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A PROBABILISTIC ANALYSIS OF THE EIGENVECTOR PROBLEM FOR DOMINANCE MATRICES OF UNIT RANK

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by

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In this paper we present a model for the probabilistic analysis of the eigenvector sealing problem for dominance matrices of unit rank. We also address the problem of subjectively assessing the decision maker's pairwise preference distribution and present an analytical technique for deciding the following types of ultimate decision questions under uncertainty: (1) What is the most (least) preferred course of action? and (2) What is the preferred ordering or ranking of the available courses of action? We also provide an analytical procedure for investigating the robustness of the decision making procedure to variations in the pairwise preference distribution used to model the subjectively assessed distribution.
1. INTRODUCTION

The primary focus of applied decision analysis, broadly defined, is the development of quantitative decision models and techniques to aid decision makers in choosing between several, possibly complex, alternative courses of action. These techniques are not replacements or substitutes for the individual's own decision making processes, but rather they serve as decision support devices developed to test the coherence (see Bunn (1984)) of the decision maker's reasoning and judgment. Ideally, methodologies developed for this purpose should incorporate both qualitative and quantitative aspects of the decision problem into a framework capable of generating priorities for the various courses of action. Such a methodology, called the Analytic Hierarchy Process (AHP), has been advanced by T. L. Saaty (1977). The foundation of the AHP paradigm is a method of scaling relative importance judgments to yield a set of priority weights for the various courses of action, under certainty. More often than not, however, a decision maker faced with the problem of choosing among alternatives may find it difficult to specify with certainty his judgments regarding the relative importances of the various alternatives under comparison. The purpose of this paper then is to extend Saaty's deterministic analytical framework to allow the decision maker's pairwise preference responses to vary probabilistically.

Briefly, this paper is organized as follows. Firstly, we present a model for the probabilistic analysis of the eigenvector scaling problem for dominance matrices of unit rank. The analysis of reciprocal matrices of arbitrary rank is not presented due to severe analytical restrictions and difficulties. In Section 3 we consider the problem of subjectively assessing the decision maker's pairwise preference distribution. In Section 4 we present an analytical technique for deciding the following types of ultimate
decision questions under uncertainty: (1) What is the most (least) preferred course of action? and (2) What is the preferred ordering or ranking of the available courses of action? In both Sections 3 and 4 considerable attention is given to the question of the sensitivity of the ultimate decision to variations in the pairwise preference distribution used to model or approximate the subjectively assessed distribution. The analytical procedure presented in Section 4 provides the necessary framework for investigating the robustness of the decision making procedure. Section 5 contains concluding remarks.
2. PROBABILISTIC ANALYSIS OF THE EIGENVECTOR SCALING PROBLEM

Consider now the problem of assigning priorities to a set of decision elements $E = \{E_1, \ldots, E_{K+1}\}$ under uncertainty. That is, rather than the assignment of a precise numerical value from an intensity response scale, we have the assignment of a random variable to each pairwise comparison response. Therefore, let the random variable $X_{ij}$ represent the decision maker's relative importance intensity response judgment resulting from the comparison of elements $E_i$ and $E_j$. Furthermore assume that the $X_{ij}$ are positive random variables that satisfy the property of cardinal transitivity, i.e., $X_{ij} = X_{ik}X_{kj}$ for $i, j, k = 1, \ldots, K+1$. Hence the matrix of pairwise comparisons $X$ has a rank of unity and we can write

$$X = \{X_{ij}\}_{i,j=1,\ldots,K+1} = V(V^{-1})$$

where $V = (V_1, \ldots, V_{K+1})'$ is the principal eigenvector of $X$ and $V^{-1} = (V_1^{-1}, \ldots, V_{K+1}^{-1})$. In the spirit of the deterministic AHP paradigm, the random variables $V_i$ represent the unknown absolute importances or weights of the decision elements $E_i$. Hence, we can regard the $V_i$ as independent positive random variables.

Given the above framework, the representation of uncertainty depends upon the proper selection of a family of distributions to reflect the decision maker's preference judgments $X_{ij}$. The basis of such a selection rests upon three general criteria. Firstly, the family of distribution functions should be rich enough to capture a wide variety of preference responses. Secondly, the family of distributions should be closed under the transformations implied by the property of cardinal transitivity. For instance, the distribution functions of the random variables $X_{ij}$ and $X_{ji} = X_{ij}^{-1}$ must belong to the same family of distributions. Thirdly, the selection of a family of distributions should lead to tractable distributional analysis of the eigenvector problem. These
criteria are sufficiently satisfied if we assume that the $v_i$ are distributed as standard gamma variates having the density

$$f(v_i) = \frac{1}{\Gamma(\alpha_i)} v_i^{\alpha_i-1} e^{-v_i}, \quad i = 1, \ldots, K+1 \quad (2)$$

for $v_i > 0$, and zero elsewhere, and where $\alpha_i > 0$. Therefore, the joint density of the principal eigenvector $V = (v_1, \ldots, v_{K+1})$ is readily obtained from density (2) via the assumption of the independence of the $v_i$, yielding

$$f(v_1, \ldots, v_{K+1}) = \frac{1}{\prod_{i=1}^{K+1} \Gamma(\alpha_i)} \prod_{i=1}^{K+1} v_i^{\alpha_i-1} \exp\left(-\sum_{j=1}^{K+1} v_j\right) \quad (3)$$

We will now discuss the derivation of the density functions of the column vectors of the pairwise comparison matrix $X$, as well as the principal right eigenvector of $X$.

2.1 The Distribution of the $j^{th}$ Column Vector

The joint density function of the $j^{th}$ column vector $X_j$ of the matrix $X$ can be obtained from density (3) by applying the transformation suggested by equation (1), i.e.,

$$T: X_{ij} = \frac{v_i}{v_j} \quad i, j = 1, \ldots, K+1, i \neq j$$

The inverse of the transformation $T$ is

$$T^{-1}: v_i = X_{ij} v_j \quad i, j = 1, \ldots, K+1, i \neq j \quad (4)$$

and the Jacobian of the transformation $T$ is

$$J = v_j^K \quad (5)$$

Thus application of the transformation $T$ (i.e., substituting equations (4)
into density (3) and multiplying the result by the absolute value of (5) and then integrating the resultant expression with respect to \( V_j \) yields the density of the \( j \)th column vector \( X_j = (X_{ij}, \ldots, X_{(j-1)j}, X_{(j+1)j}, \ldots, X_{(K+1)j})' \), i.e.,

\[
\frac{1}{B(\alpha_1, \ldots, \alpha_K)} \prod_{i \neq j} \frac{x_{ij}}{(1+x_{ij})^{\alpha_j+1}}
\]

(6)

for \( x_{ij} > 0 \) and zero elsewhere, where \( B(\alpha_1, \ldots, \alpha_K) = \Gamma(\alpha_1) \ldots \Gamma(\alpha_K)/\Gamma(\alpha_1+\ldots+\alpha_K) \). Tiao and Guttman (1965) refer to a distribution having density (5) as K-variate inverted Dirichlet distribution \( \text{ID}(\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_{K+1}; \alpha_j) \).

The marginal densities of the \( X_{ij} \) can be obtained by direct integration of density (5) or from result 2.22 of Tiao and Guttman (1965) yielding

\[
g(x_{ij}) = \frac{1}{B(\alpha_i, \alpha_j)} x_{ij}^{\alpha_i-1} (1+x_{ij})^{-(\alpha_i+\alpha_j)}
\]

(7)

for \( x_{ij} > 0 \), where \( B(\alpha_i, \alpha_j) = \Gamma(\alpha_i)\Gamma(\alpha_j)/\Gamma(\alpha_i+\alpha_j) \). Raiffa and Schlaifer (1961, p. 221) refer to a distribution having density (6) as an inverted-beta-2 distribution \( \text{IB-2}(\alpha_i, \alpha_j) \). The means, variances and covariances of the \( X_{ij} \) (see Tiao and Guttman (1964), p. 795) are

\[
E(X_{ij}) = \frac{\alpha_i}{\alpha_j-1}, \quad \alpha_j > 1 \text{ (all i)}
\]

(8a)

\[
V(X_{ij}) = \frac{\alpha_i(\alpha_i+\alpha_j-1)}{(\alpha_j-1)^2(\alpha_j-2)}, \quad \alpha_j > 2 \text{ (all i)}
\]

(8b)

and

\[
\text{cov}(X_{ij}, X_{kj}) = \frac{\alpha_i\alpha_k}{(\alpha_j-1)^2(\alpha_j-2)}, \quad \alpha_j > 2 \text{ (i\#k)}
\]

(8c)

, respectively. Clearly, the properties of the inverted-beta-2 distribution describe a family of distributions with different shapes, locations, and
dispersions. This suggests that a wide variety of preference responses can be modeled with this family of distributions, thereby satisfying the property of richness alluded to earlier.

2.2 The Distribution of the Priority Vector

Because the rank of the pairwise comparison matrix $X$ is unity, any normalized column of $X$ will yield the priority vector $Y = (Y_1, \ldots, Y_{K+1})'$ associated with the set of decision elements $E = \{E_1, \ldots, E_{K+1}\}$. Thus, the distribution of the vector random variable $Y = (Y_1, \ldots, Y_{K+1})'$ can be derived, without loss of generality, from the $(K+1)^{th}$ column vector of $X$ by applying the transformation

$$T: Y_i = X(i)_{K+1}/Y_{K+1}, \quad i = 1, \ldots, K$$

$$Y_{K+1} = 1 + \sum_{i=1}^{K} X(i)_{K+1}$$

The inverse of the transformation $T$ is

$$T^{-1}: X(i)_{K+1} = Y_i Y_{K+1}, \quad i = 1, \ldots, K$$

$$Y_{K+1} = (1 - \sum_{i=1}^{K} Y_i)^{-1}$$ \hfill (9)

and the Jacobian of the transformation $T$, using a variation of a result given in Aitken (1956, p. 133), is

$$J = (1 - \sum_{i=1}^{K} Y_i)$$ \hfill (10)

Upon applying the transformation $T$ (that is, substituting equations (9) into density (6) and multiplying the resultant expression by the absolute value of the Jacobian (10)) we find that the density function of the priority vector $Y = (Y_1, \ldots, Y_{K+1})'$ is
at any point in the simplex \( S_K = \{ (y_1, \ldots, y_K) : y_i > 0 \text{ for } i = 1, \ldots, K, \text{ and } (y_1 + \ldots + y_K) = 1 \} \) in \( \mathbb{R}^K \), and zero elsewhere. Wilks (1962) refers to a distribution having density (10) as the \( K \)-variate Dirichlet distribution \( D(\alpha_1, \ldots, \alpha_K; a_{K+1}) \).

It follows from property 7.7.1 of Wilks (1962) that the \( Y_i \) are distributed as beta variates with parameters \( \alpha_i \) and \( \rho_i = (\alpha_1 + \ldots + \alpha_{K+1}) - \alpha_i \), having the density function

\[
\pi(y_i) = \frac{1}{B(\alpha_i, \rho_i)} \frac{\alpha_i^{\alpha_i-1}}{y_i^{\alpha_i}} (1 - y_i)^{\rho_i-1}
\]

for \( 0 < y_i < 1 \), and \( \alpha_i > 0 \). The means, variances and covariances (see Wilks (1962), p. 179) of the \( Y_i \) are

\[
E(Y_i) = \frac{\alpha_i}{\alpha_i + \rho_i} \quad \text{(all } i) \tag{13a}
\]

\[
V(Y_i) = \frac{\alpha_i \rho_i}{(\alpha_i + \rho_i)^2 (\alpha_i + \rho_i + 1)} \quad \text{(all } i) \tag{13b}
\]

and

\[
cov(Y_i, Y_j) = \frac{-\alpha_i \alpha_j}{(\alpha_i + \rho_i)^2 (\alpha_i + \rho_i + 1)} \quad \text{if } i \neq j, \tag{13c}
\]

respectively. The properties of the marginal priority distribution functions prove to be extremely useful for they provide a foundation for the systematic analysis of uncertainty in the pairwise comparison process. We will address this issue in a subsequent section of this paper when we investigate the sensitivity of the decision making procedure as a function of the parameters characterizing the distribution of the pairwise comparison responses.
3. THE QUANTIFICATION OF PAIRWISE PREFERENCE JUDGMENTS

In this section we address the important question of how the decision maker can quantify his pairwise preference judgments as a probability distribution. This is indeed a thorny issue because the relationship between subjective preference judgments and a mathematical function is not at all obvious. Furthermore, a subjectively assessed probability distribution need not be a member of any particular family of distributions, i.e., it may not follow any known mathematical function precisely. Despite this, however, it may be possible to find a member of the inverted-Dirichlet family of distributions that is a "good fit" to the subjectively assessed probability distribution. This is where the property of richness of the inverted-Dirichlet family of distributions is so very important. The Dirichlet class of distribution functions is especially well-suited for this task because of its successful use in the assessment of prior probability distributions in Bayesian statistical analysis. In this area of application, the Dirichlet class of distributions has provided both a convenient and realistic model of uncertainty in many situations. Therefore, it can be said, with some confidence, that unless the subjective probability distribution is quite irregular in nature there is a good chance that a member of the inverted-Dirichlet family of distributions will approximate it reasonably well. In this regard, the assessor would normally try a number of members of the family of distributions to see if any of them prove satisfactory as a surrogate distribution. We should note that the notions "good fit" and "satisfactory" will be made more precise in the next section when we consider the sensitivity of the decision making procedure to variations in the surrogate distribution. The remainder of this section will be devoted, however, to the task of assessing the distribution of pairwise comparison responses.
Several techniques have been proposed to aid the assessor in assessing subjective probability distributions (refer to Bunn (1984)). The most prominent of these techniques is the histogram and fractile assessment procedures. According to Bunn (1984), "No single 'best' assessment procedure has yet emerged. Experimental evidence seems well divided over the relative merits of the fractile or histogram methods." Given these circumstances, we will, for expository purposes only, concentrate on the fractile assessment method.

The task we now face is the fractile assessment of the joint distribution function of an arbitrary column vector of the matrix of pairwise comparison responses. Without loss of generality we will consider the assessment of the K-variate inverted-Dirichlet distribution \( iD(\alpha_1, \ldots, \alpha_K; \alpha_{K+1}) \) associated with the \((K+1)^{th}\) column vector of the matrix \(X\). Theoretically, this task can be accomplished by assessing the K marginal inverted-beta-2 distributions \( i\beta_2(\alpha_i, \alpha_{K+1}) \) of the response variates \(X_{(i)K+1}\); if the values of the parameter \(\alpha_{K+1}\) common to all of these assessments are identical, then we are finished with our task. In practice, however, the assessor will find that this procedure will most likely produce K different values of the common parameter \(\alpha_{K+1}\). Nevertheless, the K assessments should be made and a typical \(\alpha_{K+1}\) value can be selected and used as a basis to reassess the \(\alpha_i\) values, \(i = 1, \ldots, K\). Here again, the sensitivity of the decision making procedure to the selection of the \(\alpha_i\) and \(\alpha_{K+1}\) values should be investigated.

The assessment of the \(k^{th}\) fractile of the inverted-beta-2 distribution \(i\beta_2(\alpha_i, \alpha_{K+1})\) involves finding that value of the pairwise comparison response variate \(X_{(i)K+1}\), denoted as \(i\beta_2(\alpha_i, \alpha_{K+1})^k\), such that

\[
\text{pr}[X_{(i)K+1} \leq i\beta_2(\alpha_i, \alpha_{K+1})^k] = k, \quad 0 < k < 1
\]
For the purpose of consistency and coherence it is convenient to assume that the $X_{(i)K+1}$ are restricted to a range of values defined by a suitable relative importance intensity response scale, such as the Saaty scale (refer to Saaty (1977)). As an example of the approach consider the assessment of the 0.5th fractile of the distribution $i-\beta e-2(\alpha_i, \alpha_{K+1})$. In response to a comparison of the relative importance of decision elements $E_i$ and $E_{K+1}$, $i \neq K+1$, the assessor would determine that point on the relative importance intensity scale that the decision maker feels is equally likely to be exceeded or not exceeded, i.e., the median of the pairwise response distribution. Other fractiles of this distribution can be assessed in a similar fashion.

In general, the parameters of the approximating preference distribution can be ascertained from a knowledge of only two fractiles of the subjectively assessed distribution. The question now remaining to be answered is how to translate the fractiles of the inverted-beta-2 distribution into parameter values $\alpha_i$ and $\alpha_{K+1}$.

The parameters $\alpha_i$ and $\alpha_{K+1}$ of the inverted-beta-2 distribution can be derived from the fractiles of the assessed distribution by using standard tables of the incomplete beta function (refer to Pearson (1968)). This is accomplished by applying the following transformation (see LaValle (1970), p. 257):

$$\beta(\rho, v)^k = \frac{i\beta_2(\rho, v)^k}{1+i\beta_2(\rho, v)^k}$$  \hspace{1cm} (13)

where $\beta(\rho, v)^k$ and $i\beta_2(\rho, v)^k$ are the $k^{th}$ fractiles of the beta distribution $\beta e(\rho, v)$ and the inverted-beta-2 distribution $i-\beta e-2(\rho, v)$, respectively. Relationship (13) suggests that once the intensity response value $i\beta_2(\rho, v)^k$ is assessed, the assessor can then locate an equivalent value $\beta(\rho, v)^k$. 
corresponding to the $k^{th}$ fractile of the beta distribution $\beta e(\rho, v)$. With this information the assessor can then utilize the standard tables of the incomplete beta function to isolate candidate values for the parameters $\alpha_1$ and $\alpha_{K+1}$. Numerical values for these parameters can be selected from the set of candidate values after a second fractile is assessed. Aside from the use of relationship (13) in the assessment procedure, the fractile assessment of the inverted-beta-2 distribution directly follows that of the beta distribution. Accounts of the assessment of the beta distribution as well as the K-variate Dirichlet distribution can be found in the literature of applied decision analysis; particularly lucid accounts are given in Bunn (1984), LaValle (1972), and Winkler (1972).
4. FURTHER CONSIDERATIONS OF UNCERTAINTY: SENSITIVITY ANALYSIS

In this section we consider the problem of analyzing the sensitivity of the decision procedure to variations in the induced approximating priority distribution. Before presenting an analytical framework for dealing with this problem, it is appropriate to define what we mean by the term sensitivity. Recall that the assessment of a pairwise preference distribution may produce different distributions as candidates for the subjectively assessed distribution. The decision maker in this situation may well be interested in how the ultimate decision of the decision making procedure is influenced by variations in the selection of a candidate priority distribution. If very slight variations in the distribution are likely to cause the decision to be changed, the decision making procedure is said to be sensitive to the pairwise preference distribution. Obversely, insensitivity implies that the choice of an induced priority distribution is not crucial and the ultimate decision may be the same for a wide variety of pairwise preference distributions. If there is any doubt in a particular situation, it is prudent for the decision maker to investigate the sensitivity of the decision making procedure.

To properly conduct an analysis of the sensitivity of the decision making procedure we first need to enumerate the ultimate decisions that the procedure is likely to produce. In the present decision making framework, i.e., the allocation of priorities among competing alternatives, the decision maker is faced with deciding: (1) What is the most preferred alternative?, (2) What is the least preferred alternative? or (3) What is the preferred ranking of the alternatives? Analytically, what is called for here is a general procedure for analyzing the sensitivity of the rankings of the alternatives based upon the properties of the priority distributions. At first glance, the properties (13) of the marginal priority distributions appear to provide
the necessary tools for such an enterprise; however, they fall short of the mark because they provide only partial information. That is, they provide separate pieces of information that must be compared and somehow combined to bring insight into the problem. What is required, then, is a single piece of integrated information. To find this measure of information we must explore further the properties of the joint priority distribution.

Let us assume that the priority vector \( Y = (Y_1, \ldots, Y_{K+1})' \) associated with the set of decision elements \( E = \{E_1, \ldots, E_{K+1}\} \) is distributed as the \( K \)-variate Dirichlet distribution \( D(a_1, \ldots, a_K; a_{K+1}) \) having density (11). Correspondingly, assume that the \((K+1)\)th column vector of the pairwise comparison matrix \( X \), denoted as \( Z_1 = (Z_1, \ldots, Z_K, 1)' \), is distributed as the \( K \)-variate inverted Dirichlet distribution \( iD(a_1, \ldots, a_K; a_{K+1}) \) having density

\[
8(z_1, \ldots, z_k) = \frac{1}{B(a_1, \ldots, a_{K+1})} \prod_{j=1}^{K} \frac{a_j}{z_j} (1 + \sum_{j=1}^{K} z_j)^{-a_{K+1}}
\]

If we denote \( Y_{[1]} \leq \ldots \leq Y_{[K+1]} \) as the ordered set of priority variates associated with vector \( Y = (Y_1, \ldots, Y_{K+1})' \), then we can define the most (least) preferred decision element as the element having the largest (smallest) priority variate. Since the rank of the matrix \( X \) is unity, we can equivalently define the most (least) preferred decision element as the element having the largest (smallest) pairwise preference variate for a given column vector of \( X \). Therefore, we can write the probability that \( Y_{[K+1]} = Y_k \), using density (14) of the \((K+1)\)th column vector, as follows:

\[
\Pr[(1 - \sum_{j \neq k} Y_j \leq Y_k) \cap (Y_j \leq Y_k)] = \frac{1}{B(a_1, \ldots, a_{K+1})} \int \cdots \int z_k^{a_k-1} \tau_1^{a_1} \cdots \tau_K^{a_K} \prod_{j=1}^{K-1} \tau_j^{a_j} \int \cdots \int d\tau_j d\tau_k (15)
\]
Similarly, we can write the probability that $Y_{[1]} = Y_k$ as follows:

$$
\Pr[(Y_k \leq 1 - \sum_{j \neq k}^{K} Y_j) \cap (Y_k \leq Y_j)]
$$

$$
= \frac{1}{B(a_1, \ldots, a_{K+1})} \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{a_k - 1}{z_k \tau_k} \left(1 + \sum_{j=1}^{K} \frac{a_j K a_j - 1}{\tau_j} \right) \prod_{j \neq k} \tau_j d\tau_k
$$

(16)

In like fashion, we can write the probability associated with an arbitrary ordering of the priority variates, denoted as $Y_{(1)} \leq \ldots \leq Y_{(K+1)}$, as follows:

$$
\Pr[\bigcap_{j=1}^{K} (Y_{(j)} \leq Y_{(j+1)})]
$$

$$
= \frac{1}{B(a_1, \ldots, a_{K+1})} \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{a(k)}{z(k-1) z(2)} \left(1 + \sum_{j=1}^{K} \frac{a(j) K a(j) - 1}{\tau(j)} \right) \prod_{j=1}^{K+1} \tau(j) d\tau(j)
$$

(17)

where the subscripts $((1), \ldots, (K+1))$ correspond to exactly one of the $(K+1)!$ arrangements of the integers $(1, \ldots, K+1)$.

Using the results established by Dennis (1986(a)) we can write the following explicit expressions for probabilities (15) and (16) as

$$
\Pr[Y_k = \max \{Y_1, \ldots, Y_{K+1}\}] = 1 - \sum_{i_1} P_{i_1} + \sum_{i_1 < i_2} P_{i_1 i_2}
$$

$$
+ \ldots + (-1)^m \sum_{i_1 \ldots i_m} \ldots + (-1)^k P_{i_1 \ldots (k-1)(k+1) \ldots K+1}
$$

(18)

and

$$
\Pr[Y_k = \min \{Y_1, \ldots, Y_{K+1}\}] = P_{1 \ldots (k-1)(k+1) \ldots K+1}'
$$

(19)

respectively, where $\sum_{i_1 < \ldots < i_m}$ denotes the summation over all integers $i_1, \ldots, i_m$ where (1) $1 \leq i_j \leq K+1$, $j = 1, \ldots, m$ where $i_j \neq k$, and (2) $i_1 < \ldots < i_m$, and where
\begin{equation}
\begin{split}
    p_{i1...im} = \sum_{n_{i1}=0}^{a_{i1}-1} \ldots \sum_{n_{im}=0}^{a_{im}-1} \frac{\Gamma(\alpha_{k} + \sum_{j=1}^{m} n_{ij})}{\Gamma(\alpha_{k}) \prod_{j=1}^{m} n_{ij}} \frac{1}{n_{i1}! n_{i2}! \ldots n_{im}!} \end{split}
\end{equation}

(20)

for integer valued \( \alpha_i > 0 \).

An explicit expression for probability (17) cannot be concisely written; however, a set of recursion formulas for the evaluation of (17) is found in Dennis (1986(a)). But, for the sake of brevity, we will only indicate here that the evaluation of probability (17) follows from successive application of the following integration formula established by Tiao and Guttman (1965):

\begin{equation}
\begin{split}
    \int_{0}^{a} (1+x+t)^{-(\alpha+n)} t^{n-1} dt = (1+x)^{-\alpha} B(\alpha, n) - \sum_{j=1}^{\infty} \frac{(n-1)!}{j!} (1+x)^{-(\alpha+j)} B(\alpha+j, n-j)
\end{split}
\end{equation}

where \( n \) is a positive integer and \( <x, \alpha, a> \) are positive quantities.

From the above, it is clear that we have developed an analytical framework for reaching an ultimate decision, and within this framework we also have an apparatus for investigating the sensitivity of the decision making procedure to variations in the pairwise preference distribution. To illustrate the use of this analytical approach, consider an example of a decision maker faced with establishing priorities under uncertainty for a set of alternatives \( A = \{A_1, A_2, A_3\} \). Let us assume that the decision maker has determined that his priority vector \( Y = (Y_1, Y_2, Y_3) \) associated with the set \( A = \{A_1, A_2, A_3\} \) is distributed as the Dirichlet distribution \( D(\alpha_1, \alpha_2; \alpha_3) \). We now seek to develop explicit expressions for deciding the following questions: (1) Which one of the alternatives has the highest priority ranking?, (2) Which one of the alternatives has the lowest priority ranking? and (3) What is the preferred ranking of the set of alternatives?
We will now answer questions (1) and (2) jointly. To find the most (least) preferred alternative we must search for the alternative that maximizes probability (15) (probability (16)). To demonstrate the technique, let us calculate these probabilities with respect to alternative $A_1$. Using expressions (18) and (19) we can write

$$\Pr[Y_2 < Y_1, 1-(Y_1+Y_2) < Y_1] = 1 - (P_2 + P_3) + P_{23}$$

$$= 1 - \sum_{n_2=0}^{a_2-1} \frac{\Gamma(a_1 + n_2)}{\Gamma(a_1) n_2!} \frac{1}{2} a_1 + n_2 - \sum_{n_3=0}^{a_3-1} \frac{\Gamma(a_1 + n_3)}{\Gamma(a_1) n_3!} \frac{1}{2} a_1 + n_3$$

$$+ \sum_{n_2=0}^{a_2-1} \sum_{n_3=0}^{a_3-1} \frac{\Gamma(a_1 + n_2 + n_3)}{\Gamma(a_1) n_2! n_3!} \frac{1}{3} a_1 + n_2 + n_3$$

$$= 1 - \frac{a_2-1}{\Gamma(a_1)} \frac{a_3-1}{\Gamma(a_1)} \frac{1}{2} a_1 + \frac{1}{3} a_1 + n_2 + n_3$$

(22)

and

$$\Pr[Y_1 < Y_2, Y_1 < 1-(Y_1+Y_2)] = P_{23} = \sum_{n_2=0}^{a_2-1} \sum_{n_3=0}^{a_3-1} \frac{\Gamma(a_1 + n_2 + n_3)}{\Gamma(a_1) n_2! n_3!} \frac{1}{3} a_1 + n_2 + n_3$$

(23)

Likewise, we can calculate these same probabilities for alternatives $A_2$ and $A_3$. Having done this we can easily compare the magnitude of these probabilities and discover which one of the alternatives is the most (least) preferred.

To answer question (3) we must calculate probability (17) for each of the $3!$ arrangements of the set of alternatives $A = \{A_1, A_2, A_3\}$; the arrangement yielding the largest probability is therefore deemed to be the most preferred. To illustrate the process, consider the evaluation of probability (16) for the case of $K=2$, i.e.,

$$\Pr[Y_{(1)} < Y_{(2)}, Y_{(2)} < 1-(Y_{(1)}+Y_{(2)})]$$

$$= \frac{1}{B(a_{(1)}, a_{(2)}, a_{(3)})} \int \int \int \frac{a_{(1)}-1 a_{(2)}-1}{(1 + \tau_{(1)} + \tau_{(2)})^{a_{(1)} + a_{(2)} + a_{(3)}}} \tau_{(1)}^{z_{(2)}} \tau_{(2)} d\tau_{(1)} d\tau_{(2)}$$

(24)
It can be shown (see Dennis (1986(a))) that the application of integration formula (21) in probability (24) yields (dropping the parenthesis in the subscripts)

\[
pr[Y_1 \leq Y_2, Y_2 \leq 1-(Y_1+Y_2)]
= \sum_{n_3=0}^{a_3-1} \frac{\Gamma(a_2+n_3)}{\Gamma(a_2)n_3!} \left( \frac{1}{2} \right)^{a_2+n_3} - \sum_{n_1=0}^{a_1-1} \sum_{n_3=0}^{a_3-1} \frac{\Gamma(a_2+n_1+n_3)}{\Gamma(a_2)n_1!n_3!} \left( \frac{1}{3} \right)^{a_2+n_1+n_3}
\]

Upon reordering the subscripts in expression (25) we can develop formulas for calculating the probabilities associated with the remaining possible arrangements of the set of alternatives. The results of these calculations will then serve to identify the preferred ordering of the alternatives.

To determine the sensitivity of the decision making procedure, the decision maker can vary the parameters \(a_1\) characterizing the pairwise preference distribution to see whether or not the ultimate decision changes. Obviously, this analysis entails the recalculation of probabilities (23), (24) and (25) for each new priority distribution. If the ultimate decision remains fixed with respect to the choice of an approximating priority distribution, we have established a robust decision model coherent with the subjectively assessed pairwise preference distribution. If the ultimate decision is not invariant to the choice of a priority distribution, then it may be that we cannot approximate the subjectively assessed preference distribution sufficiently well to merit further analysis using this decision analytic approach.
5. CONCLUSION

In this paper we have presented a probabilistic framework for the study of the effects of uncertainty in pairwise comparison process underlying the AHP paradigm. This framework provides a formal procedure to aid the individual decision maker in the task of assigning priorities to a set of alternatives under uncertainty. The procedure is both practical and flexible and can be utilized within the general AHP framework of assigning priorities in hierarchically structured decision problems of some complexity (refer to Dennis (1986(b)). It is hoped that applied decision analysts will find this study useful in practical applications of the AHP decision methodology in those situations where the decision problems are complicated by the elements of uncertainty and ambiguity.
REFERENCES


Dennis, S. Y., "Some probabilities associated with the ordering of unknown multinomial cell probabilities" (unpublished working paper, submitted for publication, 1986(a)).

Dennis, S. Y., "A probabilistic approach to priority assignment in hierarchically structured decision problems" (unpublished working paper, submitted for publication, 1986(b)).


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