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Bicriterion Decision Making Under Uncertainty: An Interactive Approach

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ABSTRACT

Uncertainty presents unique difficulties in optimization problems. Decision Makers (DMs) are faced with risky situations requiring analysis of multiple outcomes in each solution. Very few direct choice (interactive) methods are capable of addressing problems with probabilistic outcomes. We present a general algorithm which will allow for uncertainty. The method is appropriate for use in a multiple criteria framework with a discrete number of possible outcomes, but is explored and developed in the context of a bicriterion problem using a two stage mathematical programming model.

1. Introduction

Determination of an optimal solution to a problem, or selection of a preferred alternative, is a product of model development, model (parameter) measurement, and model optimization. Alteration of any of the three tasks can lead to a change in the resulting decision. If a problem has multiple criteria and the alternatives have uncertain outcomes, the difficulties in model development, measurement and optimization are compounded. Interactive methods may not be proper under uncertainty because the concept of an efficient frontier is lost when referring to expected utility.

Many papers exist on development of utility models under risk and multiple criteria. Techniques for utility construction appear in Keeney and Raiffa [1976] and approaches to ease the task of measuring utility functions are discussed in [Klein, et al 1982] and reviewed in [Farquhar 1984]. These utility methods appear as the only major methodology under the case of uncertainty in multicriteria optimization [Zionts 1979 and Hwang and Masud 1979]. The lack of an interactive procedure for the case of uncertainty in multicriteria models excludes the advantages inherent in progressive preference articulation methods. The ease of use, speed and solution confidence [Wallenius 1975 and Klein, Moskowitz and Ravindran 1985] would be a welcome asset to situations where uncertainty is involved. In order to capitalize on the benefits of an interactive approach, many researchers have developed effective methodologies to locate efficient solutions under certainty. Among the better known methods for nonlinear problems are those of Zionts and Wallenius [1976 and 1983], Geoffrion, Dyer and Feinberg [1979], and Sadagopan and Ravindran [1982].

This paper develops an interactive method for handling a bicriteria problem under uncertainty with two uncertain outcomes. The method combines the two previous approaches by Geoffrion, Dyer, and Feinberg (GDF) [1976] and the

Paired Comparison Method (PCM) of Sadagopan and Ravindran [1982]. A general problem framework is described, followed by an algorithm and example.

2. Problem Specification

Consider the mathematical program with two criteria and two uncertain right hand side (RHS) vectors in the constraint set. Denote the problem as TSEU:

$$
\begin{aligned}
\text{MAX} \quad & f_{11}\,(X,Z) \\
& f_{12}\,(Y,Z) \\
& f_{21}\,(X,Z) \\
& f_{22}\,(Y,Z) \\
\text{S.T.} \quad & g_1\,(X,Z) < b_1\,(1) \\
& \vdots \\
& g_m\,(X,Z) < b_m(1) \\
& g_1\,(Y,Z) < b_1(2) \\
& \vdots \\
& g_m\,(Y,Z) < b_m(2)
\end{aligned}
$$

where Z represents first-stage variables that require immediate commitment, X and Y are second-stage variables that may be determined immediately prior to implementation, f(.) is a concave, differentiable objective function, and g_1, to g_m are convex, differentiable constraint functions.

As can be seen, the incorporation of uncertainty has increased the dimensionality of the problem. The PCM could not solve this problem without the necessary extensions to handle four criteria. The GDF Method could solve this problem by treating different outcomes of the same objective as separate objective functions, but the complexity of the problem has increased the dimensionality and the severity of the criteria tradeoff requests of the DM. By presenting the different outcomes as separate objectives to the DM when requesting tradeoff information, the true state of uncertainty is not properly represented. This may lead to a bias on the part of the DM regarding trade-offs across outcomes or across probabilities.

The problem is a two outcome generalization of others appearing in the literature. Let us begin with the simple two-stage model from Dantzig [1963] or Wagner [1969] of one criterion and two right-hand-side vectors of known probabilities ($b^{(1)}$ with probability p_1, and $b^{(2)}$ with probability p_2). The problem assumes that several decision variables require a commitment at the present time. These variables cannot be changed at a later date, however there also exist variables that are flexible. The flexible variables can be changed after the true right-hand-side levels are known. The variables requiring an immediate commitment are called the first-stage variables. The flexible variables are called the second-stage variables.

A formulation for the two-stage decision would be

TS: $\text{Max } p_1 f(X,Z) + p_2 f(Y,Z)$

subject to:

$$\begin{array}{ccc} g_1(X,Z) & \leq & b_1^{(1)} \\ \vdots & & \vdots \\ g_m(X,Z) & \leq & b_m^{(1)} \\ g_1(Y,Z) & \leq & b_1^{(2)} \\ \vdots & & \vdots \\ g_m(Y,Z) & \leq & b_m^{(2)} \end{array}$$

where Z is the first stage variable vector, X is the second-stage vector associated with the first RHS ($b^{(1)}$), Y is the second-stage vector associated with the second RHS ($b^{(2)}$), $b^{(i)}$ is the i^{th} RHS vector of length m, $f(\cdot)$ is a concave, differentiable objective function, and g_1 to g_m are convex, differentiable functions.

Note that the formulation will determine a complete decision vector (X,Y,Z). Initially the decision variables in the Z vector would be implemented. The second-stage variables X would be the plan for the first RHS and Y for the second RHS. Another feature is the use of expected values in the objective function. The Z vector is identical in each functional computation, but the second-stage variables differ, and each has a known probability of being implemented. Thus all the information needed to compute expected values is available. Because of the use of expected values, the incorporation of more than two discrete options is a simple extension.

The two-stage (TS) model is appropriate in the context of utility theory only if the decision maker's utility function is linear, and has known weights on each outcome. This means that no transformation of the criterion outcomes are made prior to the taking of expectations. If the utility function is known, but not of a simple form, then the model modifications would be to simply change the objective to compute utilities of the function prior to the expected value computations. That is, Max $p_1U(f(X,Z)) + p_2U(f(Y,Z))$. This process would require a tedious utility function measurement as described in [Keeney and Raiffa, 1976]. If, however, we assume no knowledge about the utility function then the two uncertain outcomes must be separated. The separation of the outcomes suggests the use of one of the interactive methods discussed earlier. The formulation for this model (TSU) would appear:

TSU: Max $f_1 = f(X,Z)$

$f_2 = f(Y,Z)$

subject to the same constraint set in TS.

As can be seen, the resulting formulation is similar to a bicriterion problem. There is only one criterion, but the level has two possible outcomes. To solve the problem for maximum expected utility is to find X, Y, and Z such that the certainty equivalent to the implied lottery is maximized. It would be possible to use the PCM for this problem by setting each outcome as a separate criterion and solving as a two-stage problem. The GDF method could also handle this problem in a similar fashion.

A moderate twist to the model in problem TSEU is when the uncertainties in a bicriterion model are in the objective function instead of the right hand side. Allow only 2 uncertain outcomes but associate them with parameters in the objective function. The formulation for the two-stage objective is:

$$
\begin{aligned}
\text{TSO:}\quad \text{Max}\ & f_{11}(X,Z) \\
& f_{12}(X,Z) \\
& f_{21}(X,Z) \\
& f_{22}(X,Z) \\
\text{subject to:}\ & \\
& g_1(X,Z) \leq b^{(1)} \\
& \quad\vdots \\
& g_m(X,Z) \leq b_m{}^{(1)}
\end{aligned}
$$

where $b^{(1)}$ are the certain RHS constants and f_{ij} is the function for the i^{th} criterion at the j^{th} parameter level (outcome). The GDF method could handle this problem in the same fashion as problem TSEU. The existing PCM could handle the problem only by utilizing expected values. It is also important to note that for this case, the decision variables are now all first-stage variables, and the true state of nature need not be determined in order to make the decisions.

In a final variation, Bard [1983] discusses a Bi-level Programming Problem that is an extension of the two-stage model allowing for the objective functions to be different functions rather than different possible occurrences of the same function. When two levels of management have conflicting objective functions the top management objective may be satisfied prior to the lower management objective.

3. An Interactive Algorithm Under Uncertainty

The purpose of this section is to develop a method that will solve the above problems without increasing the dimensionality of the vector maximization and reduce the number of criteria requiring tradeoff information. Features from both the PCM and GDF Method will be used to provide a method which presents the DM with the easiest questions possible.

The approach will utilize tradeoff information as does the GDF, to provide a direction to travel in the feasible region. The method will not require specific tradeoff values of the DM for all criteria across all outcomes, but will estimate tradeoffs using paired comparisons within each outcome. Thus, the questions asked of the DM will be structured after the soft interactions of the PCM.

3.1 Problem Structuring

Consider partitioning problem TSEU into two bicriterion mathematical programs:

BMP 1:	BMP 2:
Max $f_{11}(X,Z)$	Max $f_{12}(Y,Z)$
$f_{21}(X,Z)$	$f_{22}(Y,Z)$
subject to	subject to
$g(X,Z) < b^{(1)}$	$g(Y,Z) < b^{(2)}$

BMP 1 represents the problem to be solved when the RHS is $b^{(1)}$. BMP 2 is when the RHS is $b^{(2)}$. Each problem will have an associated payoff vector set (V_1 and V_2) such that

$$V_1 = \{V | f_{11}(X,Z),\ f_{21}(X,Z) = V \text{ for some } X,Z \text{ where } g(X,Z) < b^{(1)}\}$$

$$V_2 = \{V | f_{12}(Y,Z),\ f_{22}(Y,Z) = V \text{ for some } Y,Z \text{ where } g(Y,Z) < b^{(2)}\}.$$

BMP 1 and BMP 2 now provide an opportunity to examine each outcome independently. However, feasibility may be lost if BMP1 is solved and outcome 2 is the true state of nature. Thus a link, in the form of minimum achievement levels and duplicated constraints must be made between BMP1 (Outcome 1) and BMP2 (Outcome 2). These levels are denoted a_{11} and a_{21} for the minimum achievement levels of $f_{11}(X,Z)$ and $f_{21}(X,Z)$. They will permit the separate optimization of the outcomes while maintaining feasibility. The selection of the specific levels are discussed during the development of the procedure. The resulting problems are conditional bicriterion mathematical programs (CBMP) and would appear as:

$\text{CBMP1}(a_{12},\ a_{22})$:

$$\begin{aligned}
\text{Max } & f_{11}(X,Z) \\
& f_{21}(X,Z) \\
\text{Subject to} & \\
& g(X,Z) < b^{(1)} \\
& g(Y,Z) < b^{(2)} \\
& f_{12}(Y,Z) > a_{12} \\
& f_{22}(Y,Z) > a_{22}
\end{aligned}$$

$\text{CBMP2}(a_{11},\ a_{21})$

$$\begin{aligned}
\text{Max } & f_{12}(Y,Z) \\
& f_{22}(Y,Z) \\
\text{Subject to} & \\
& g(X,Z) < b^{(1)} \\
& g(Y,Z) < b^{(2)} \\
& f_{11}(X,Z) > a_{11} \\
& f_{21}(X,Z) > a_{21}
\end{aligned}$$

The a_{ij} are determined at each iteration of the algorithm proposed later in this section. Each of the above problems will have an associated payoff set that is a reduction of the V_1 and V_2 sets due to the extra constraints in CBMP1 and CBMP2. These reduced payoff sets will be determined by the tightness of the minimum achievement levels.

Formally,

$$\begin{aligned}
\bar{V}_1 = \bar{V}_1(a_{12},\ a_{22}) = \{V \mid\ & f_{11}(X,Z),\ f_{21}(X,Z) = V \\
& \text{for some } X,Z \text{ where } g(X,Z) < b^{(1)},\ g(Y,Z) < b^{(2)}, \\
& f_{12}(Y,Z) > a_{12}, \text{ and } f_{22}(Y,Z) > a_{22}\}
\end{aligned}$$

$\bar{V}_2 = \bar{V}_2(a_{11}, a_{12}) = \{V | f_{12}(Y,Z), f_{22}(Y,Z) = V$

for some Y,Z where $g(Y,Z) \leq b^{(2)}$

$g(X,Z) \leq b^{(1)}$, $f_{11}(X,Z) \geq a_{11}$, and $f_{21}(X,Z) \geq a_{21}\}$.

Note that $\bar{V}_1$ is contained in V and $\bar{V}_2$ is contained in V_2. Therefore, an efficient solution to CBMP1 may not be efficient to BMP1 but a feasible solution to CBMP1 will be feasible for BMP1 (figure 1). This point becomes important, because it is through the manipulation of the minimum achievement levels, and thus the manipulation of CBMP1(a_{12}, a_{22}), CBMP2(a_{11},a_{21}), $\bar{V}_1$, and $\bar{V}_2$ that the algorithm moves toward a final solution. Since in the process of maximizing expected utility the conditional bicriterion formulations will be used, we introduce the concepts of efficiency and mutual efficiency.

Definition: A solution X^o, Y^o, $Z^o \in S$ is efficient if $f_k(X,Y,Z) > f_k(X^o, Y^o, Z^o)$ for some X, Y, Z $\in$ S implies that $f_j(Z,Y,Z) < f_j(X^o, Y^o, Z^o)$ for at lease one other index j. Consider the set of efficient solutions to CBMP1 (a_{12}, a_{22}) and the set of efficient solutions to CBMP2(a_{11},a_{21}). X^*, Y^*, Z^* is mutually efficient to CBMP1 (a_{12}, a_{22}) and CBMP2 (a_{11}, a_{21}) if and only if X^*, Y^*, Z^* belongs to both sets of efficient solutions.

Mutual effeciency (ME), like efficiency, is a property to ensure that a solution under consideration is non-dominated, and thus a candidate for the most satisfactory solution. This implies three properties, the latter two of which will be subsequently proven. The first property is that, by definition of ME and efficiency, a non-dominated solution is present for each problem CBMP1 and CBMP2. This indicates that when a DM is presented a ME solution to CBMP1 and CBMP2 he is assured that no attribute improvements can be made within any outcome without detrimenting another attribute within the same outcome or at least one attribute in the remaining outcome.

A second property is that a ME solution to CBMP1 and CBMP2 is an efficient solution to the general problem TSEU. This implies that a simple method which utilizes the partitioning of TSEU into CBMP1 and CBMP2 will have an efficient point in TSEU. Thus finding a ME solution to CBMP1 and CBMP2 yields an efficient solution to TSEU.

Theorem 1: X^*, Y^*, Z^* is efficient to TSEU iff a decision variable vector X^*, Y^*, Z^* is ME to CBMP1 and CBMP2.

Requirements of Feasibility. Denote the feasible region for TSEU as S and the feasible regions for CBMP1 and CBMP2 and S_1 and S_2. Any $(X,Y,Z) \in S_1$ or S_2 is also an element of S since S1 and S2 are subsets of S. Thus any (X,Y,Z) feasible to CBMP1 and CBMP2 is feasible to TSEU. To reverse the consideration, any point $(X,Y,Z) \in S$ will yield values for $f_{11}(X,Z)$, $f_{21}(X,Z)$, $f_{12}(Y,Z)$, and $f_{22}(Y,Z)$. Let these values be a_{11}, a_{21}, a_{12}, and a_{22} respectively and (X,Y,Z) becomes $\in S_1$ and S_2 by definition.

Proof: Sufficieny. Let (X^*,Y^*,Z^*) be ME to CBMP1 and CBMP2. If (X^*,Y^*,Z^*) is not efficient to TSEU, then there exists a point $(X^\circ, Y^\circ, Z^\circ)$ such that $fij(X^\circ, Z^\circ) > fij(X^*,Z^*)$ for $a_{11}(i,j)$ and at least one of the following:

$$f_{11}(X^\circ,Z^\circ) > f_{11}(X^*,Z^*) = a_{11},$$

$$f_{21}(X^\circ,Z^\circ) > f_{21}(X^*,Z^*) = a_{21},$$

$$f_{12}(Y^\circ,Z^\circ) > f_{12}(Y^*,Z^*) = a_{12}, \text{ or}$$

$$f_{22}(Y^\circ,Z^\circ) > f_{22}(Y^*,Z^*) = a_{22}.$$

This however, means that (X^*, Y^*, Z^*) is not efficient to either CBMP1 (a_{12},a_{22}) or CBMP2 (a_{11},a_{21}). Thus ME is contradicted.

Necessity: Let (X^*,Y^*,Z^*) be efficient to TSEU. For CBMP1 to not be efficient would imply an $(X^\circ,Y^\circ,Z^\circ)$ such that $f_{11}(X^\circ,Z^\circ) > f_{11}(X^*,Z^*)$ or

$f_{21}(X^{o},Z^{o}) > f_{21}(X^{*},Z^{*})$. This contradicts efficiency of TSEU, thus (X^{*},Y^{*},Z^{*}) is efficient to CBMP1. A similar argument holds for CBMP2. Since CBMP1 a CBMP2 are both efficient at (X^{*},Y^{*},Z^{*}) and feasible, (X^{*},Y^{*},Z^{*}) is mutually efficient to CBMP1 and CBMP2.

A final property is that when the constraints on the objective functions within the conditional bicriterion mathematical programs are tight and an efficient solution to CBMP1 is found for an efficient payoff in CBMP2 is known, a ME solution is also found. This permits verification of the ME property within an algorithm.

Theorem 2. X^{*},Y^{*},Z^{*} is ME to CBMP1 and CBMP2 if (a_{12}, a_{22}) is an efficient payoff in CBMP2, X^{*},Y^{*},Z^{*} is efficient to CBMP1 (a_{12}, a_{22}), $f_{12}(Y^{*},Z^{*}) = a_{12}$, and $f_{22}(Y^{*},Z^{*}) = a_{22}$.

Proof: Since (a_{12},a_{22}) is an efficient payoff vector in CBMP2 there is no feasible Y in CBMP2 such that $\{f_{12}(Y,Z^{*}) > a_{12}$ and $f_{22}(Y,Z^{*}) > a_{22}\}$ or $\{f_{12}(Y,Z^{*}) > a_{12}$ and $f_{12}Y,Z^{*}) > a_{22}\}$. Thus by definition X^{*},Y^{*},Z^{*} is efficient to CBMP2 and by construction to CBMP1 and is therefore ME to CBMP1 and CBMP2. The argument must be repeated for the reversal of CBMP1 and CBMP2 in order to be complete.

3.2 A Stepwise Technique

Using the efficiency concepts, the development of an algorithm that maintains mutual efficiency at each iteration is desired. In order to accomplish this task a method will be developed that utilizes the above properties. The method will start by finding a mutually efficient solution using existing MCDM techniques. Once a starting point is determined, the algorithm will use tradeoffs to estimate utility functions for the decision maker. With estimates of the utility functions, the method will explore the feasible region of one outcome while relaxing the criterion levels in the second.

Theorem 1 will be implied in the exploration to retain feasibility and to insure an efficient solution is secured to TSEU when an ME solution to CBMP1 and CBMP2 is secured. The exploration is conducted to insure that the conditions implying ME in Theorem 2 are preserved. Thus, a new efficient solution to TSEU is found that is used as a starting point for the next iteration. The steps are as follows:

Step 0: Initialize the Iterations:

Solve CBMP1 (minimum achievements) using the PCM. This is denoted as outcome 1 (Q1), with solution (X,Y,Z). Set $a_{11} = f_{11}(X,Z)$ and $a_{21}(X,Z)$. Solve CBMP2 (a_{11},a_{21}) using the PCM. Denote this as outcome 2 (Q2), with new solution (X^{o},Y^{o},Z^{o}). Set $a_{12} = f_{12}(Y^{o},Z^{o})$ and $a_{22} = f_{22}(Y^{o},Z^{o})$. The starting points would be as shown in Figure 1.

Figure 1

In initializing the method, the PCM is recommended to determine starting values. This is to insure accurate estimations of the linear utility in Step 1. In addition, by starting at the preferred solution in each outcome, the method may start closer to the preferred solution across all outcomes. Mutual efficiency will exist at this point in the algorithm because no better solution exists for Q1 from a utility viewpoint, and Q2 from CBMP2(a_{11},a_{21}) is defined by the efficient vector (X^*,Y^*,Z^*).

Step 1: Generate Local Tradeoffs:

Approximate the slope for a linear utility function for each outcome. Ask the DM how much increase does he expect in criterion 2 in Q1 for a unit decrease in criterion 1. Let the response be λ_{11}. Set $\lambda_{12} = 1$. Ask the same question in Q2, let the response be λ_{21}. Set $\lambda_{22} = 1$.

The tradeoff questions are similar to those asked when measuring value functions (Kenney and Raiffa 1976). If the wish is to eliminate questions of the DM at this step, the tradeoffs may be approximated in a fashion requiring no interaction. Within each outcome solve the problem

$$
\begin{aligned}
P_\lambda:\quad & \text{MAX } f_{1j} \\
& \text{subject to:}\quad g(X,Z) \leq b^{(1)} \\
& \qquad\qquad\qquad g(Y,Z) \leq b^{(2)} \\
& \qquad\qquad\qquad f_{1j}(X,Y,Z) \geq a_{1j} \\
& \qquad\qquad\qquad f_{2j}(X,Y,Z) \geq a_{2j} - 1,
\end{aligned}
$$

where the last constraint permits a unit drop in the level of the second criterion and the solution is $(X^\lambda,Y^\lambda,Z^\lambda)$. This will provide an adjacent efficient point that can be used to determine the values by setting $\lambda_{1j} = f_{1j}(X^\lambda,Y^\lambda,Z^\lambda) - f_{1j}(X^o,Y^o,Z^o)$. Since the DM selected the solution (X^o,Y^o,Z^o) by using the PCM, the line estimated will be close to that provided by a direct tradeoff value given a consistent DM. The tradeoffs derived are in terms of the criteria, not the decision variables, as shown in Figure 2.

Figure 2

Step 2: Determine Direction to Travel:

The directional problem may be derived from CBMP2. Let the vector (a_{11},a_{21}) be the incumbent outcome Q1 generated in either Step 0 or Step 3. Let r range from 0 to 2π radians with 0 as arbitrary due north. Any change of distance = 1 in the criteria levels may thus be represented:

$$
\bar{Q}1 = \begin{matrix} a_{11} + \cos(r) \\ a_{21} + \sin(r). \end{matrix}
$$

Since we are looking for the direction of maximal expected utility increase, we are solving the general objective function: Max $p[U(\bar{Q}1) - U(Q1)] + (1-p)[(\bar{Q}2) - U(Q2)]$, where $\bar{Q}1$ is the outcome i from the direction search step of the current iteration, $U(Q2)$ is a constant, and $U(\bar{Q}1) - U(Q1)$ reduces to $U(\cos(r), \sin(r))$ due to linearity assumptions, and $U(\bar{Q}2)$ is determined by the $CBMP2(a_{11}, a_{21})$ objective function. Thus the directional problem becomes:

DIR: $\text{Max } p[\lambda_{11}\cos(r) + \lambda_{21}\sin(r)] + (1-p)\ \{\lambda_{12}f_{12}(Y,Z) + \lambda_{22}f_{22}(Y,Z)\}$

Subject to:

$$0 \leq r \leq 2\pi$$

$$g(X,Z) \leq b^{(1)}$$

$$g(Y,Z) \leq b(2)$$

$$f_{11}(X,Z) \geq a_{11} + \cos(r)$$

$$f_{21}(X,Z) \geq a_{21} + \sin(r)$$

where λ_{ij} represent linear utility weights on the criteria.

Using the λ's determined in Step 1 as linear utility weights on the criteria, solve problem DIR. Figure 3 shows the workings of problem DIR. As r varies direction with constant radius of length = 1, in Q1, the estimated linear utility function in Q2 will shift outward (or inward) to its best value. Since r is a variable, the maximum increase in expected estimated utility is found by DIR for a unit change in the criteria. The utility will increase for any positive value of the objective to problem DIR, so global optimality to problem DIR may not be necessary.

Figure 3

Step 3: Find the Distance of Travel

Solve CBMP2 $(a_{11} + \alpha(\cos(r)), a_{21} + \alpha(\sin(r)))$ at several values of α from 0 to a maximum distance. Denote the solutions as (X^d, Y^d, Z^d). The optimization could be accomplished by using the PCM for each distance on Q2 (this would be tedious) or by using the linear approximations (λ_{ij}'s) from Step 1 to solve:

$$\text{D: MAX} \quad \lambda_{12}f_{12}(X,Y,,Z) + \lambda_{22}f_{22}(X,Y,Z)$$

subject to:

$$g(\cdot) \leq b(\cdot)$$
$$f_{11}(X,Y,Z) \geq a_{11} + \alpha\cos(r)$$
$$f_{21}(X,Y,Z) \geq a_{21} + \alpha\sin(r)$$

for each α selected.

For each solution present a lottery to the DM. Each lottery would be $\langle f_{11}(X^d,Y^d,Z^d), f_{21}(X^d,Y^d,Z^d)\rangle$ as Q1 with probability p, and $\langle f^{12}(X^d,Y^d,Z^d), f_{22}(X^d,Y^d,Z^d)\rangle$ as Q2 with probability (1-p). If $\alpha = 0$ is the preferred distance then stop. Else set $a_{ij} = f_{ij}(X^d,Y^d,Z^d)$ of the preferred lottery for all i and j, and return to Step 1. The distance determination would progress as shown in Figure 4.

It may also be appropriate to verify mutual efficiency by making certain the constraints in CBMP2 that generated the chosen lottery are tight. If not, correct by adding the slack values to the minimum achievement levels associated with Q1 before presenting the outcomes to the DM. It is also possible to correct by solving CBMP2$(f_{12}(Y,Z), f_{22}(Y,Z))$ with the PCM where Y and Z are the solutions from the CBMP2 that generated the preferred lottery.

Figure 4

3.3 More Than Two Criteria, Outcomes

As the number of criteria increases, the number of conditional problems remain the same. The changes would involve adding objectives and constraints to each conditional problem. As outcomes are added, the number of conditional problems increases. The major stumbling block for larger problems would be the amount of information provided to the DM at each iteration. In addition to more than two criteria, the DM may be faced with viewing lotteries with many outcomes. Also, the simplistic direction and distance finding problems would have to be replaced by methods that handle more complexity.

4. Applications

Two Problems Illustrate the Use of the Method.

4.1 Uncertainty in the Right Hand Sides: Example

A common occurence, is the uncertainty of the RHS in a specific mathematical program. In a production mix framework, resource suppliers are not always dependable, machines break down, or cash flow may be strained. This leads to differing feasible regions and differing production plans. This corresponds to problem TSEU.

No first-stage variables appear in this example problem, not being important in the demonstration of the algorithm. Consider a raw material intensive production problem. Assume that resource requests are being filled, and that the three raw materials will have a constraint vector of (in thousands of units):

RM	1 = 30	with .5	= 25	with .5
	2 = 15	probability,	= 20	probability.
	3 = 40	or	= 45	

Only two products are made from these raw materials, desks (x_1) and chairs (x_2). Two objectives exist at the corporate level, increased market share and increased profit. The BMP1 associated with this problem is:

f_1 = profit (p) = $x_1 + 3x_2$

Max

f_2 = market share (MS) = $x_1 + x_2$

Subject to:

$x_1 < RM1$

$x_2 < RM2$

$x_1 + 2x_2 < RM3$

$x_1, x_2 > 0$ and in thousands of units.

In addition to these constraints, the company has placed a limit on the flexibility of the variables. The flexibility limits may be expressed by the addition of the following constraints using:

$|x_1 - y_1| = \beta_1$

$|x_2 - y_2| = \beta_2$

$|x_1 + 2x_2 - y_1 - 2y_2| = \beta_3$

$\beta_1 + \beta_2 + \beta_3 < 5.$

β_i represents the absolute deviation in resource usage for raw material i. Total deviations are restricted to be less than 5000 units in this example. These represent estimates of the purchasing department regarding the ability to secure differing resources in a limited time frame.

At any iteration of this algorithm, a variation on CMBP1(a_{12}, a_{22}) is used in the direction finding and distance steps. The BMP_λ using Geoffrion's method for this problem would appear:

BMP_λ Max $\lambda_1(_1 + 3x_2) + \lambda_2(x_1 + x_2)$

Subject to:

$x_1 \leq 25$

$x_2 \leq 20$

$x_1 + 2x_2 \leq 45$

$y_1 \leq 30$

$y_2 \leq 15$

$y_1 + 2y_2 \leq 40$

$|x_1 - y_1| = \beta_1$

$|x_2 - y_2| = \beta_2$

$|x_2 - 2x_2 - y_1 - 2y_2| = \beta_3$

$\beta_1 + \beta_2 + \beta_3 \leq 5$

$y_1 + 3y_2 \geq 12$

$y_1 + y_2 \geq 22$

$x_1, y_1, x_2, y_2 \geq 0$ and in thousands of units.

With the specific production problem outlined above, let us apply our algorithm. Assume the decision makers true utility is: $U = (MS/40) + (p/60) - (MS/40)^2 - (p/60)^2$. Using a quadratic mathematical program, the optimal utility occurs at MS = 35, π = 55 for v_1, and MS = 42.5, p = 47.5 for v_2. The optimal plan under the first RHS vector occurs at $x_1 = 25$, $x_2 = 7.5$. The optimal plan under the second RHS vector is $y_1 = 25$, $y_2 = 10$. The beta constraint is tight.

<u>Step 0</u>: Utilize the PCM on CBMP2(0,0). The solution out is p = 46 and MX = 34. Denote this as Q2. Set $a_{12} = 46$ and $a_{22} = 34$ and use BMP_λ to get an efficient solution to define Q1. This may occur at p = 57 and MS = 27.

<u>Step 1</u>: At this step we request local tradeoff information. Using the decision maker's true utility, we find an increase in MS of .54 will compensate for a unit decrease in for outcome 2 (Q2). The tradeoff for Q1 is 1.86.

Step 2: Construct problem DIR using the tradeoff information to approximate the utility with a linear function. (Scale λs to sum to 1.)

DIR:

$$\text{MAX} \quad .5[.65\cos(r) + .35\sin(r)] + .5[.35f_1(x) + .65f_2(x)]$$

Subject to the same constraint set as in the previous example plus

$X_1 + 3X_2 \geq 46 + \cos(r)$, and

$X_1 + X_2 \geq 34 + \sin(r)$.

The best direction is found at 4.975 radians.

Step 3: Using the direction indicated, the distance lotteries are generated for several distances. The distance problem is:

$$\text{MAX} \ .35f_1(x) + .65f_2(x)$$

subject to the same constraint set as the direction problem in Step 2, but set $r = 4.975$ and multiply the trigonometric functions by the distance. The lotteries generated are in Table 1.

TABLE 1

Distance Lotteries for the Production Example

Distance	Lottery $p \equiv .5$, $(1-p) \equiv .5$	
	$(\pi$, MS)	
1	<(33,47);	(34.5,53.5)>
2	<(32,48);	(35,55)>
3	<(31,49);	(35,55)>
5	<(30,50);	(35,55)>

The decision maker would prefer the lottery at the distance of 2. However, the slack variable for the market share criterion in $\bar{V}_1$ has become positive. Using the remedy of solving CBMP1($f_{12}(Y,Z)$, $f_{22}(Y,Z)$) by the PCM, where

Y and Z are from the solution to the BMP of the preferred lottery, optimality is achieved. Optimality would not be confirmed until Step 2 of the next iteration. In this example, where all the conditions for the algorithm are met, the exact optimal is reached. Three PCMs were performed, two tradeoff conditions were evaluated, and one series of lotteries was presented to the decision maker.

4.2 Uncertainty in the Objective Function

In order to examine the methods ability to handle a variety of problem structures, an integer acceptance sampling model for quality control was used as a test problem. Two criteria in quality control settings are Average Outgoing Quality (AOQ) and Average Lot Inspection Cost (ALIC). The constraint set is composed only of items natural to the problem: 1) sample size is between 0 and lot size, and 2) acceptance level is between 0 and sample size. These features will not change under uncertainty. The feature that may readily change is the true lot fraction defective. A different value for lot fraction defective will yield a different AOQ and ALIC. Thus when prior percent defective becomes a distribution rather than a point estimate, difficulties arise. The model is described in detail in [Moskowitz, Ravindran, Klein, and Eswaren 1982].

One method of handling the uncertainty aspect would be to take expected AOQ and ALIC values. These expected values may be optimized by the PCM or preference assessment methods [Moskowitz, et al, 1982]. A second, more appropriate approach is to consider the risk by utilizing a utility measure. Maximum expected utility would be the objective and would correspond to problem TS. The RHS values are deterministic, but the parameters in the objective function are uncertain, so that the function $f_{11}(X,Z)$ may not necessarily be equivalent to $f_{12}(X,Z)$. All variables in this problem are first-stage variables since the true state of nature is not determined except under complete sampling.

Let us use the specific problem:

f_1 = AOQ = $(N-n)P_A P_D/N$

f_2 = ALIC - $nC_I + (N-n)(1-P_A)C_I$

when n = sample size,

PA = Probability of Acceptance =

f(hypogeometric distribution, n,

acceptance level, Pd)

N = lot size = 100,

C_I = Cost of inspection per unit = \$12,

P_d = Percent defective = 10% with p = .25

15% with (1-p) = .75

Assume the decision maker's true utility function is U = 2 - .3(ALIC/1200)^2 - AOQ^2. In order to permit comparison, the maximum expected utility as determined by complete enumeration occurs at:

AOQ (10%)	=	.09437
ALIC (10%)	=	67.57
AOQ (15%)	=	.1392
ALIC (15%)	=	86.40
n	=	5
c	=	2

It should be noted that this Quality Control Model does not fit the convexity condition required for global optimality, but the algorithm will hopefully prove to be robust. Optimization was performed using an integer modification to Box's complex search [Box 1965].

The methodology terminated after two iterations with a final solution of:

AOQ (10%)	=	.09498
ALIC (10%)	=	60.29
AOQ (15%)	=	.14227
ALIC (15%)	=	61.8
n	=	5
c	=	3,

or, not far from the optimal.

5. Conclusion

The methodology for an interactive procedure under uncertainty is developed. The process relies heavily on the theory and methodology of the Paired Comparison Method and the Geoffrion, Dyer and Feinberg method. The general development handles uncertainty in either the objective or the RHS. In order to implement the procedure, a concept of mutual efficiency is defined, and theory determining the existence of mutual efficiency is developed. A limiting factor is the ability of a Decision Maker to handle lotteries involving multiple outcomes and multiple criteria. These limits suggest research into ways to decompose the questions, into proper questioning methodologies, and possible decision aids to help the DM better visualize the outcomes.

REFERENCES

Bard, J. F., "An Efficient Point Algorithm for a Linear Two-Stage Optimization Problem," Operations Research, Vol. 31-4, 1983.

Box, M. J., "A New Method of Constrained Optimization and a Comparison with Other Methods," Computer Journal, Vol. 8, 42-52, 1965.

Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, Princeton, NJ, 1963.

Farquhar, P. H., "Utility Assessment Methods," Management Science, Vol. 30-11, 1984.

Geoffrion, A. M., Dyer, J. S. and Feinberg, A., "An Interactive Approach to Multi-Criterion Optimization with an Application to the Operation of an Academic Department," Management Science, Vol. 19, 357-368, 1976.

Hwang, C. L., Masud, A. S., Multiple Objective Decision Making = Methods and Applications, Springer-Verlag, New York, 1979.

Kenney, R. L., and Raiffa, H., Decisions with Multiple Objectives: Preferences and Value Tradeoffs, John Wiley and Sons, New York, 1976.

Klein, G., Moskowitz, A., and Ravindran, A., "A Comparative Study of Prior vs. Progressive Articulation of Preference Approaches for Bicriterion Problems," to appear in NRLQ, 1985.

Moskowitz, H., Ravindran, A., Klein, G. and Eswaren, P. K., "A Bicriterion Acceptance Sampling Model," TSMS Optimization in Statistics, Zanakis, Ed.; 1982.

Sadagopan, S. and Ravindran, A., "Interactive Solution of Bicriteria Mathematical Programs," Naval Research Logistics Quarterly, Vol. 29-3, 1982.

Wagner, H. M., Principles of Operations Research, Prentice-Hall, New Jersey, 1969.

Wallenius, J., "Comparative Evaluation of Some Interactive Approaches to Multicriteria Optimization," Management Science, Vol. 21, 1387-1396, 1975.

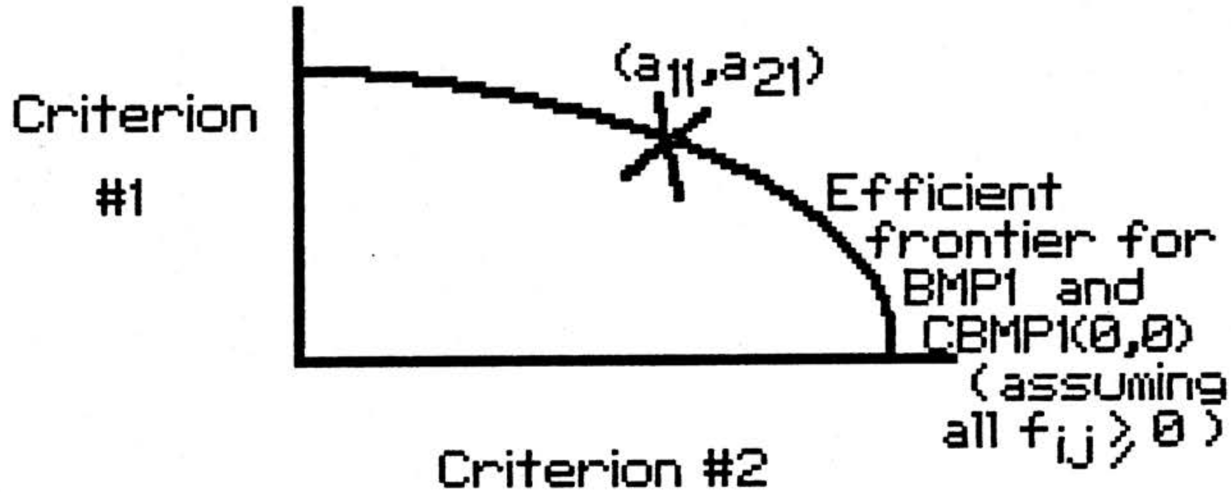
Zionts, S., "Methods for Solving Management Problems Involving Multiple Objectives," WP 400, SUNY at Buffalo, 1979.

Zionts, S. and Wallenius, J., "An Interactive Programming Method for Solving the Multiple Criteria Problem," Management Science, Vol. 22-6, 1976.

Zionts, S. and Wallenius, J., "An Interactive Multiple Objective Linear Programming Method for a Class of Underlying Nonlinear Utility Functions," Management Science, Vol. 29-5, 1983.

LIST OF FIGURES

Q1 - Outcome 1



Q2 - Outcome 2

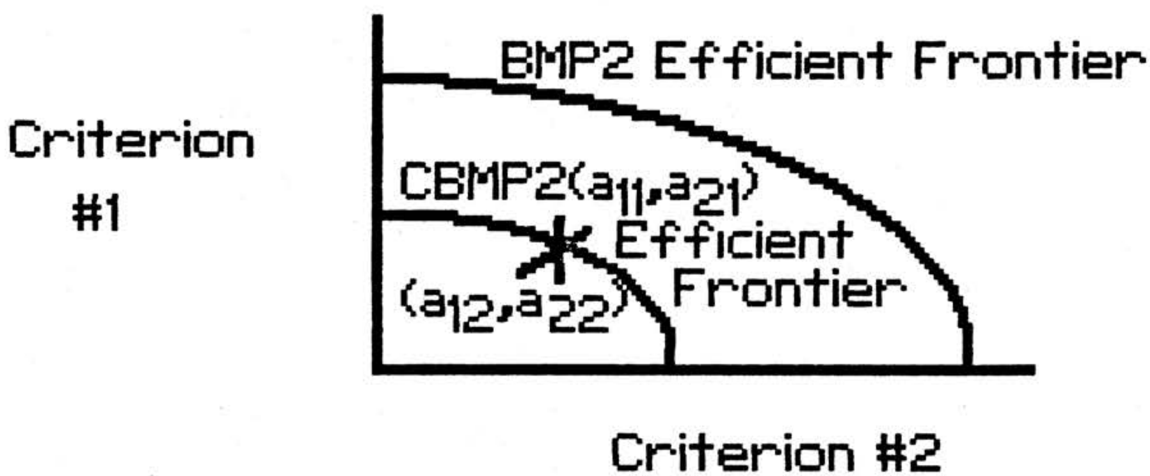


FIGURE 1: Starting Criteria

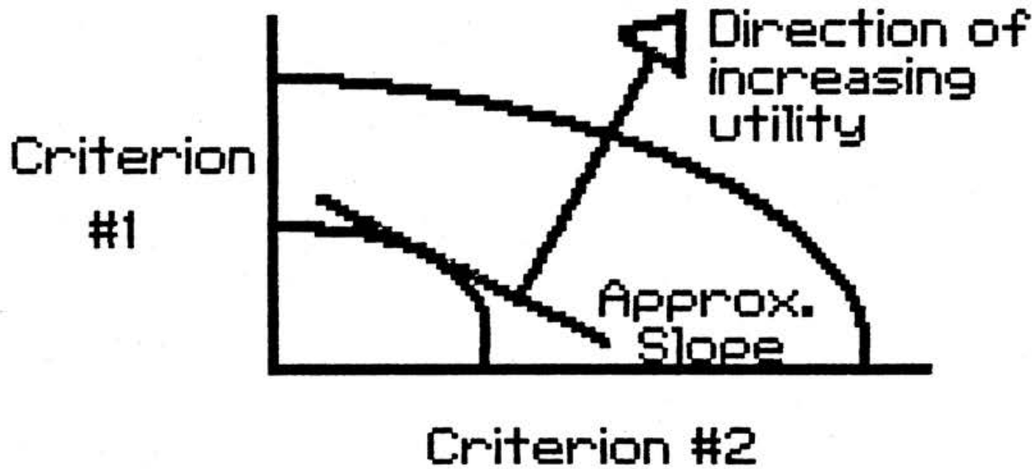


FIGURE 2: Approximating Utility over the criteria

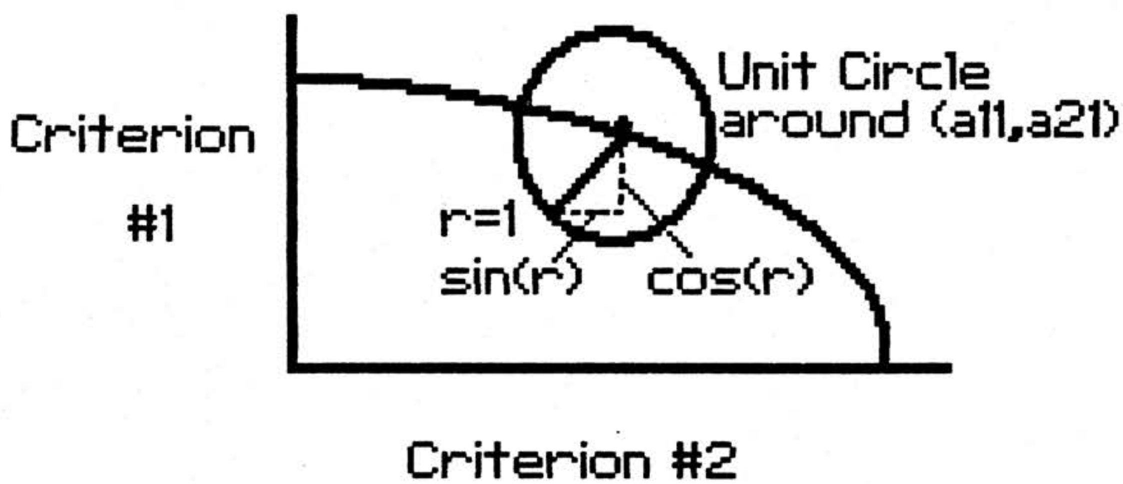
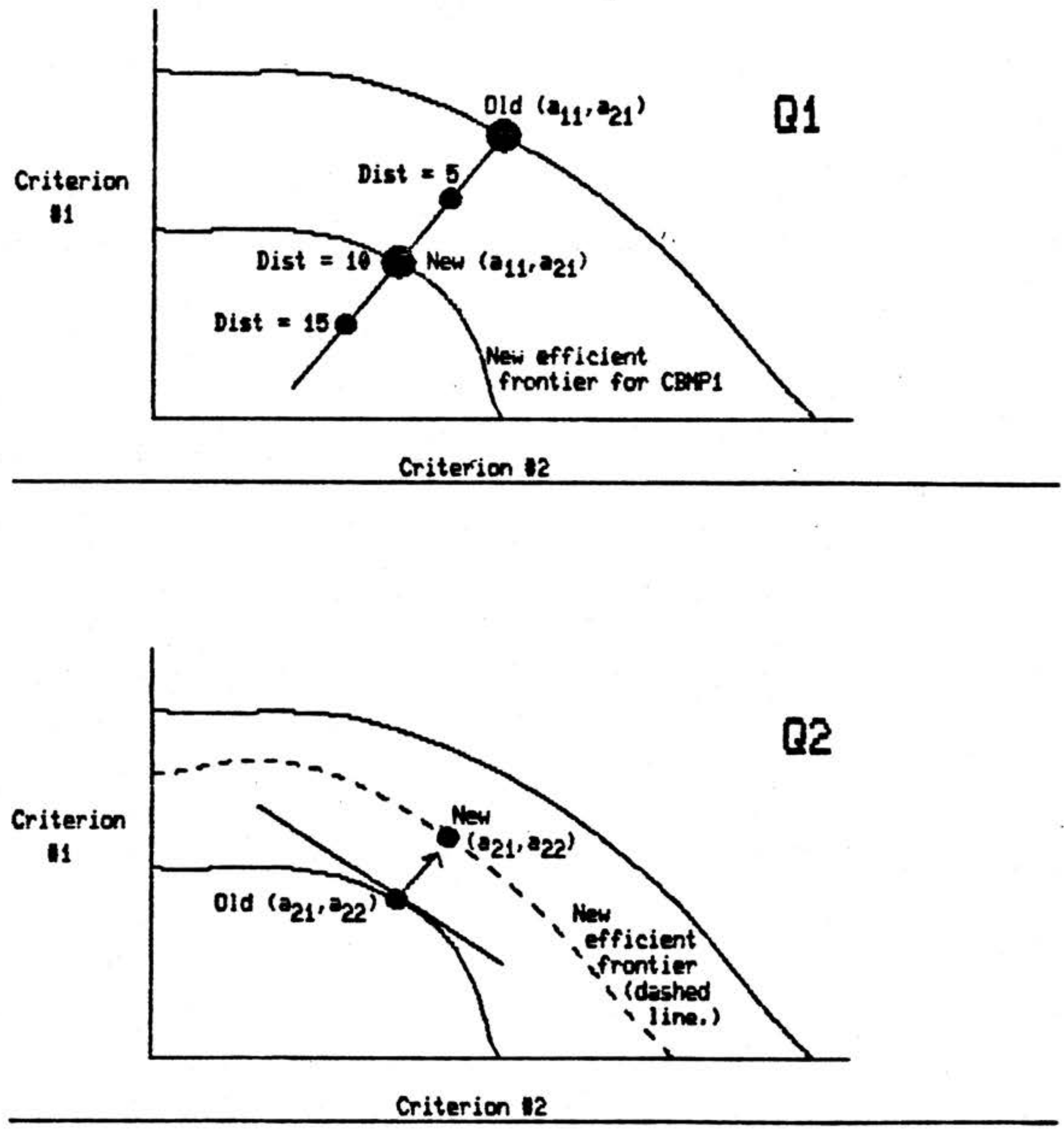


FIGURE 3: Direction Finding

FIGURE 4: Distance Determination (preferred dist is 10)



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