal to s, the futures contract on a coupon bond becomes a cash coupon bond and thus (10) reduces to (4). Some useful properties of the stochastic quasi-duration of the futures contract on a coupon bond can be stated as follows:

Lemma 2: The stochastic quasi-duration of the futures contract on a coupon bond, D_f, as defined in (10) has the following properties:

(i) $\lim_{c \to 0} D_f = D_F$

(ii) $\lim_{t \to s} D_f = D_B$

(iii) $\frac{\partial D_f}{\partial (s-t)} < 0$

(iv) $\frac{\partial D_f}{\partial C} < 0$

(v) $\frac{\partial D_f}{\partial n} \gtrless 0.$

Proof: (i) This result is obvious, since as $C \to 0$, $f \to F$.

(ii) Since as $t \to s$, $f \to B$. As a result, as $t \to s$, $D_f \to D_B$.

(iii) This can be directly derived from the result of (ii) in lemma 1.

(iv)

$$\frac{\partial D_f}{\partial C} = \frac{1}{f^2} \cdot F(t,n\delta) \cdot \eta(s-t) \cdot e^{-(\kappa+\lambda)(s-t)} \cdot$$

$$\Sigma F(t,i\delta)\left[\frac{G(i\delta)}{G(i\delta) + \eta(s-t)} - \frac{G(n\delta)}{G(n\delta) + \eta(s-t)}\right]$$

Since $G'(\tau) > 0$, $G(i\delta) > 0$, $\eta(s-t) > 0$, and $n \geq i$, the quantity in the bracket is negative. As a result, $\frac{\partial D_f}{\partial C} < 0$.

(v) If we can show that $\frac{\partial D_B}{\partial n} \gtrless 0$, it is easily proved that $\frac{\partial D_f}{\partial n} \gtrless 0$. Assuming that the coupon payments are continuous, D_B can be rewritten as

$$D_B \equiv G^{-1}(Z)$$

where

$$Z = \frac{C\int_0^n A(t)\exp[-rG(t)]G(t)dt + A(n)\exp[-rG(n)]G(n)}{C\int_0^n A(t)\exp[-rG(t]dt + A(n)\exp[-rG(n)]}$$

$$= \frac{H}{B}$$

$$\frac{\partial Z}{\partial n} = \frac{1}{B^2}\{G'(n)A(n)\exp[-rG(n)]B$$

$$+ CA(n)\exp[-rG(n)][BG(n)-H]$$

$$- rG'(n)A(n)\exp[-rG(n)](BG(n) - H)\}$$

$$+ A'(n)\exp[-rG(n)][BG(n)-H] \qquad (11)$$

Since $A'(n) < 0$,[9] $G'(n) > 0$, and $[BG(n) - H] \geq 0$, the first two terms in the right hand side are positive, while the last two terms are negative. As a result, the sign of $\partial Z/\partial n$ can be negative, positive, or zero. Notice that $\frac{\partial D_B}{\partial Z} = \frac{\partial G^{-1}(Z)}{\partial Z} > 0$. The sign of $\frac{\partial D_B}{\partial n} = \frac{\partial D_B}{\partial Z} \cdot \frac{\partial Z}{\partial n}$ will depend on the sign of $\partial Z/\partial n$. The proof of $\partial D_f/\partial n \gtrless 0$ can be directly obtained from the result of $\partial D_B/\partial n$. Q.E.D.

If $C = 0$, then $(BG(n) - H) = 0$, and $\frac{\partial Z}{\partial n} > 0$. Therefore, $\frac{\partial D_B}{\partial n} > 0$, if $C = 0$. However, if $C \neq 0$, the stochastic duration of a cash coupon bond and the stochastic quasi-duration of a futures contract on the coupon bond need not be an increasing function of maturity n. The result in the former case has been pointed out by CIR (1979) without proof. The maximum for the quasi-duration is at the point n with $\partial Z/\partial n = 0$. Two corollaries to lemma 2 have been derived in Appendix.

Since the properties of the stochastic duration of a futures contract appear to be complicated, simulation analysis is employed to demonstrate that their practical application is not as restrictive as it looks once the parameters of the interest rate process in (3) are estimated. For illustration, we have simulated the stochastic durations of financial futures contracts using the parameter values in (3) estimated by CIR (1979). Using a time series of the weekly auction rates on 91-day Treasury bills for 1967-1976, CIR has estimated κ = .692, μ = 5.623%, and σ^2 = .00608.

Table 1 presents the simulation results on stochastic durations of futures contracts on discount bonds and coupon bonds with varying coupon rates and time periods. We have assumed $\mu = r$ and λ (liquidity premium) = 0 to see only the effects of uncertainty. We have also used the reversion parameter, κ = .692, in order to highlight the effect of interest rate process with drift affecting the shape as well as the location of the yield curve, as opposed to the random walk with zero drift affecting the location only.[10]

Insert Table 1 about here

Table 1 demonstrates that the stochastic quasi-duration of futures contracts on bonds decreases as coupon rate increases, which is consistent with the duration of cash bonds. It also shows that as s-t becomes longer for the given period of Δ, the stochastic duration becomes smaller, which is consistent with lemma 1 and 2. This result is also intuitively plausible, since the futures contract as of time t with the maturity date s on a bond maturing at time T can be viewed conceptually as a portfolio going long in the bond with the maturity date T and at the same time going short in the bond maturing at time s.[11] In addition, the results in Table 1 are consistent with the notion of CIR (1979) and the results of lemma 2 that the stochastic duration of a cash coupon bond and the quasi-duration of a coupon bond futures need not be an increasing function of maturity.

Insert Table 2 about here

However, Table 1 is not directly comparable to CIR (1979) because of the different underlying securities. Table 2 presents an indirect comparison between the stochastic duration of cash bonds reported in CIR (1979) and the stochastic quasi-duration of futures contracts on the same bonds when the time period until the maturity of the futures contracts is extremely short. As expected, under this circumstance, they are quite similar.

It is important to note that the stochastic quasi-duration of financial futures developed in this paper is sensitive to the reversion parameter. Table 3 demonstrates the sensitivity of the stochastic duration to the reversion parameter κ. This clearly indicates that the effectiveness of the stochastic duration for practical applications critically depends on correct estimates of parameters in the interest rate process specified in (3).

Insert Table 3 about here

Once the aforementioned stochastic durations for cash bonds and the futures on the bonds are estimated, they can be utilized to calculate the hedge ratios in the immunization strategies with financial futures.

Since the stochastic quasi-duration for financial futures developed in this paper allows for parallel shifts as well as reshapings in the yield-curve, it must be a better risk measure and will provide a more effective means in immunization strategies for bond portfolio management.

3. Conclusion

The concept of duration has been commonly used as a measure of basis risk of bonds. However, the usefulness of the traditional duration and its extentions is restrictive because they are valid only for parallel market shifts in the entire yield curve. Since the prices of financial futures contracts are very sensitive to yield-curve reshapings, the traditional duration does not provide enough usefulness in immunization strategies with financial futures. We have developed stochastic quasi-duration of a financial futures contract as a proxy for its dynamic measure of risk, based on a more plausible interest rate process allowing changes in shape as well as location of the yield curve suggested by CIR (1979). The simulation results confirm the validity of the aforementioned stochastic duration as a risk measure for financial futures.

As Bierwag, Kaufman and Toevs (1983) have pointed out that using futures contracts to change the duration of a portfolio is of great usefulness in high interest rate environments. An important and interesting area for further research would be how to combine stochastic durations of both cash portfolio and futures contracts to derive an optimal immunization strategies for financial institutions with various investment planning horizons. In addition, the empirical test of the properties of the stochastic quasi-duration of a futures contract deserves a further research.

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Footnotes

[1]See Bacon and Williams (1976), Chance (1982, 1983), Ederington (1979), Hill and Schneeweis (1980), and Kolb and Chiang (1981, 1982).

[2]In a recent paper, Bierwag, Kaufman and Toevs (1983) pointed out the importance of using futures contracts to change the duration of a portfolio. However, they have not addressed the issue of what constitutes a proper measure of duration for a futures contract.

[3]Under assumptions of some generally known shifts of the yield curve, several measures of duration have been developed recently. In the most recent paper, Gultekin and Rogalski (1984) have compared the usefulness of alternative duration specifications. However, their non-Macauley duration measures are not measured in units of time.

[4]See Boquist, Racette and Schlarbaum (1975), Bierwag (1977), Bierwag and Kaufman (1979), Chance (1982, 1983), Khang (1979), Kolb and Chiang (1981, 1983), and Williams and Pfeiger (1982).

[5]See Kolb and Chiang (1982) and Chance (1982) for an application of the concept of Macaulay's duration to futures contracts.

[6]The "performance bond" in futures trading is different from the margin in stock or bond trading, and the opportunity cost of futures trading is zero if Treasury bills are posted as margin. However, this convention does not change the fact that a futures contract is a capital asset, which has equilibrium value and price, and the fact that any gains and losses in futures trading are subject to income taxes. Furthermore, the purpose of using financial futures in bond portfolio management is precisely because the futures can be used to change the duration of the hedged portfolio of bonds.

With bonds or bills, the price term in the definition of duration is the dollars an investor must pay to acquire the bond or the bill. With futures, one essentially pays <u>no</u> dollars to acquire the price fluctuations inherent in owning a futures contract. However, one should not argue that the duration of a futures contract is infinite (zero into anything is infinity). One should recognize that the futures has a price quote which implies a delivery price, and thus define the duration of a futures contract as its expected price change (for a given yield change) relative to its delivery price. CIR (1981) and Ingersoll (1982) also point out that although not the price of an asset, a futures price satisfies the same equilibrium condition as asset prices. A futures contract can be interpreted as a portfolio yielding positive and negative cash flows (see Little (1984)): "A long position implies an outflow at the delivery date and subsequent inflows from the delivered instrument" (pp. 285). Also, a futures contract can be regarded as a "futures bond," a contract which when initiated today guarantees the prevailing futures price at a specified later point in time (see Ball and Torous (1984)). In any case, the duration of a futures contract can be defined as the duration of an asset.

[7]All arguments about futures contracts (including derivation of stochastic duration) have been done also for forward contracts. The results on forward contracts are not reported here but will be available upon request.

[8]Note that the duration of a futures contract on the discount bond is not equivalent to the duration of the discount bond itself which is equal to the maturity. Also, the correctness of (8) can be easily checked by deriving the duration of cash discount bond with the maturity, T-s

$$D_s \equiv G^{-1} \frac{C(T)P(T-s)G(T-s)}{C(T)P(T-s)}$$

$$= G^{-1}[G(T-s)]$$

$$= T-s$$

[9]$$A'(n) = \left(\frac{2\kappa\mu}{\sigma^2}\right) \left\{\frac{2\gamma\exp[(\gamma+\kappa+\lambda)\tau/2}{(\gamma+\kappa+\lambda)[\exp(\gamma\tau)-1]+2\gamma}\right\}^{\left(\frac{2\kappa\mu}{\sigma^2} - 1\right)}$$

$$\cdot \frac{\gamma(\gamma+\kappa+\lambda)\exp[(\gamma+\kappa+\lambda)\tau/2][\exp(\gamma\tau)-1](\kappa+\lambda-\gamma)}{\{(\gamma+\kappa+\lambda)[\exp(\gamma\tau)-1]+2\gamma\}^2}$$

$$< 0,$$

since all terms are positive except $(\kappa+\lambda-\gamma)$ which is negative.

[10]See CIR (1979) for the effect of κ.

[11]See Little (1984) for the interpretation of futures contracts in much the same way.

Appendix

Corollaries to Lemma 2

Corollary 2.1 (See Hopewell and Kaufman (1973) for a similar argument in the traditional measure of duration.)

(i) The necessary (not sufficient) condition for the traditional duration of a coupon bond to be an increasing function of maturity is that the coupon rate is greater than or equal to the interest rate. That is $C \geq r$.

(ii) The maximum traditional duration of a coupon bond is at the point where the maturity is equal to its duration plus $1/(r-c)$.

(iii) The necessary (not sufficient) condition for the stochastic duration of a coupon bond to be an increasing function of maturity is

$$[C-rG'(n)]A(n) - A'(n) \geq 0.$$

Proof: (i) From the definition of the traditional duration, it is obvious that $A(n) = 1$, $A'(n) = 0$, $G(n) = n$, and $G'(r) = 1$ in equation (11). Substituting these values into equation (11) yields

$$\frac{\partial z}{\partial n} = B\exp(-rn) + (C-r)(nB-H)\exp(-rn). \tag{A1}$$

The first term on the right hand side of equation (A1) is positive, and $(nB-H) > 0$. Consequently, if $(C-r) \geq 0$, we have $\partial z/\partial n > 0$, and $\partial D_B/\partial n > 0$.

(ii) To maximize the duration, set $\partial z/\partial n$ in equation (A1) to zero. Then we can obtain

$$n = \frac{1}{r-C} + \frac{H}{B},$$

where H/B is the duration.

(iii) This result is obvious from equation (11). Q.E.D.

The necessary condition for the stochastic duration to be an increasing function of maturity is complicated.

However, if $A'(n)$ approaches zero, then $C > rG'(n)$ would be the condition. Additionally, if $G'(n) \to 1$, $C > r$ would be the necessary condition which is the same for the traditional duration.

<u>Corollary 2.2</u>. If the maturity approaches infinity, then both the traditional and the stochastic durations of a coupon bond would be approaching a constant. Namely,

$$\lim_{n\to\infty} D_B = k.$$

Proof: To prove that $\lim_{n\to\infty} D_B = k$ is equivalent to prove that $\lim_{n\to\infty} \partial z/\partial n = 0$ in equation (11) or (A1).

(i) The traditional duration case:

From equation (A1), it is easy to prove that

$$\lim_{n\to\infty} \frac{\partial z}{\partial n} = \lim_{n\to\infty} B \cdot \exp(-rn) + (C-r)(nB-H)\exp(-rn)$$

$$= 0.$$

(ii) The stochastic duration case:

From equation (11)

$$\lim_{n \to \infty} G'(n) = 0,$$

$$\lim_{n \to \infty} A(n) = 0,$$

$$\lim_{n \to \infty} A'(n) = 0.$$

Thus, we can obtain that $\lim_{n \to \infty} \partial z/\partial n = 0$ in equation (11). Q.E.D.

Table 1

Stochastic Duration of Futures Contracts on Discount Bonds and Coupon Bonds*

s-t (Year)	T-s=Δ (Year)	Coupon Rates 0%	4%	6%	8%
.25	.25	.2072	.2072	.2072	.2072
.25	.50	.4079	.4058	.4047	.4037
.25	.75	.6013	.5947	.5916	.5886
.25	1.00	.7866	.7733	.7671	.7611
.25	1.50	1.1304	1.0966	1.0815	1.0673
.25	1.75	1.2876	1.2404	1.2196	1.2004
.25	2.00	1.4345	1.3719	1.3449	1.3204
.25	5.00	2.4144	2.1488	2.0613	1.9916
.25	10.00	2.6186	2.2591	2.1733	2.1128
.25	15.00	2.6252	2.2469	2.1793	2.1354
.25	20.00	2.6254	2.2345	2.1819	2.1499
.50	.25	.1722	.1722	.1722	.1722
.50	.50	.3348	.3331	.3323	.3315
.50	.75	.4872	.4821	.4797	.4773
.50	1.00	.6289	.6189	.6142	.6097
.50	1.25	.7599	.7435	.7360	.7288
.50	1.50	.8799	.8559	.8451	.8350
.50	1.75	.9890	.9566	.9422	.9289
.50	2.00	1.0875	1.0459	1.0278	1.0112
.50	5.00	1.6541	1.5167	1.4688	1.4296
.50	10.00	1.7512	1.5752	1.5299	1.4971
.50	15.00	1.7542	1.5689	1.5331	1.5094
.50	20.00	1.7543	1.5623	1.5345	1.5173
.75	.25	.1434	.1434	.1434	.1434
.75	.50	.2761	.2747	.2740	.2734
.75	.75	.3978	.3938	.3918	.3900
.75	1.00	.5086	.5008	.4972	.4937
.75	1.25	.6088	.5963	.5906	.5852
.75	1.50	.6986	.6808	.6728	.6652
.75	1.75	.7786	.7550	.7445	.7348
.75	2.00	.8495	.8197	.8067	.7947
.75	5.00	1.2285	1.1412	1.1100	1.0843
.75	10.00	1.2884	1.1787	1.1497	1.1285
.75	15.00	1.2902	1.1747	1.1518	1.1365
.75	20.00	1.2903	1.1705	1.1527	1.1416
1.00	.25	.1197	.1197	.1197	.1197
1.00	.50	.2285	.2274	.2268	.2263
1.00	.75	.3267	.3235	.3219	.3205
1.00	1.00	.4147	.4086	.4057	.4029
1.00	1.25	.4930	.4833	.4789	.4747
1.00	1.50	.5622	.5486	.5424	.5366
1.00	1.75	.6229	.6051	.5971	.5897

Table 1 (cont.)

s-t (Year)	T-s=Δ (Year)	Coupon Rates 0%	4%	6%	8%
1.00	2.00	.6760	.6538	.6440	.6350
1.00	5.00	.9484	.8875	.8654	.8471
1.00	10.00	.9895	.9138	.8934	.8785
1.00	15.00	.9907	.9110	.8949	.8841
1.00	20.00	.9908	.9080	.8955	.8877
1.25	.25	.1000	.1000	.1000	.1000
1.25	.50	.1897	.1887	.1883	.1879
1.25	.75	.2695	.2669	.2657	.2645
1.25	1.00	.3402	.3353	.3330	.3308
1.25	1.25	.4024	.3948	.3912	.3879
1.25	1.50	.4567	.4461	.4412	.4367
1.25	1.75	.5040	.4901	.4839	.4782
1.25	2.00	.5449	.5278	.5203	.5133
1.25	5.00	.7490	.7042	.6879	.6743
1.25	10.00	.7790	.7236	.7086	.6976
1.25	15.00	.7799	.7216	.7097	.7017
1.25	20.00	.7799	.7194	.7102	.7044

*The value of parameters used in this table are $r = \mu = 5.623\%$, $\sigma^2 = .00608$ and $\kappa = .692$.

Table 2

Stochastic Duration of Futures Contracts on Coupon Bonds When s-t is Short Relative to Δ*

s-t (Year)	T-s=Δ (Year)	Coupon Rates 4% Futures	CIR	6% Futures	CIR	8% Futures	CIR
.01	5	3.67	3.81	3.41	3.52	3.23	3.34
.01	10	4.10	4.29	3.79	3.93	3.60	3.73
.01	15	4.05	4.24	3.81	3.95	3.66	3.81
.01	20	4.00	4.18	3.82	3.96	3.71	3.86

*Assumed that $\mu = r = 5.623\%$, $\sigma^2 = .00608$ and $\kappa = 0.692$. The column, CIR presents the stochastic duration of cash coupon bonds with time to maturity, Δ, which was calculated by CIR (1979).

Table 3

Stochastic Duration of Futures Contracts on Discount Bonds for Different Values of κ

s-t (Year)	T-s=Δ (Year)	κ = .001	κ = .100	κ = .692
.25	.25	.2499	.2437	.2072
.25	.50	.4997	.4872	.4079
.25	1.00	.9990	.9733	.7866
.25	4.00	3.9867	3.8640	2.2281
.25	20.00	19.5546	17.3795	2.6254
.50	.25	.2498	.2376	.1722
.50	.50	.4994	.4747	.3348
.50	1.00	.9980	.9473	.6289
.50	4.00	3.9734	3.7341	1.5590
.50	20.00	19.1352	15.5427	1.7543
.75	.25	.2497	.2316	.1434
.75	.50	.4991	.4625	.2761
.75	1.00	.9970	.9221	.5086
.75	4.00	3.9602	3.6099	1.1684
.75	20.00	18.7390	14.1363	1.2903
1.00	.25	.2496	.2258	.1197
1.00	.50	.4987	.4507	.2285
1.00	1.00	.9960	.8977	.4147
1.00	4.00	3.9471	3.4911	.9066
1.00	20.00	18.3639	13.0022	.9908
1.25	.25	.2495	.2201	.1000
1.25	.50	.4984	.4391	.1897
1.25	1.00	.9950	.8739	.3402
1.25	4.00	3.9341	3.3772	.7184
1.25	20.00	18.0078	12.0560	.7799

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