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SUMMARY

This paper deals with certain probabilities associated with the ordering of the components of a vector of unknown multinomial cell probabilities. The analysis is based upon the assumption that the random multinomial parametric vector is distributed as a Dirichlet distribution. Explicit expressions for the *a priori* probability that a given event is the most probable or the least probable multinomial event are developed. In addition, recursion formulas are developed that permit the determination of the *a priori* probability associated with an arbitrary ordering of the unknown cell probabilities.

Some key words: Dirichlet distribution, inverted-Dirichlet distribution, least probable multinomial event, most probable multinomial event, most probable ordering of multinomial events.

1. INTRODUCTION

1.1 Purpose of Research

Many problems concerning the Bayesian analysis of data generating processes involve the multinomial distribution. Unfortunately, sparsity of data may make it impossible to design an optimal sampling procedure for selecting the most or the least probable multinomial event; nevertheless, there is a need to provide some useful information regarding these events for the purposes of supporting decision making. One obvious and useful piece of information is the probability that a given event is the most (least) probable event prior or posterior to sample information; however, explicit expressions for these probabilities are not to be found in the literature. This paper then deals with the development of explicit expressions for these probabilities. In addition, we develop a set of recursion formulas useful for constructing an explicit expression for the probability associated with an arbitrary ordering of the unknown multinomial cell probabilities; recursion formulas for this probability are presented in lieu of an explicit expression in order to minimize notational complexity.

1.2 Background and Preliminaries

Suppose that $Y = (Y_1,\ldots,Y_{K+1})$ is a vector of observations having a multinomial distribution with parameter N and random parametric vector $P = (P_1,\ldots,P_{K+1})$, where $P_1+\ldots+P_K \leq 1$; hence, the kernel of the likelihood function of the vector of observations is of the form

$$\prod_{j=1}^{K} p_j^{y_j}(1 - \sum_{j=1}^{K} p_j)^{N-y_o}$$

where $y_o = y_1+\ldots+y_K$. By definition a conjugate prior density is a normalized likelihood function; hence, a natural conjugate family of distributions for observations having a multinomial distribution is the Dirichlet family of distributions. Thus, the kernel of the prior density of a Dirichlet distribution with parametric vector $\alpha = (\alpha_1,\ldots,\alpha_{K+1})$ is of the form

$$\prod_{j=1}^{K} p_j^{\alpha_j-1} (1 - \sum_{j=1}^{K} p_j)^{\alpha_{K+1}-1}$$

for any point in the simplex $S_K = \{(p_1,\ldots,p_K): p_j > 0 \text{ for } j = 1,\ldots,K$ and $p_1+\ldots+p_K \leq 1\}$ in R^K, and zero elsewhere, where the α_j are positive real quantities.

Unfortunately, the evaluation of probability integrals involving the Dirichlet distribution are in general very difficult to carry out directly; primarily because of the simplex constraint placed upon the components of the vector random variable $P = (P_1,\ldots,P_{K+1})$. This difficulty, however, is easily overcome through a transformation of random variables suggested by Tiao and Guttman (1965); namely,

$$T: \quad Q_i = P_i(1 - \sum_{j=1}^{K} P_j)^{-1}, \quad i=1,\ldots,K$$

The inverse transformation of T is

$$T^{-1}: \quad P_i = Q_i(1 + \sum_{j=1}^{K} Q_j)^{-1}, \quad i=1,\ldots,K$$

and the Jacobian J of the transformation is found to be

$$J = (1 + \sum_{j=1}^{K} Q_j)^{-(K+1)}$$

Upon application of the transformation T it follows that the vector random variable $Q = (Q_1,\ldots,Q_K)$ is distributed as an inverted-Dirichlet distribution with density function

$$\frac{1}{B(\alpha_1,\ldots,\alpha_{K+1})} \prod_{j=1}^{K} q_j^{\alpha_j - 1} \quad (1 + \sum_{j=1}^{K} q_j)^{-\sum_{j=1}^{K+1} \alpha_j}$$

for $0 < q_j < \infty$, $j = 1,\ldots,K$, in R^K, and zero elsewhere, and where $B(\alpha_1,\ldots,\alpha_{K+1}) = \Gamma(\alpha_1)\ldots\Gamma(\alpha_{K+1})/\Gamma(\alpha_1+\ldots+\alpha_{K+1})$.

If we now denote $P_{[1]} \leq \ldots \leq P_{[K+1]}$ as the ordered set of unknown cell probabilities associated with the random parametric vector $P = (P_1,\ldots,P_{K+1})$, then we can define the most (least) probable multinomial event as the event with the largest (smallest) cell probability. Utilizing the transformational relationship between the Dirichlet distribution and its multivariate analog, the inverted-Dirichlet distribution, we can write the probability that $P_{[K+1]} = P_k$ as

$$\mathrm{pr}[(1 - \sum_{\substack{j=1 \\ j \neq k}}^{K+1} P_j \leq P_k) \bigcap_{\substack{j=1 \\ j \neq k}}^{K} (P_j \leq P_k)]$$

$$= \frac{1}{B(\alpha_1,\ldots,\alpha_{K+1})} \int_1^{\infty} \int_0^{q_k} \ldots \int_0^{q_k} w_k^{\alpha_k - 1} (1 + \sum_{j=1}^{K} w_j)^{-\sum_{j=1}^{K+1} \alpha_j} \prod_{\substack{j=1 \\ j \neq k}}^{K} w_j^{\alpha_j - 1} \, dw_j dw_k \quad (1)$$

Similarly, we can write the probability that $P_{[1]} = P_k$ as

$$\mathrm{pr}[(P_k \le 1-\sum_{\substack{j=1\\ j\neq k}}^{K+1} P_j)\bigcap_{\substack{j=1\\ j\neq k}}^{K}(P_k \le P_j)]$$

$$= \frac{1}{B(\alpha_1,\ldots,\alpha_{K+1})}\int_0^1\int_{q_k}^{\infty}\cdots\int_{q_k}^{\infty} w_k^{\alpha_k-1}(1+\sum_{j=1}^{K} w_j)^{-\sum_{j=1}^{K+1}\alpha_j}\prod_{\substack{j=1\\ j\neq k}}^{K} w_j^{\alpha_j-1}\,dw_j\,dw_k \quad (2)$$

In like fashion, the probability associated with an arbitrary ordering of the unknown cell probabilities $P = (P_1,\ldots,P_{K+1})$, i.e., $P_{(1)} \le \cdots \le P_{(K+1)}$, can be written as

$$\mathrm{pr}[\bigcap_{j=1}^{K}(P_{(j)} \le P_{(j+1)})]$$

$$= \frac{1}{B(\alpha_{(1)},\ldots,\alpha_{(K+1)})}\int_0^1\int_0^{q_{(K)}}\int_0^{q_{(K-1)}}\cdots\int_0^{q_{(2)}}(1+\sum_{j=1}^{K} w_{(j)})^{-\sum_{j=1}^{K+1}\alpha_{(j)}}\prod_{j=1}^{K} w_{(j)}^{\alpha_{(j)}-1}\,dw_{(j)} \quad (3)$$

where the subscripts $((1),\ldots,(K+1))$ correspond to exactly one of the $(K+1)!$ permutations of the integers $(1,\ldots,K+1)$.

2. THE MOST PROBABLE EVENT

The results of this section are summarized in the following lemma:

Lemma 1. If the vector random variable $P = (P_1,\ldots,P_{K+1})$ has a Dirichlet distribution with parametric vector $\alpha = (\alpha_1,\ldots,\alpha_{K+1})$, where the α_j are positive integers, then the probability that the kth variate is the largest Dirichlet variate is

$$\mathrm{pr}[P_k = \max\{P_1,\ldots,P_{K+1}\}] = 1 - \sum_{i_1} \pi_{i1} + \sum_{i_1<i_2} \pi_{i1i2}$$

$$+ \ldots + (-1)^m \sum_{i_1<\ldots<i_m} \pi_{i1\ldots im} + \ldots + (-1)^K \pi_{1\ldots(k-1)(k+1)\ldots K+1}$$

where $\sum_{i_1<\ldots<i_m}$ denotes the summation over all integers $i_1,\ldots,i_m$, where (i) $1 \leq i_j \leq K+1$, $j = 1,\ldots,m$ ($i_j \neq k$) and (ii) $i_1<\ldots<i_m$, and where

$$\pi_{i1\ldots im} = \sum_{x_{i1}=0}^{\alpha_{i1}-1} \cdots \sum_{x_{im}=0}^{\alpha_{im}-1} \frac{\Gamma(\alpha_k + \sum_{j=1}^{m} x_{ij})}{\Gamma(\alpha_k) \prod_{j=1}^{m} x_{ij}!} \left(\frac{1}{m+1}\right)^{\alpha_k} \prod_{j=1}^{m} \left(\frac{1}{m+1}\right)^{x_{ij}}$$

To illustrate the application of the above lemma, consider the Dirichlet distributed vector variate $P = (P_1,P_2,P_3,P_4 = 1-(P_1+P_2+P_3))$ having the parametric vector $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. If the parameters α_j are positive integers, then the probability that the kth variate, where k=3, is the largest Dirichlet variate is given by

$$\mathrm{pr}[P_3 = \max\{P_1,P_2,P_3,P_4\}] = 1 - (\pi_1+\pi_2+\pi_4) + (\pi_{12}+\pi_{14}+\pi_{24}) - \pi_{124}$$

$$= 1 - \sum_{x_1=0}^{\alpha_1-1} \frac{\Gamma(\alpha_3+x_1)}{\Gamma(\alpha_3)x_1!}\left(\frac{1}{2}\right)^{\alpha_3+x_1} - \sum_{x_2=0}^{\alpha_2-1} \frac{\Gamma(\alpha_3+x_2)}{\Gamma(\alpha_3)x_2!}\left(\frac{1}{2}\right)^{\alpha_3+x_2}$$

$$- \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_4)}{\Gamma(\alpha_3)x_4!}(\tfrac{1}{2})^{\alpha_3+x_4} + \sum_{x_1=0}^{\alpha_1-1} \sum_{x_2=0}^{\alpha_2-1} \frac{\Gamma(\alpha_3+x_1+x_2)}{\Gamma(\alpha_3)x_1!x_2!}(\tfrac{1}{3})^{\alpha_3+x_1+x_2}$$

$$+ \sum_{x_1=0}^{\alpha_1-1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_1+x_4)}{\Gamma(\alpha_3)x_1!x_4!}(\tfrac{1}{3})^{\alpha_3+x_1+x_4} + \sum_{x_2=0}^{\alpha_2-1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_2+x_4)}{\Gamma(\alpha_3)x_2!x_4!}(\tfrac{1}{3})^{\alpha_3+x_2+x_4}$$

$$- \sum_{x_1=0}^{\alpha_1-1} \sum_{x_2=0}^{\alpha_2-1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_1+x_2+x_4)}{\Gamma(\alpha_3)x_1!x_2!x_4!}(\tfrac{1}{4})^{\alpha_3+x_1+x_2+x_4}$$

We will now formally prove Lemma 1, without loss of generality, for the case of k=K. The evaluation of probability (1) follows directly from successive application of the following formula established by Taio and Guttman (1965):

$$\int_0^a (1+x+t)^{-(\alpha+n)} t^{n-1} dt =$$

$$(1+x)^{-\alpha} B(\alpha,n) - \sum_{j=0}^{n-1} \binom{n-1}{j} a^j (1+x+t)^{-(\alpha+j)} B(\alpha+j,n-j) \qquad (4)$$

where n is a positive integer, and $<x,\alpha,a>$ are positive quantities, and where $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

For simplicity of notation define

$$J_{r\ldots K} = \frac{1}{B(\alpha_r,\ldots,\alpha_{K+1})} \int_1^\infty \int_0^{q_K} \cdots \int_0^{q_K} (1+\sum_{j=1}^{K} w_j)^{-\sum_{j=r}^{K+1}\alpha_j} \prod_{j=r}^{K} w_j^{\alpha_j-1} dw_j$$

and

$$J_{r\ldots K}^{i_1\ldots i_k} = \sum_{x_{i1}=0}^{\alpha_{i1}-1} \cdots \sum_{x_{ik}=0}^{\alpha_{ik}-1} \frac{\Gamma(\alpha_{K,k})}{\Gamma(\alpha_K)\prod_{j=1}^{k} x_{ij}!}$$ (continued on next page)

$$\frac{1}{B(\alpha_r,\ldots,\alpha_{K,r},\alpha_{K+1})}\int_1^\infty \int_0^{q_K}\cdots\int_0^{q_K} \frac{w_K^{\alpha_{K,k}-1}\prod_{j=r}^{K-1} w_j^{\alpha_j-1}\, dw_j\, dw_K}{\{1+\sum_{j=r}^{K-1} w_j+(k+1)w_K\}^{\sum_{j=r}^{K-1}\alpha_j+\alpha_{K,k}+\alpha_{K+1}}} \quad (5)$$

where $i_1,\ldots,i_k$ is an arbitrary sequence of positive integers of length k, $1 \leq k \leq r-1$, such that $1 \leq i_1<\ldots<i_k \leq r-1$, and where $\alpha_{K,k} = \alpha_K+x_{i1}+\ldots+x_{ik}$.

For $\alpha_1,\ldots,\alpha_K$ positive integers, successive application of integration formula (4) in probability integral (1) yields the following result:

$$J_{1\ldots K} = J_K - \sum_{i_1} J_K^{i_1} + \sum_{i_1<i_2} J_K^{i_1 i_2} + \ldots + (-1)^m \sum_{i_1<\ldots<i_m} J_K^{i_1\ldots i_m} + \ldots + (-1)^K J_K^{1\ldots K-1} \quad (6)$$

where $\sum_{i_1<\ldots<i_m}$ denotes the summation over all integers $i_1,\ldots,i_m$ where (i) $1 \leq i_j \leq K+1$ for $j = 1,\ldots,K-1,K+1$ and (ii) $i_1<\ldots<i_m$.

From definitional formula (5) we have that

$$J_K^{i_1\ldots i_k} = \sum_{x_{i1}=0}^{\alpha_{i1}-1}\cdots\sum_{x_{ik}=0}^{\alpha_{ik}-1} \frac{\Gamma(\alpha_{K,k})}{\Gamma(\alpha_K)\prod_{j=1}^{k} x_{ij}!}\left[\frac{1}{B(\alpha_{K,k},\alpha_{K+1})}\int_1^\infty \frac{q_K^{\alpha_{K,k}-1}\, dq_K}{\{1+(1+k)q_K\}^{\alpha_{K,k}+\alpha_{K+1}}}\right] \quad (7)$$

It is well known, see for example Olkin and Sobel (1965), that for any real $r>0$ and positive integer s

$$\frac{1}{B(r,s)}\int_\theta^1 t^r(1-t)^{s-1}dt = \frac{1}{B(r,s)}\int_{\theta/(1-\theta)}^\infty u^{r-1}(1+u)^{-(r+s)}du = 1 - I_\theta(r,s) \quad (8)$$

where

$$I_\theta(r,s) = \theta^r \sum_{x=0}^{s-1} \frac{\Gamma(r+x)}{\Gamma(r)x!}(1-\theta)^x \tag{9}$$

Thus, using equations (8) and (9) we can write equation (7), upon rearranging, as

$$J_K^{i_1 \ldots i_k} = \pi_{i_1 \ldots i_k} - \pi_{i_1 \ldots i_{k(K+1)}} \tag{10}$$

for $1 \leq k \leq K-1$, where

$$\pi_{i1\ldots ik} = \sum_{x_{i1}=0}^{\alpha_{i1}-1} \cdots \sum_{x_{ik}=0}^{\alpha_{ik}-1} \frac{\Gamma(\alpha_{K,k})}{\Gamma(\alpha_K) \prod_{j=1}^{k} x_{ij}!} \left(\frac{1}{k+1}\right)^{\alpha_{K,k}}$$

and where

$$\pi_{i1\ldots ik(K+1)} = \sum_{x_{i1}=0}^{\alpha_{i1}-1} \cdots \sum_{x_{ik}=0}^{\alpha_{ik}-1} \sum_{x_{K+1}=0}^{\alpha_{K+1}-1} \frac{\Gamma(\alpha_{K,k}+x_{K+1})}{\Gamma(\alpha_K) \prod_{j=1}^{k} x_{ij}! x_{K+1}!} \left(\frac{1}{k+2}\right)^{\alpha_{K,k}+x_{K+1}}$$

Substituting equation (10) into equation (6) yields, upon rearranging, the desired result, i.e.,

$$J_{1\ldots K} = 1 - \sum_{i_1} \pi_{i1} + \sum_{i_1<i_2} \pi_{i1i2} + \cdots + (-1)^m \sum_{i_1<\cdots<i_m} \pi_{i1\ldots im}$$

$$+ \cdots + (-1)^K \pi_{1\ldots(K-1)(K+1)}$$

3. THE LEAST PROBABLE EVENT

The results of this section are summarized in the following lemma:

Lemma 2. If the vector random variable $P = (P_1,\ldots,P_{K+1})$ has a Dirichlet distribution with parametric vector $\alpha = (\alpha_1,\ldots,\alpha_{K+1})$, where the α_j are positive integers, then the probability that the kth variate is the smallest Dirichlet variate is

$pr[P_k = \min\{P_1,\ldots,P_{K+1}\}] =$

$$\sum_{x_1=0}^{\alpha_1-1} \cdots \sum_{x_{k-1}=0}^{\alpha_{k-1}-1} \sum_{x_{k+1}=0}^{\alpha_{k+1}-1} \cdots \sum_{x_{K+1}=0}^{\alpha_{K+1}-1} \frac{\Gamma(\alpha_k + \sum_{\substack{j=1\\ j\neq k}}^{K+1} x_j)}{\Gamma(\alpha_k) \prod_{\substack{j=1\\ j\neq k}}^{K+1} x_j!} \left(\frac{1}{K+1}\right)^{\alpha_k} \prod_{\substack{j=1\\ j\neq k}}^{K+1} \left(\frac{1}{K+1}\right)^{x_j}$$

To demonstrate the application of the above lemma, consider once again the Dirichlet vector variate $P = (P_1,P_2,P_3,P_4=1-(P_1+P_2+P_3))$ having the parametric vector $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. If the parameters α_j are positive integers, then we can write the probability that the kth event, where k=3, is the smallest Dirichlet variate as

$pr[P_3 = \min(P_1,P_2,P_3,P_4)] =$

$$\sum_{x_1=0}^{\alpha_1-1} \sum_{x_2=0}^{\alpha_2-1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_1+x_2+x_4)}{\Gamma(\alpha_3)x_1!x_2!x_4!}\left(\frac{1}{4}\right)^{\alpha_3+x_1+x_2+x_4}$$

We will now formally prove Lemma 2, without loss of generality for the case of k=K. The evaluation of probability (2) follows directly from successive application of the following variation of formula (4):

$$\int_a^\infty (1+x+t)^{-(\alpha+n)} t^{n-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} a^j (1+x+a)^{-(\alpha+j)} B(\alpha+j,n-j) \qquad (11)$$

where n is a positive integer, and <x,α,a> are positive quantities and where $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

For simplicity of notation define

$$L^{1\ldots k}_{k+1\ldots K} = \sum_{x_1=0}^{\alpha_1-1} \cdots \sum_{x_k=0}^{\alpha_k-1} \frac{\Gamma(\alpha_{K,k})}{\Gamma(\alpha_K)\prod_{j=1}^{k} x_j!}$$

$$\cdot \frac{1}{B(\alpha_{k+1},\ldots,\alpha_{K,k},\alpha_{K+1})} \int_0^1 \int_{q_K}^{\infty} \cdots \int_{q_K}^{\infty} \frac{w_K^{\alpha_{K,k}-1} \prod_{j=k+1}^{K-1} w_j^{\alpha_j-1}\, dw_j\, dw_K}{\{1+\sum_{j=k+1}^{K-1} w_j+(k+1)w_K\}^{\alpha_{K,k}+\alpha_{K+1}}}$$

where $\alpha_{K,k} = \alpha_K+x_1+\ldots+x_k$, for $1 \le k \le K-1$.

For $\alpha_1,\ldots,\alpha_K$ positive integers, successive application of integration formula (11) in probability integral (2) yields the following result:

$$L_{1\ldots K} = L_K^{1\ldots K-1}$$

where

$$L_K^{1\ldots K-1} = \sum_{x_1=0}^{\alpha_1-1} \cdots \sum_{x_{K-1}=0}^{\alpha_{K-1}-1} \frac{\Gamma(\alpha_{K,K-1})}{\Gamma(\alpha_K)\prod_{j=1}^{K-1} x_j!} \left(\frac{1}{K}\right)^{\alpha_{K,K-1}} \left\{\frac{1}{B(\alpha_{K,K-1},\alpha_{K+1})} \int_0^K \frac{z^{\alpha_{K,K-1}} dz}{(1+z)^{\alpha_{K,K-1}+\alpha_{K+1}}}\right\} \tag{12}$$

It is well known that for any real $r>0$ and positive integer s

$$\frac{1}{B(r,s)} \int_0^{\theta} \frac{u^{r-1} du}{(1+u)^{r+s}} = I_{\theta/(1+\theta)}(r,s) = \left(\frac{\theta}{1+\theta}\right)^r \sum_{x=0}^{s-1} \frac{\Gamma(r+x)}{\Gamma(r)x!} \left(\frac{1}{1+\theta}\right)^x \tag{13}$$

Hence, using relationship (13) in equation (12) yields the desired result

$$L_{1\ldots K} = \sum_{x_1=0}^{\alpha_1-1} \cdots \sum_{x_{K-1}=0}^{\alpha_{K-1}-1} \frac{\Gamma(\alpha_{K,K-1}+x_{K+1})}{\Gamma(\alpha_K)\prod_{j=1}^{K-1} x_j!\,x_{K+1}!}\left(\frac{1}{K+1}\right)^{\alpha_{K,K-1}+x_{K+1}}$$

4. THE MOST PROBABLE ORDERING OF EVENTS

In this section we derive a set of recursion formulas useful for the development of an explicit expression for the probability associated with an arbitrary ordering of the components of a vector random variable having a Dirichlet distribution; specifically the probability implied by equation (3). The derivation of the recursion formulas underlying the evaluation of probability integral (3) follows directly from repeated integration using relationship (4); we once again assume that the components of the parametric vector α are positive integers. For clarity of presentation we drop the parenthesis in subscripts of probability (3) and proceed without loss of generality to evaluate the probability that $P_1 \leq \ldots \leq P_{K+1}$.

For simplicity of notation define

$$M_{s \ldots K} = \int_0^1 \int_0^{q_K} \int_0^{q_{K-1}} \cdots \int_0^{q_{s+1}} \frac{\left(1+\sum_{j=s}^{K} w_j\right)^{-\sum_{j=1}^{K+1} \alpha_j} \left(\prod_{j=s}^{K} w_j^{\alpha_j - 1}\, dw_j\right)}{B(\alpha_s, \ldots, \alpha_{K+1})}, \quad s=1,\ldots,K$$

$$M^k_{s \ldots K} = \int_0^1 \int_0^{q_K} \int_0^{q_{K-1}} \cdots \int_0^{kq_{s+1}} \frac{\left(1+\sum_{j=s}^{K} w_j\right)^{-\left(\sum_{j=1}^{K+1} \alpha_j + x_{s-1}\right)} w_s^{\alpha_s + x_{s-1} - 1} \left(\prod_{j=s+1}^{K} w_j^{\alpha_j - 1}\, dw_j\right) dw_s}{B(\alpha_s + x_{s-1}, \alpha_{s+1}, \ldots, \alpha_{K+1})}$$

for $s = 2,\ldots,K-1$ and $2 \leq k \leq s$,

$$\pi_{r \ldots s} = \prod_{j=r}^{s} \left\{ \sum_{x_j=0}^{\alpha_j + x_{j-1} - 1} \frac{\Gamma(\alpha_{j+1} + x_j)}{\Gamma(\alpha_{j+1})\, x_j!} \left(\frac{1}{j+1}\right)^{\alpha_{j+1}} \left(\frac{j}{j+1}\right)^{x_j} \right\}, \quad s=1,\ldots,K-1, \; r<s \tag{14}$$

and

$$\pi_s = \sum_{x_s=0}^{\alpha_s-1} \frac{\Gamma(\alpha_{s+1}+x_s)}{\Gamma(\alpha_{s+1})x_s!}(\tfrac{1}{2})^{\alpha_{s+1}+x_s}, \quad s=1,\ldots,K-1 \tag{15}$$

For $\alpha_1,\ldots,\alpha_{K+1}$ positive integers, successive application of formula (4) in equation (3) yields the following recursion formulas:

$$\text{(i)} \quad M_{s\ldots K} = M_{s+1\ldots K} - \pi_s M^2_{s+1\ldots K}, \qquad s=1,\ldots,K-1 \tag{16}$$

$$\text{(ii)} \quad \pi_{s-(k+1)\ldots s-1} M^k_{s\ldots K} = \pi_{s-k+1\ldots s-1} M_{s+1\ldots K} - \pi_{s-k+1\ldots s} M^{k+2}_{s+1\ldots K}, \qquad s=2,\ldots,K-1,\ 2 \le k \le s \tag{17}$$

where

$$M_K = I_{\frac{1}{2}}(\alpha_K,\alpha_{K+1}) = \sum_{x_{K+1}=0}^{\alpha_{K+1}-1} \frac{\Gamma(\alpha_K+x_{K+1})}{\Gamma(\alpha_K)x_{K+1}!}(\tfrac{1}{2})^{\alpha_K+x_{K+1}} \tag{18}$$

and

$$M^k_K = I_{k/k+1}(\alpha_K+x_{K-1},\alpha_{K+1}) = \sum_{x_{K+1}=0}^{\alpha_{K+1}-1} \frac{\Gamma(\alpha_K+x_{K-1}+x_{K+1})}{\Gamma(\alpha_K+x_{K-1})x_{K+1}!}(\tfrac{k}{k+1})^{\alpha_K+x_{K-1}}(\tfrac{1}{k+1})^{x_{K+1}} \tag{19}$$

using relationship (9).

To illustrate the constructive application of equations (14) and (17), consider the vector random variable $P = (P_1,P_2,P_3,P_4=1-(P_1+P_2+P_3))$ having a Dirichlet distribution with parametric vector $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. If the parameters α_j are positive integers we can write, using recursion formulas (16)-(17), the probability that $P_1 \le P_2 \le P_3 \le P_4$ as follows:

$$\text{pr}[P_1 \le P_2, P_2 \le P_3, P_3 \le P_4] = M_3 - \pi_2 M_3^2 - \pi_1 M_3 + \pi_{12} M_3^3 \tag{20}$$

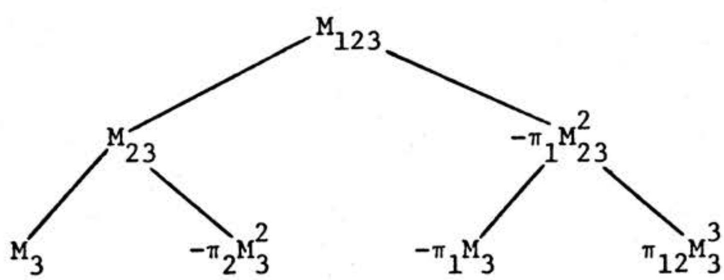
Using relationships (14)-(15) and (18)-(19) in probability (20) we have, upon rearranging, the result

$$pr[P_1 \le P_2, P_2 \le P_3, P_3 \le P_4] = \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_4)}{\Gamma(\alpha_3)x_4!}\left(\frac{1}{2}\right)^{\alpha_3+x_4}$$

$$- \sum_{x_2=0}^{\alpha_2-1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_2+x_4)}{\Gamma(\alpha_3)x_2!x_4!}\left(\frac{1}{3}\right)^{\alpha_3+x_2+x_4} - \sum_{x_1=0}^{\alpha_1-1} \frac{\Gamma(\alpha_2+x_1)}{\Gamma(\alpha_2)x_1!}\left(\frac{1}{2}\right)^{\alpha_2+x_1} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_4)}{\Gamma(\alpha_3)x_4!}\left(\frac{1}{2}\right)^{\alpha_3+x_4}$$

$$+ \sum_{x_1=0}^{\alpha_1-1} \frac{\Gamma(\alpha_2+x_1)}{\Gamma(\alpha_2)x_1!}\left(\frac{1}{2}\right)^{\alpha_2+x_1} \sum_{x_2=0}^{\alpha_2+x_1-1} \frac{\Gamma(\alpha_3+x_2)}{\Gamma(\alpha_3)x_2!}\left(\frac{1}{3}\right)^{\alpha_3}\left(\frac{2}{3}\right)^{x_2} \sum_{x_4=0}^{\alpha_4-1} \frac{\Gamma(\alpha_3+x_2+x_4)}{\Gamma(\alpha_3+x_2)x_4!}\left(\frac{3}{4}\right)^{\alpha_3+x_2}\left(\frac{1}{4}\right)^{x_4} \quad (21)$$

The application of the recursion formulas in the above calculation of the probability M_{123} can be visualized with the help of the following tree diagram:



where in equation (20), the root of the tree is set equal to the sum of its terminal branches. The tree is easily extended to yield explicit expressions for any of the probabilities defined by equation (3). For instance, if the root of the tree is M_{1234} then it is easily verified using equations (16) and (17) that the probability that $P_1 \le P_2 \le P_3 \le P_4 \le P_5$, where $P_5 = 1 - (P_1+P_2+P_3+P_4)$, is given by

$$\mathrm{pr}[P_1 \leq P_2, P_2 \leq P_3, P_3 \leq P_4, P_4 \leq P_5] = M_4 - \pi_3 M_4^2 - \pi_2 M_4 + \pi_{23} M_4^3$$

$$- \pi_1 M_4 + \pi_1 \pi_3 M_4^2 + \pi_{12} M_4 - \pi_{123} M_4^4 \qquad (22)$$

An explicit expression for probability (22) can then be easily obtained using relationships (14)-(15) and (18)-(19) in equation (22).

5. CONCLUSION

We have presented several useful expressions for calculating certain probabilities associated with the ordering of the components of a vector of unknown multinomial cell probabilities. These expressions provide important information in support of the selection of the most (least) probable event or the most probable ordering of multinomial events within a decision framework characterized by a high degree of subjective information and sparsity of data. Admittedly, problems of this ilk occur all too frequently in decision making situations; to this end it is hoped that this study will be of some practical value.

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