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Abstract. This paper addresses the inventory problem facing an individual warehouser who is part of a large scale distribution system that works on a consignment basis. The particular case investigated here is one that is already being used in practice. In this system, the manufacturer bears the holding and ordering costs of the consigned goods. However, to ensure that the warehouser carries sufficient stock to meet regional service needs, the manufacturer pays a sales commission that is split into two parts: the first part is a sales fee (approximately 2/3 of the total commission), and the second part is a warehousing fee (the remaining 1/3 of the commission). If the warehouser sells an item that is not in stock, then only the sales fee is received, and the warehousing fee is paid to another warehouser in the system who ships the item to the customer. Therefore, unsatisfied demand is not backlogged, and the warehousing fee becomes the cost of a "lost sale." To complicate matters, deliveries of replenishment stock involve item-specific time lags. We incorporate both of these features into a multi-item periodic review model with an order-up-to-S_i replenishment policy for each item i. Within this framework, it is shown that the warehouser's average expected loss due to stockouts is a separable convex function of S_i. Consequently, optimal replenishment levels can be readily determined using the classical methods of separable convex programming. Our consignment model is quite general in that only the cost of a lost sale is required. We illustrate our approach using real data supplied by a warehouser who is currently participating in this type of consignment system.

Keywords: Inventory; Replenishment Models; Consignment; Separable Convex Programming; Linear programming

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Introduction

Inventory held on a consignment basis has received considerable attention of late as a device for backing up the seller's costs onto the manufacturer. While this may often be the case, other manufacturers are considering consignment arrangements to reduce their involvement in the distribution process and to encourage their retailers to take a more active role in product marketing. This type of system is particularly attractive to manufacturers who wish to expand costly product lines (Culgin, 1996) or maintain ownership of their inventory for legal reasons (Fagel, 1996). Although still regarded as an experimental model in most circles, a number of large organizations have recently developed and implemented their own consignment systems. One such system was brought to the authors' attention by a warehouser who had just finalized a consignment agreement with a billion-dollar manufacturer of large-scale industrial electric equipment. The multi-item inventory problem facing this warehouser is the topic of the current paper.

Perhaps the most notable feature of this consignment system is its simplicity. On the front end, the manufacturer agrees to bear the incremental costs associated with holding, shipping and ordering of the consigned goods. On the back end, the customer pays for the shipping and handling costs following purchase from the warehouser. The warehouser's responsibility is thus limited to warehousing the goods and distributing them to the buyer. A sales commission is paid by the manufacturer for every item sold, but the total amount of the commission depends on whether or not the item is currently held in stock by the warehouser. The commission is thus split into two parts: one part is a *sales fee* which is given regardless of whether the warehouser has the item in stock; the other part is a *warehousing fee* which is paid if the warehouser has the unit in stock. This fee structure deters the warehouser from minimizing his local service obligations and focusing solely on his role as a regional agent (or broker) for the manufacturer. If a unit is not in stock, then another warehouse in the distribution system supplies the item and receives the warehousing fee.

This arrangement has a number of characteristics that make it interesting for practitioners and modelers alike. First, the warehouser's inventory problem does not suffer from the normal difficulties associated with estimating costs. Under this consignment system, the warehouser's primary cost is the shortage cost, which is given by a clearly defined penalty applied to the warehouser's commission. Consequently, we will assume throughout this paper that the warehouser's objective is to minimize the cost of shortages (lost warehousing fees). From a modeler's perspective, this problem can be shown to possess surprisingly good structure for analysis and subsequent computations. Indeed, one of the significant contributions of this paper is its novel method of generating a separable convex program.

Apart from the absence of an incremental holding cost, our periodic review consignment model incorporates five features which, *collectively*, distinguish it from other periodic review models:

I. Multiple products constrained by resources
II. No backlogging of unsatisfied demand (i.e., the lost sales case)
III. Lags in delivery
IV. No specific distributional assumptions about demand (including independence)
V. An efficient, reliable, and easily implemented solution procedure

Properties I-IV are driven by our case; property V is needed to ensure that our model contributes to inventory management in practice. In reviewing the literature, we were unable to find any periodic review model which included all five of the features listed above, let alone one which addressed the specialized case of consignment. Distantly related models that include multiple products are proposed by Veinott (1965) and Evans (1967). Veinott considers lost sales and delivery lags separately but explicitly avoids their occurrence together because of complexity issues. Evans assumes lost sales but does not consider delivery lags. Both works avoid special distributional assumptions, but neither offers a computational procedure.

Periodic review models that address both lost sales and delivery lags simultaneously have been presented in Arrow, Karlin and Scarf (1958), Gaver (1959), Bartmann and Beckmann (1992), and Vendemia, Patuwo and Hung (1995). The models discussed in these works assume a single product and independence of demand over time. The last two works offer sketches of some computational procedures, but the lack of closed form expressions for the various cost functions makes them too complex for adaptation to the multi-product case with resource constraints.

As noted by Ehrhardt (1985), methods for calculating optimal (s, S) policies generally require that the demand distribution be completely specified. Moreover, a prohibitive amount of work is typically needed for even the single item case. Freeland and Porteus (1980) and Porteus (1985) have developed some very efficient heuristics that estimate optimal (s, S) parameters in periodic review models with delivery lags. However, these methods were developed for the case of a single item with full backlogging and independent demands. Like the numerical procedures cited earlier, these approaches do not seem to generalize (in an easy way) to our multi-product consignment problem with resource constraints.

The remainder of our paper is divided into five sections. Section 2 introduces a simple model to address the consignment problem when lead times are negligible. The analysis presented is straightforward, but it serves to highlight the important details of the consignment problem before launching into the more realistic situation (and the one encountered in our own application) where deliveries are lagged. Multi-period versions of the model that incorporate these lag times are presented in sections 3 and 4. The model is illustrated in section 5 using portions of real data supplied by our warehouser. The final section summarizes our results and points to a potentially extensive area of new inventory research.

§2 Replenishment Models for Inventory under Consignment: Negligible Lead Times

The warehouser's consignment problem when lead times are negligible requires a minimal amount of notation which is presented in table 1 and described next. The total storage capacity (C) available to the warehouser may be treated as a parameter if the warehouser does not view expansion as an option, or as a decision variable with an appropriate cost otherwise. In either case, space imposes a constraint on the warehouser. Much of the storage capacity is in the form of heavy industrial shelving, but this type of storage is not suitable for all items. For example, large 300 hp electric motors weighing approximately 1.5 tons can only be stored on the floor, and other moderately sized motors can only be shelved in limited quantities per shelf. This means that categories of space (k=1, 2,...., K) are needed to account for items with specific weight and storage requirements. The total amount of category k space available is denoted by $C^{(k)}$. Motors come in one of ten standard sizes, thus the amount of category k space used by item i ($C_i^{(k)}$) is easily measured.

In most consignment models, the warehouser would be restricted by how much merchandise (B, in dollars) he could hold. In our case, this bound is inessential because the manufacturer wanted to invest a substantial amount in regional inventory to establish their market presence. In most other situations, however, it is likely the manufacturer would restrict the value of the consigned merchandise. The precise amount might become part of future negotiations, and in this case our model could measure how this bound impacts losses due to stockouts.

The manufacturer schedules shipments through an outside freight company and pays for the cost of shipping. Because they have the opportunity to control shipping costs and regional inventory levels, the warehouser cannot expect to receive a continuous supply of "small" shipments to replenish his stock. A periodic review model is appropriate for this situation. Moreover, while the warehouser could expect a new shipment on a regular basis (in our case, weekly), each item involves a delivery lead-time. Most high volume products are replenished in one week, but many of the larger and less frequently purchased items take 3 or 4 weeks. Some of the biggest items take even longer. Nevertheless, for completeness we begin with the case where delivery lead times are negligible. This assumption means that the warehouser can make adjustments in his order up until the shipment is sent, a condition that commonly occurs when orders are handled using electronic communications (see Nahmias and Smith 1994). This assumption simplifies the analysis considerably over the case where time lags are present (sections 3 and 4).

Since the warehouser's only significant incremental cost is the loss incurred when he sells an item that must be supplied from another warehouse's stock, his primary concern is to avoid losing the warehousing fee (WF_i) that could be collected from the anticipated (historical) demand for item *i*. Consequently, the warehouser seeks stock levels for each item that would — given his space and budget restrictions — minimize his expected loss associated with stockouts. Moreover, it is assumed throughout this paper that these inventory levels are managed using simple order-up-to-S_i control policies, a standard approach in cases where ordering costs are insignificant (see Love, 1979).

TABLE 1. Notation for Section 2

VARIABLE	MEANING
C	Total warehouse capacity available (may be a decision variable or a fixed parameter).
$C^{(k)}$	Amount of category *k* space available.
$C_i^{(k)}$	Amount of category *k* space required by item *i* (*i*=1,...,*n*)
B	Upper bound on dollar amount of consigned goods
$f(x)$	Probability density function describing demand over the replenishment period (continuous case)
S_i	Replenishment level for item *i* (*i*=1,...,*n*)
$L_i(S_i)$	Expected single period loss for item *i* using replenishment level S_i
n	Number of items the warehouser is contracted to sell
P_{ij}	Probability that demand is for *j* units of item *i* over the replenishment period (*i*=1,...,*n*, *j*=1,2,.......)
SF_i	Sales fee for selling item *i* (*i*=1,...,*n*)
V_i	The value of item *i* (*i*=1,...,*n*)
WF_i	Warehousing fee for item *i* (*i*=1,...,*n*)

Given a known period of time between shipments — henceforth termed the *replenishment period* — it is possible to estimate the expected loss due to stockouts. To make our presentation sufficiently clear, consider the case of a single item with warehousing fee $WF > 0$. Let $P_0, P_1, P_2, \ldots, P_j, \ldots$ denote the probabilities of receiving demand for 0, 1, 2,...., j,... units over the replenishment period, and let S denote the replenishment level. In this case, the *discrete expected loss function* for a single period takes on the form

$$L(S) = WF\sum_{j=S}^{\infty}(j-S)P_j \ . \tag{L}$$

This loss function has a number of desirable properties, the most important of which is that the incremental expected loss is a monotone nonincreasing function of the stock level.

Proposition 1. Let S, S+1, S+2 be three consecutive nonnegative integers. Then the discrete expected loss function is monotone nonincreasing and satisfies

$$L(S) - L(S+1) \geq L(S+1) - L(S+2).$$

This proposition follows immediately from a direct calculation:

$$L(S) - L(S+1) = WF\sum_{j=S}^{\infty}(j-S)P_j - WF\sum_{j=S+1}^{\infty}(j-(S+1))P_j$$

$$= WF\sum_{j=S+1}^{\infty}P_j \ .$$

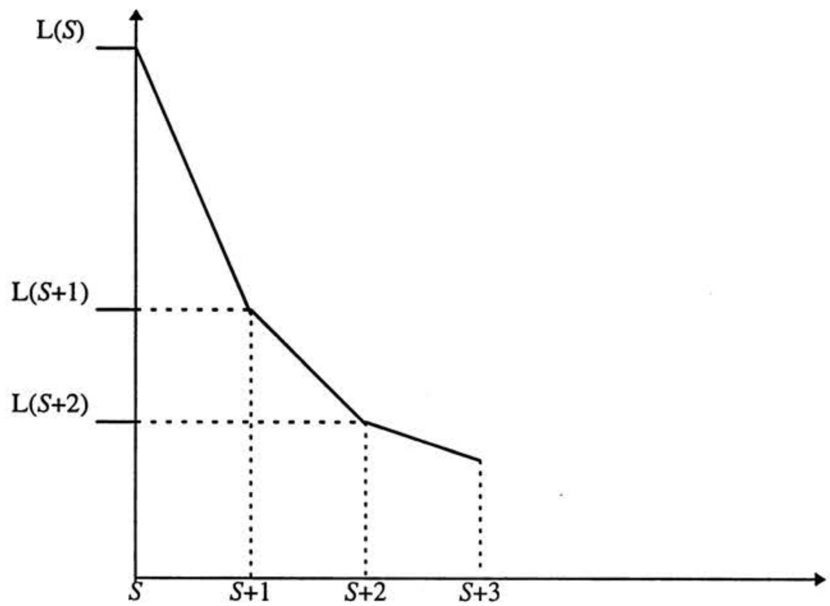
Similarly,

$$L(S+1) - L(S+2) = WF\sum_{j=S+2}^{\infty}P_j \leq WF\sum_{j=S+1}^{\infty}P_j = L(S) - L(S+1),$$

which verifies the proposition.

Proposition 1 reveals that the discrete expected loss function is a convex piecewise linear function of S as shown in figure 1. Convexity is an essential feature for a separable programming problem to be considered numerically tractable. Even though the loss function is only meaningful at integer values, we shall say it is convex if its piecewise linear interpolating form is convex.

Figure 1. The Expected Loss



The expected loss function pictured above does not have a unique minimum. This is in contrast to the classical order-up-to S cost function

$$G(S) = \int_0^S h(S-x) f(x) dx + \int_S^\infty p(x-S) f(x) dx$$

analyzed by Arrow, Karlin and Scarf (Ch. 9, 1958) for the static one period model with no ordering cost and zero lead time. Here, the first integral is the expected holding cost associated with replenishment level S, and the second integral is the expected shortage cost (both using demand density $f(x)$). The authors consider various conditions on the holding costs $h(S-x)$ and shortage costs (also termed penalty costs) $p(x-S)$ that ensure $G(S)$ is convex and satisfies $G(S) \to \infty \; as \; S \to \infty$. The latter conditions guarantee the existence of a unique minimum cost solution. The case $h(S-x) = 0$ is ruled out, possibly because $G(S)$ does not attain its infimum in many cases. Therefore, in a consignment system, the incentive to stock unlimited amounts of inventory must be counterbalanced by other pressures such as the warehouser's storage capacity or the manufacturer's willingness to subsidize holding costs.

In the case of n items with independent demands, the expected single period loss for stocking levels $S_1, S_2, ..., S_n$ is the sum of individual losses, i.e.,

$$L(S_1, S_2, ..., S_n) = \sum_{i=1}^{n} L_i(S_i).$$

It is the warehouser's objective to minimize his expected loss, but we note that in a consignment system with this type of commission structure, minimizing the expected loss is equivalent to maximizing the expected profit. This is generally regarded as a desirable property in an inventory model and can be shown to hold in certain special instances (most notably the newsboy problem; see Peterson and Silver (1979) for a proof). In our case of consignment, it is not difficult to show that the expected profit, $P(S_1, S_2, ..., S_n)$, is given by

$$P(S_1, S_2, ..., S_n) = \sum_{i=1}^{n} (SF_i + WF_i)\mu_i - L(S_1, S_2, ..., S_n) - F$$

where μ_i is the expected demand for item i and F is the fixed charge associated with maintaining the warehouse over the replenishment period. Keeping this equivalence in mind, the optimal replenishment levels for a single period can be obtained from the inventory under consignment formulation

$$\begin{aligned} \underset{S_1, S_2, ..., S_n}{Min} \quad & L(S_1, S_2, ..., S_n) && \text{(IC)} \\ & \sum_{i=1}^{n} C_i S_i \leq C \\ & \sum_{i=1}^{n} V_i S_i \leq B \\ & S_i \geq 0 \end{aligned}$$

where C_i is the amount of storage capacity needed for item i, and V_i is its value. We have assumed a single category of storage space to eliminate the superscripts for expositional clarity. The left hand sides of the constraints reflect peak on-hand inventory conditions. These levels will be experienced when lead times are negligible since the on-hand inventory for item i will equal its replenishment level S_i immediately after a new shipment is received and shelved.

As noted earlier, in many applications (including ours) the total storage capacity is divided into different categories. In this case the single storage constraint used above would be replaced by a set of constraints

$$\sum_{i=1}^{n} C_i^{(k)} y_i^{(k)} \leq C^{(k)} \text{ for } k = 1, \ldots, K$$

where k=1,....,K indexes the different storage categories, $C^{(k)}$ is the amount of type k storage available, and $C_i^{(k)}$ represents the amount of type k space required by item i *if* it can be shelved there. The variable S_i has been disaggregated into *allocation variables* $y_i^{(k)} \geq 0$ to denote the total amount of category k space used by item i. This requires adding another linear constraint of the form $S_i - \sum_k y_i^{(k)} = 0$, where it is understood that $y_i^{(k)}$ is only included in the sum if item i can be stored in type k storage. Note that items with less restrictive storage requirements are allowed to compete for space with items having more restrictive storage requirements. For example, light 10hp motors can certainly edge out heavier 300hp motors for floor space if this turns out to be a cost effective strategy.

Since (IC) and its variants as outlined above are all instances of separable convex programming (Charnes and Lemke, 1954), each item's loss function in the objective function of a minimization program can be replaced by its piecewise linear approximation

$$L_i(S_i) = L_i(0) + \sum_{j=1}^{M_i} (L_i(j) - L_i(j-1))S_{ij} \ ,$$

$$\text{where } S_i = \sum_{j=1}^{M} S_{ij} \ , \ 0 \le S_{ij} \le 1 \, .$$

The constant M_i is used here to designate a "suitably large number" since the sums are open ended. In other words, one cannot be sure how many terms are involved (a priori) even though the number is clearly finite. In practice, a value for M_i will be determined through our estimation procedure because the loss function will eventually reach zero. The resulting linear programming formulation of (IC) is

$$\text{Min} \quad \sum_{i=1}^{n} \left[L_i(0) + \sum_{j=1}^{M_i} (L_i(j) - L_i(j-1))S_{ij} \right] \qquad \text{(IC-LP)}$$

s.t.

$$\sum_{i=1}^{n} \sum_{j=1}^{M_i} A_i S_{ij} \le A$$

$$\sum_{i=1}^{n} \sum_{j=1}^{M_i} V_i S_{ij} \le B$$

$$0 \le S_{ij} \le 1$$

Again for expositional clarity, the model IC-LP does not incorporate different storage types. It also does not include the most important feature of our study: time lags in delivery. We begin with the simplest case, that of a one period lag.

§3 Replenishment Models for Inventory under Consignment: One Period Lag

Although shipments are received on a weekly basis in our application, lead times are item-specific and vary from one to four weeks. An item's lead-time is primarily influenced by its sales volume. In this section we will focus specifically on the case of items with a one period time lag (i.e., the higher volume items). As in section 2, we will continue to insist that unsatisfied demand is lost.

In the case of lagged delivery, we will show that the average expected loss due to stockouts is once again a separable convex function of the replenishment level S. Unlike the case for immediate deliveries, the term average is essential here since the function describing losses for a *single* period of the horizon is not typically convex. A multi-period model is needed to capture the effect of lagged deliveries, but the situation can be simplified somewhat by assuming a common order-delivery mechanism for multi-period order-up-to-S replenishment models without backlogging (see Gaver (1959), Arrow, Karlin and Scarf (1958)). Under this assumption, an order is placed at the beginning of each period which is sufficient to bring the existing on-hand inventory up to the replenishment level S. This order is received at the end of the time period and is combined with the period ending inventory (if any) to constitute the beginning on-hand inventory for the next time period. Such a mechanism is easily implemented in practice: simply place a new order immediately after the previous order is received and shelved.

It has been shown by various authors (e.g., Gaver 1959, Arrow Karlin and Scarf 1958) that when demand is not backlogged because customers are impatient[1] the single period expected loss due to shortages can be represented in the form

$$\int_S^\infty (y-S) f(S,y)\,dy$$

where $f(S,y)$ is the density for the period ending "deficit below base-stock level" distribution. Here, $y-S$ represents unsatisfied demand (which is considered lost). Unlike the case where lead times are negligible or the case where unfilled demands are backlogged, the density $f(S,y)$ depends on the replenishment level S. The computation of this density is a difficult exercise; for

[1]Customer impatience normally means unsatisfied demand is lost. In our consignment system, unsatisfied demand is never lost in the sense that shortages are covered by other warehouses in the system. However, customer impatience refers to demand which is not backordered and satisfied by the warehouser's future inventory. This is representative of the current consignment system.

even assuming that the period demands are *i.i.d.* random variables with a known distribution, $f(S, y)$ involves calculating the steady state distribution of period-beginning inventory levels followed by an integral convolution. Various authors have noted the difficulty of determining an optimal replenishment level *S* in the presence of even a single period time lag. A few special cases have been successfully analyzed (e.g., when period demands are *i.i.d.* exponential, see Gaver 1959, Arrow, Karlin and Scarf 1958).

We will propose a much simpler and more practical approach for our consignment model, which, for the sake of clarity, is described in terms of a single item with finite demand governed by a discrete probability distribution $P_0, P_1, \ldots, P_N$. The following notation will be helpful.

$A_t^S =$ Actual (on-hand) inventory at the beginning of period *t* assuming a replenishment level *S*.

$D_t =$ Demand *during* period *t*

$S =$ Replenishment level.

$T =$ Number of time periods (the planning horizon)

In terms of this notation, the on hand inventory at the start of period *t* can be expressed as

$$A_t^S = (S - A_{t-1}^S) + Max\left\{A_{t-1}^S - D_{t-1},\ 0\right\}$$

where $(S - A_{t-1}^S)$ is the order that arrives at the beginning of period *t* and $Max\left\{A_{t-1}^S - D_{t-1},\ 0\right\}$ is the carryover. Let $d = (d_1, d_2, \ldots, d_T)$ represent a single realization of the random demand vector $D = (D_1, D_2, \ldots, D_T)$ over periods $t = 1,2,\ldots,T$. Let us assume that $A_1^S = S$, i.e., the system begins with stock amounts at their replenishment levels. This assumption is not critical to our subsequent analysis but provides a convenient basis for comparing different replenishment policies. The average conditional loss for replenishment level S given demand $D=d$ is defined as

$$L_T(S|D=d) = WF \cdot \frac{\textit{Total Shortage over Periods } 1,\ 2,\ldots,\ T}{T}.$$

Here, the term *total shortage* refers to the total number of items that were sold by the warehouse but were not supplied by the warehouse using replenishment level S and demand vector d. Note that the subscript on the loss function now stands for the length of the planning horizon instead of the item index. We will show that $L_T(S|D=d)$ is a convex function of S, the proof of which is broken into three parts: Lemma 3.1, Theorem 3.2 and Theorem 3.3. These three results provide insight into the relationship between the loss function and the replenishment level and additionally suggest a direct computational procedure. The first lemma is rather intuitive but is included for completeness.

Lemma 3.1. Given $d=(d_1, d_2, \ldots, d_T)$ and $A_1^S = S$, then $A_t^S \le A_t^{S+1} \le A_t^S + 1$ for all $t \le T$.

Proof. The proof proceeds by induction on the number of time periods. It is trivially true for t=1. Suppose that it is true for time period T-1, i.e., $A_{T-1}^S \le A_{T-1}^{S+1} \le A_{T-1}^S + 1$. For time period T

$$A_T^S = (S - A_{T-1}^S) + Max\left\{A_{T-1}^S - d_{T-1},\ 0\right\},\quad A_T^{S+1} = (S+1-A_{T-1}^{S+1}) + Max\left\{A_{T-1}^{S+1} - d_{T-1},\ 0\right\}.$$

There are two cases to consider. Case (*i*) $A_{T-1}^{S+1} - d_{T-1} > 0$. Then by the induction hypothesis we must have $A_{T-1}^S - d_{T-1} \ge 0$, hence $A_T^S = S - d_{T-1}$ and $A_T^{S+1} = S + 1 - d_{T-1}$, i.e., $A_T^{S+1} = A_T^S + 1$. Case (*ii*) $A_{T-1}^{S+1} - d_{T-1} \le 0$. Then also $A_{T-1}^S - d_{T-1} \le 0$ by the induction hypothesis, thus

$A_T^S = (S - A_{T-1}^S)$ and $A_T^{S+1} = (S+1-A_{T-1}^{S+1})$. The induction hypothesis and the latter two equations require $A_T^S \le A_T^{S+1} \le A_T^S + 1$ and the lemma is proved.

Observe that the initial condition $A_1^S = A_1^{S+1} = Y$ (where $Y > 0$ is the initial inventory level for both replenishment levels) could have been used with only slight modifications to the proof.

Theorem 3.2. Let $d = (d_1, d_2, ..., d_T), A_1^S = S$, and $T \ge 2$. Then for the single period lag problem

(*i*) If $A_T^{S+1} = A_T^S + 1$, then stockouts were *not* reduced in period *T*-1 by using replenishment level *S*+1 instead of *S*.

(*ii*) If $A_T^{S+1} = A_T^S$, then stockouts were reduced by one unit in time period *T*-1 using replenishment level *S*+1 instead of *S*.

Proof. For period *T* we consider three mutually exclusive and collectively exhaustive scenarios: there is *positive* carryover for replenishment level *S*+1, i.e., $A_{T-1}^{S+1} - d_{T-1} > 0$ (scenario 1); there is nonpositive carryover for replenishment level *S*+1 and the previous period's initial inventory levels using *S* and *S*+1 are equal, i.e., $A_{T-1}^{S+1} - d_{T-1} \le 0$ and $A_{T-1}^{S+1} = A_{T-1}^S$ (scenario 2); there is nonpositive carryover for replenishment level *S*+1, and the previous period's inventory levels using *S* and *S*+1 are unequal, i.e., $A_{T-1}^{S+1} - d_{T-1} \le 0$ and $A_{T-1}^{S+1} = A_{T-1}^S + 1$ (scenario 3). The first two scenarios are shown to constitute part (*i*) of the theorem, the last scenario is shown to constitute part (*ii*).

For $A_{T-1}^{S+1} - d_{T-1} > 0$ (scenario 1), we observe that $A_{T-1}^S - d_{T-1} \ge 0$ (Lemma 3.1). In this case no shortages are experienced in period *T*-1, and we therefore have $A_T^{S+1} = S + 1 - d_{T-1}$ and $A_T^S = S - d_{T-1}$. Thus $A_T^{S+1} = A_T^S + 1$ and shortages are not reduced in period *T*-1. Scenario 1 relates to part (*i*) of the theorem.

For $A_{T-1}^{S+1} - d_{T-1} \le 0$ and $A_{T-1}^{S+1} = A_{T-1}^{S}$ (scenario 2), we must have $A_T^{S+1} = A_T^S + 1$ since there is no carryover for either replenishment level, but the equal on-hand inventory in the previous period means that one additional unit is contained in the order ($S + 1 - A_{T-1}^{S+1}$) placed under the S+1 replenishment policy. The condition $A_{T-1}^{S+1} = A_{T-1}^{S}$ also ensures that shortages in the previous period are not improved by the S+1 replenishment policy. Scenario 2 also relates to part (*i*) of the theorem.

For $A_{T-1}^{S+1} - d_{T-1} \le 0$ and $A_{T-1}^{S+1} = A_{T-1}^{S} + 1$ (scenario 3), observe that these initial conditions immediately imply that shortages in period T-1 are reduced by exactly 1 unit under the S+1 policy. Moreover, under this scenario $A_T^{S+1} = A_T^S$. This follows from the fact that there is no carryover under either the S+1 or S policy, and the orders placed in period T-1 are of equal size, i.e., $S + 1 - A_{T-1}^{S+1} = S - A_{T-1}^{S}$.

To summarize, scenario 3 is the only one where $A_T^{S+1} = A_T^S$, and shortages are reduced as described in part (*i*) of the theorem. Scenarios 1 and 2 result in $A_T^{S+1} = A_T^S + 1$, and shortages are not reduced as described in part (*ii*) of the theorem. This completes the proof.

Remark: Theorem 3.2 is true if the initial conditions are replaced by $A_1^S = A_1^{S+1} = Y$ (for any $Y > 0$).

We are now in a position to prove convexity of the loss function as a function of S. A wealth of convexity results appear in the inventory literature, but most of these involve convexity of cost as a function of either the period beginning inventory position or the amount ordered (e.g., Arrow, Karlin and Scarf 1958, Vendemia, Patuwo and Hung 1995). For general (s, S) reorder-point/order-up-to systems arising in renewal theory, Sahin (1990) has shown that the cost rate function is convex in the variable s provided the difference $\Delta = S - s \geq 0$ is held constant and there is full backlogging of unsatisfied demand. In the (r,q) reorder-point/order-quantity system (where r is the reorder point and q is the fixed order quantity), Zipkin (1986) has shown that the average number of stockouts per unit time is a convex function of (r,q) provided the demand density satisfies certain distributional assumptions. We could not find any results regarding convexity of the shortage function for general order-up-to S inventory systems with lost sales and delivery lags.

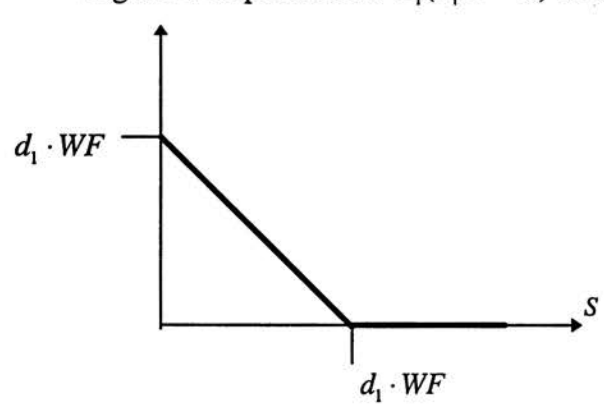
Theorem 3.3. Let $d = (d_1, d_2, ..., d_T)$ and $A_1^S = S$. Then the average conditional loss function

$$L_T(S|D = d) = WF \cdot \frac{Total\ Shortage\ over\ Periods\ 1,2,....,T}{T}$$

for a consigned item with one period lag is a monotone decreasing convex function of the replenishment level S.

Proof. Monotonicity follows directly from Lemma 3.1. Convexity will be shown using induction on the number of time periods. We may assume $WF = 1$ without loss of generality; in this case the terms "loss" and "shortage" become synonymous. For t=1, it is easily seen that $L_1(S|D = d)$ has the graph (shown in bold) in Figure 2 below.

Figure 2. Expected loss $L_1(S|D = d)$ as a function of S.



Assume the result is true for all t=1,2,....,T-1. Another way of stating the induction hypothesis is that the total shortage (the total number of items demanded which were not in stock) satisfies

$$t \cdot \left\{ L_t(S|D=d) - L_t(S+1|D=d) \right\} \geq t \cdot \left\{ L_t(S+1|D=d) - L_t(S+2|D=d) \right\} \quad (1)$$

for t=1,2,.....T-1.

In words, (1) means that the incremental change in the total shortage over t time periods is a monotone nonincreasing function of S. Note that the vector d is truncated in (1) as needed, i.e., for period t, $d = (d_1, d_2, \ldots, d_t)$.

To prove the theorem, it suffices to show that

$$T \cdot \left\{ L_T(S|D=d) - L_T(S+1|D=d) \right\} \geq T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\} \quad (2)$$

where $d = (d_1, d_2, \ldots, d_T)$

To prove that (2) is true given (1), we first break the problem into four mutually exclusive and collectively exhaustive cases as summarized below:

Case I	Case III
$A_T^{S+1} = A_T^S$ and $A_T^{S+2} = A_T^{S+1}$	$A_T^{S+1} = A_T^S + 1$ and $A_T^{S+2} = A_T^{S+1}$
Case II	**Case IV**
$A_T^{S+1} = A_T^S$ and $A_T^{S+2} = A_T^{S+1} + 1$	$A_T^{S+1} = A_T^S + 1$ and $A_T^{S+2} = A_T^{S+1} + 1$

Observe that for any value of S, $T \cdot L_T(S|D=d)$ can be expressed as the total shortage in periods 1,2,...,T-1 plus any additional shortages incurred in period T:

$$T \cdot L_T(S|D=d) = (T-1) \cdot L_{T-1}(S|D=d) + \text{Shortage in Period } T. \tag{3}$$

In a similar fashion, $T \cdot L_T(S|D=d)$ can be expressed as the sum of the total shortages in periods 1,2,....T-2 plus any additional shortages incurred over the last two periods:

$$T \cdot L_T(S|D=d) = (T-2) \cdot L_{T-2}(S|D=d) + \text{Shortages in periods } T\text{-1}, T. \tag{4}$$

<u>Case I</u> $A_T^{S+2} = A_T^{S+1} = A_T^S$, thus the shortages incurred in period T are the same for stocking policies S, S+1, and S+2. Applying (3) with S, S+1, S+2 and taking successive differences yields

$$T \cdot \left\{L_T(S|D=d) - L_T(S+1|D=d)\right\} = (T-1) \cdot \left\{L_{T-1}(S|D=d) - L_{T-1}(S+1|D=d)\right\} \tag{5}$$

and

$$T \cdot \left\{L_T(S+1|D=d) - L_T(S+2|D=d)\right\} = (T-1) \cdot \left\{L_{T-1}(S+1|D=d) - L_{T-1}(S+2|D=d)\right\}. \tag{6}$$

It follows from (5), (6) and the induction hypothesis that

$$T \cdot \left\{L_T(S|D=d) - L_T(S+1|D=d)\right\} \geq T \cdot \left\{L_T(S+1|D=d) - L_T(S+2|D=d)\right\},$$

which proves Case I.

<u>Case II</u> $A_T^{S+1} = A_T^S$ and $A_T^{S+2} = A_T^{S+1} + 1$. By Theorem 3.2, $A_T^{S+1} = A_T^S$ implies that shortages are reduced in period T-1 by one unit using policy S+1 instead of policy S. Shortages in period T are unchanged. Applying (4) with S, S+1 and taking successive differences yields

$$T \cdot \{(L_T(S|D=d) - L_T(S+1|D=d)\} = (T-2)\{L_{T-2}(S|D=d) - L_{T-2}(S+1|D=d)\} + 1 \tag{7}$$

Also by Theorem 3.2, $A_T^{S+2} = A_T^{S+1} + 1$ implies that shortages are not reduced in period T-1 using S+2 instead of S+1. Shortages in the final period may be reduced by at most one unit in using an S+2 level instead of S+1. Applying (4) with S+1, S+2 and taking differences

$$T \cdot \{(L_T(S+1|D=d) - L_T(S+2|D=d)\} = (T-2)\{L_{T-2}(S+1|D=d) - L_{T-2}(S+2|D=d)\} + \delta \cdot 1 \tag{8}$$

where $\delta = 1$ if there are shortages in period T that are improved using an S+2 replenishment level instead of S+1, and $\delta = 0$ otherwise. In either case ($\delta = 0$ or 1), it is clear from (7), (8) and the induction hypothesis for t=T-2 that

$$T \cdot \{L_T(S|D=d) - L_T(S+1|D=d)\} \geq T \cdot \{L_T(S+1|D=d) - L_T(S+2|D=d)\},$$

which proves Case II.

<u>Case III</u> $A_T^{S+1} = A_T^S + 1$ and $A_T^{S+2} = A_T^{S+1}$. The condition $A_T^{S+2} = A_T^{S+1}$ requires

$$T \cdot \{L_T(S+1|D=d) - L_T(S+2|D=d)\} = (T-1) \cdot \{L_{T-1}(S+1|D=d) - L_{T-1}(S+2|D=d)\} \tag{9}$$

The condition $A_T^{S+1} = A_T^S + 1$ requires

$$T \cdot \{L_T(S|D=d) - L_T(S+1|D=d)\} \geq (T-1) \cdot \{L_{T-1}(S|D=d) - L_{T-1}(S+1|D=d)\} \tag{10}$$

Equations (9), (10) and the induction hypothesis for t=T-1 imply

$$T \cdot \{L_T(S|D=d) - L_T(S+1|D=d)\} \geq T \cdot \{L_T(S+1|D=d) - L_T(S+2|D=d)\},$$

which proves Case III.

<u>Case IV</u> $A_T^{S+1} = A_T^S + 1$ and $A_T^{S+2} = A_T^{S+1} + 1$. Applying (3) with S, S+1 and taking successive differences yields

$$T \cdot \{L_T(S|D=d) - L_T(S+1|D=d)\} = (T-1) \cdot \{L_{T-1}(S|D=d) - L_{T-1}(S+1|D=d)\} + \delta \tag{11}$$

where $\delta = 1$ if there is a period T shortage that is improved using an S+1 replenishment level instead of S, $\delta = 0$ otherwise. In a similar fashion,

$$T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\} = (T-1) \cdot \left\{ L_{T-1}(S+1|D=d) - L_{T-1}(S+2|D=d) \right\} + \gamma \tag{12}$$

where $\gamma = 1$ if there is a shortage in period T that is improved using S+2 instead of S+1, $\gamma = 0$ otherwise. But observe that $\gamma = 1$ implies $\delta = 1$. Equations (11), (12) and the induction hypothesis once again require

$$T \cdot \left\{ L_T(S|D=d) - L_T(S+1|D=d) \right\} \geq T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\},$$

which completes the proof of Case IV and Theorem 3.3.

It is helpful to illustrate these results with a numerical example. We selected a 12-period horizon and tested replenishment levels S = 0, 1, 2,....., 10. The observed demands d_t (t=1,2,...,12) listed at the top of Table 2 are for a 10hp electric motor. The inventory level refers to the period-beginning inventory level, assuming the replenishment level S and the initial condition $A_1^S = S$. It is worth repeating that the shortage incurred in a *single* period of the horizon is not usually a convex function of S, as can be verified by the example in Table 2 (shortages in periods 2, 3, 4, 6, 8, 9, 10, 11, and 12 are nonconvex). Theorem 3.1 only guarantees that the *sum* (or average) of these shortages over any horizon (1,2,....,t) is a convex function of S.

TABLE 2. Shortages as a Function of S_i: One Period Lag

	Demands →	4	2	10	1	3	1	2	5	6	4	4	3	
Replenishment Level (S_i) ↓														Total Shortage ↓
0	Inventory Level →	0	0	0	0	0	0	0	0	0	0	0	0	
	Stockouts →	4	2	10	1	3	1	2	5	6	4	4	3	45
1	Inventory Level →	1	0	1	0	1	0	1	0	1	0	1	0	
	Stockouts →	3	2	9	1	2	1	1	5	5	4	3	3	39

2	Inventory Level →	2	0	2	0	2	0	2	0	2	0	2	0	
	Stockouts →	2	2	8	1	1	1	0	5	4	4	2	3	33
3	Inventory Level →	3	0	3	0	3	0	3	1	2	1	2	1	
	Stockouts →	1	2	7	1	0	1	0	4	4	3	2	2	27
4	Inventory Level →	4	0	4	0	4	1	3	2	2	2	2	2	
	Stockouts →	0	2	6	1	0	0	0	3	4	2	2	1	21
5	Inventory Level →	5	1	4	1	4	2	4	3	2	3	2	3	
	Stockouts →	0	1	6	0	0	0	0	2	4	1	2	0	16
6	Inventory Level →	6	2	4	2	5	3	5	4	2	4	2	4	
	Stockouts →	0	0	6	0	0	0	0	1	4	0	2	0	13
7	Inventory Level →	7	3	5	2	6	4	6	5	2	5	3	4	
	Stockouts →	0	0	5	0	0	0	0	0	4	0	1	0	10
8	Inventory Level →	8	4	6	2	7	5	7	6	3	5	4	4	
	Stockouts →	0	0	4	0	0	0	0	0	3	0	0	0	7
9	Inventory Level →	9	5	7	2	8	6	8	7	4	5	5	5	
	Stockouts →	0	0	3	0	0	0	0	0	2	0	0	0	5
10	Inventory Level →	10	6	8	2	9	7	9	8	5	5	6	6	
	Stockouts →	0	0	2	0	0	0	0	0	1	0	0	0	3

It is a simple matter to extend $L_T(S|D=d)$ to a function for the T-period expected average loss:

$$L_T(S) = WF\sum_d L_T(S|D=d)\cdot P(D=d)\,. \tag{13}$$

Observe that $L_T(S)$ does not assume independence of the random variables D_t (t=1,2,....,T). The T-period expected average loss is a finite nonnegative combination of convex functions and therefore convex. Thus order-up-to S replenishment models with a one period time lag have certain desirable theoretical properties that complement their ease of implementation in practice.

The loss function can be further extended to include multiple products. In this case we define the conditional loss for item i with replenishment level S_i given demand vectors $d_1, d_2, \ldots, d_n$ as

$$L_{i,T}(S_i | D_1 = d_1, D_2 = d_2, \ldots, D_n = d_n) = \frac{\textit{Total Shortage of Item i over Periods } 1,\ 2, \ldots,\ T}{T},$$

where now $D_l = (D_{l,1}, D_{l,2}, \ldots, D_{l,T})$ and $d_l = (d_{l,1}, d_{l,2}, \ldots, d_{l,T})$ for items $l = 1,\ 2, \ldots, n$. For the calculations in Lemma 3.1, Theorem 3.2, and Theorem 3.3, only the single demand vector $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,T})$ is needed in connection with item *i*. However, all possible demand vectors are needed when forming the expected average loss over all *n* items and *T* time periods:

$$L_T(S_1, \ldots, S_n) = \sum_{d_1, d_2, \ldots, d_n} \sum_{i=1}^{n} WF_i \cdot L_{i,T}(S_i | D_1 = d_1, \ldots, D_n = d_n) \cdot P(D_1 = d_1, \ldots, D_n = d_n). \quad (14)$$

Even if demands are dependent across time periods or products, the expected average loss is a nonnegative combination of convex functions and therefore convex. Unfortunately, the evaluation of $L_T(S)$ given in each of the extensions (13) and (14) presents a formidable combinatorial challenge for even relatively short horizons and low product demands.

A more satisfactory approach is obtained by determining optimal replenishment levels using forecasts of future demands. In this case the observed demands d_t in Theorem 3.3 are replaced by forecasts f_t for future periods. Historical demand sequences can be substituted for the f_t to build an empirical (convex!) estimate of the true expected loss function. Optimal replenishment levels (S_i^*) computed using this *empirical loss function* can be thought of as sample estimates which directly incorporate any special features of the time series (e.g. autocorrelation, product demand dependencies) without recourse to assumed parametric structures. This approach will be discussed in detail after completing the general analysis for multiple period delivery lags.

§4 The Case for k-Period Lags

The situation where orders placed in period t do not arrive until period $t+k$ in an order-up-to-S replenishment model without backlogging can be handled in a manner similar to that presented in section 3. We shall refer to this problem as simply the *k-lag* consignment problem. All proofs are provided in the appendix.

A new variable is needed to handle the outstanding orders:

$O_t^S =$ The number of units ordered at the start of period t to arrive for use at the start of period $t+k$.

With this additional variable we can prove the following analog to Lemma 3.1. The assumption $A_1^S = S$ with no outstanding orders is for ease of exposition only. It can be replaced throughout this section with the joint assumptions (*i*) $A_1^S = A_1^{S+1} = Y$ for all policies S and (*ii*) outstanding orders satisfy $O_j^S = O_j^{S+1}$ for $j = -k+2, \ldots\ldots, 0$ and all policies S.

Lemma 4.1. Suppose $d = (d_1, d_2, \ldots., d_T)$ and $A_1^S = S$ with no outstanding orders. Then $A_t^S \le A_t^{S+1} \le A_t^S + 1$ and $O_t^S \le O_t^{S+1} \le O_t^S + 1$ for all $t \le T$.

The next theorem generalizes the result of Theorem 3.2.

Theorem 4.2. Let $d = (d_1, d_2, \ldots., d_T)$, and $T \ge 2$. Then for the k-period ($k \ge 2$) lag problem

(*i*) If $A_T^{S+1} = A_T^S + 1$, then total stockouts over the preceding k time periods were *not* reduced using replenishment level S+1 instead of S.

(*ii*) If $A_T^{S+1} = A_T^S$, then total stockouts over the preceding k time periods were reduced by one unit using replenishment level S+1 instead of S.

The next theorem is the analog to Theorem 3.2. This is an important result for consignment problems where the review period is quite short (say 1 week) relative to the delivery lead-time (say 3 or 4 weeks).

Theorem 4.3. Let $d = (d_1, d_2, ..., d_T)$, $A_1^S = S$ with no outstanding orders. Then the average conditional loss function

$$L_T(S|D = d) = WF \cdot \frac{\mathit{Total\ Shortage\ over\ Periods}\ 1,2,....,T}{T}$$

for a consigned item with k-period lag is a monotone decreasing convex function of the replenishment level S.

§5 An Illustration

In this section we illustrate the technique proposed in the last three sections using ten items (i=1, 2,..., 10) extracted from the full data set supplied by our warehouser for years 1995-1996. These items were among the top twenty best selling motors. An analysis of the full problem (approximately 360 different motors) is underway and will be detailed in a later report.

There are three types of storage space available: floor space; 11 foot sections of adjustable shelving ("long" shelves); and 8 ft. sections of adjustable shelving ("short" shelves). Because the analysis presented here is for illustrative purposes only, we will consider only two types of storage, floor space (k=1) and long shelving (k=2).

The ten standardized package sizes noted earlier are determined by the motor's frame dimensions. Those from the five smallest frame sizes are packaged in heavy duty cardboard boxes that can be stacked up to three high. Those from the five largest frame sizes are bolted to individual pallets and cannot be stacked. Four of the five box sizes and one of the five palletized sizes are represented in our sample. The sole palletized model in the sample (weighing 1000 lbs) must be stored on the floor. The remaining nine models in the sample come in boxes and can be kept on the shelves or on the floor. For efficient packing and subsequent location/retrieval, the warehouser stores motors together according to frame size. Once it is determined how deep the motors can be shelved (either two or three deep per shelf), it is easy to determine an item's storage requirement $C_i^{(k)}$ in terms of *linear* shelf space. The motor's width or length (whichever gives the best shelving orientation) is divided by 6 or 9 depending on whether it can be stacked two deep by three high or three deep by three high. These storage requirements are summarized in Table 3 below.

TABLE 3. Storage Requirements, $C_i^{(k)}$ (in inches).

Item →	1	2	3	4	5	6	7	8	9	10
Floor	34	4.7	4.7	4.7	2.4	2.4	2.1	2.1	1.4	1.4
Long Shelving	NA	4.7	4.7	4.7	2.4	2.4	2.1	2.1	1.4	1.4
Retail Value	$2059	$433	$482	$349	$227	$189	$147	$129	$108	$258

The first 26 weeks of sales from 1996 (T=26) are used to build our empirical estimate of the average expected loss function for each of the ten motors selected. These figures include drop shipments made to customers when an item was not in stock (hence resulting in a lost warehousing fee). Although these numbers are clearly surrogates for true demand, the warehouser agreed that subsequent record keeping would include more accurate demand information. We assumed a one week (one period) delivery lag for the boxed motors (items 2 through 10) and a two week lag for the palletized motor (item 1). The sales patterns are shown in Table 4 below.

TABLE 4. 26 Weeks of Sales

	Week																									
ITEM ↓	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	0	0	0	0	0	3	0	0	0	10	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0
2	2	0	0	0	0	0	1	0	0	0	5	12	0	0	11	4	0	1	0	0	0	0	0	0	12	0
3	0	0	2	0	0	0	0	0	40	0	40	2	0	0	0	0	0	16	2	0	4	0	0	0	0	0
4	0	0	4	0	2	7	1	0	0	1	0	14	0	0	1	1	0	1	0	0	0	2	7	1	2	1
5	0	4	10	1	0	6	0	2	3	1	3	6	0	3	3	3	0	5	1	1	0	2	0	3	5	0
6	0	1	4	2	20	2	1	3	1	3	1	4	1	2	2	1	3	0	2	0	1	3	7	3	5	2
7	6	4	4	0	10	4	0	9	5	2	3	6	6	7	8	2	6	1	6	11	3	4	2	6	4	4
8	25	4	0	0	1	13	0	29	0	1	0	20	0	1	1	0	13	0	0	0	2	7	1	0	4	0
9	0	0	4	2	0	0	0	16	0	2	1	2	4	1	0	0	4	0	0	0	1	2	0	5	0	0
10	0	0	0	2	0	0	0	0	0	0	0	0	0	2	0	2	0	4	9	0	0	0	0	0	0	2

We selected the total amount of storage available in each category to suit the abbreviated list of items used in our illustration: $C^{(1)} = 120$ (equivalent to one section of floor space under a long shelf), and $C^{(2)} = 240$ (two sections of long shelves). The historical demands listed above are used in Theorem 4.3 to calculate empirical estimates of the incremental losses. The linear programming problem addressing our stocking problem becomes

$$\text{Min} \quad \sum_{i=1}^{10} \left[L_i(0) + \sum_{j=1}^{M_i} (L_i(j) - L_i(j-1)) S_{ij} \right]$$

s.t.

$$\sum_{i=1}^{10} C_i^{(1)} y_i^{(1)} \le C^{(1)} \qquad \text{(floor space)}$$

$$\sum_{i=2}^{10} C_i^{(2)} y_i^{(2)} \le C^{(2)} \qquad \text{(shelf space)}$$

$$\sum_{j=1}^{M_i} S_{ij} = y_i^{(1)} \qquad i = 1$$

$$\sum_{j=1}^{M_i} S_{ij} = y_i^{(1)} + y_i^{(2)} \qquad i = 2, 3, \ldots, 10$$

$$0 \le S_{ij} \le 1, \quad y_i^{(1)}, y_i^{(2)} \ge 0.$$

The optimal stocking levels S_i^* (i=1,...,10), complete with their allocations to the different storage types ($y_i^{(k)*}$ k=1,2), are summarized in Table 5 below.

TABLE 5 Stocking Results, S_i^*

Item →	1	2	3	4	5	6	7	8	9	10
Floor $y_i^{(1)*}$	0	0	19.27	0	0	0	0	14	0	0
Long Shelving $y_i^{(2)*}$	NA	12	13.25	9	8	8	14	0	6	2
Repl. Level S_i^*	0	12	32.52	9	8	8	14	14	6	2

With the exception of the palletized motor, all models are stocked. This is consistent with practical advice given to the warehouser from other consignees in the motor distribution business. The warehousing fees accumulated using these replenishment levels total \$2913.58, which represents a little over 60% of the warehousing fees potentially available (\$4814.35) for the items and periods analyzed. Not surprisingly, the dual multipliers for the two types of storage are both \$4.51 per inch. We note that two issues involving implementation of our illustrative solution, that of fractional units and shelf packing, do not pose much of a problem in practice. With only 6 different motor sizes to be shelved, at most 6 "transitional" sections would contain items of different dimensions. Finally, observe that 95 motors are stocked at full capacity, and peak storage is achieved at the start of period 8 when 92 of these are in stock.

§6 Conclusions

We have presented a model that addresses the situation of inventory under consignment when there are multiple items, lost sales, delivery lags, and uncertain demand distributions. Using simple order-up-to-S_i control policies for each item, we have demonstrated that the loss function due to shortages is a separable convex function of the S_i. A numerically tractable solution procedure based on an empirical estimate of this loss function has been presented.

A somewhat lengthy list of new issues remain to be explored. Foremost among these is the accuracy of our empirical loss function relative to the true expected loss function. Some convergence results for large T or some other type of sensitivity analysis would be beneficial. Other possible topics include: using the model in conjunction with forecasts of future demands; using the model to calculate appropriate storage capacities (i.e., warehouse selection); devising more sophisticated computing informatics for large scale consignment systems; investigating the model under specialized distributional assumptions. These are currently being pursued by the authors.

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Appendix
Proofs of Lemma 4.1, Theorem 4.2 and Theorem 4.3

PROOF OF LEMMA 4.1. The proof is once again by induction on T. The case for T=1 is straightforward and therefore omitted. Assume the truth of the proposition for all $t \le T-1$ $(T \ge 2)$. Before proceeding with the general induction step, observe that for general t

$$A_t^S = O_{t-k}^S + Max\left\{A_{t-1}^S - d_{t-1},\ 0\right\} \tag{A1}$$

and

$$O_t^S = S - \left[A_t^S + O_{t-1}^S + \cdots + O_{t-k+1}^S\right], \tag{A2}$$

where we define $O_{t-k}^S = 0$ if $t-k \le 0$. Taking successive differences between S+1 and S in (A2) yields

$$O_t^{S+1} - O_t^S = 1 + (A_t^S - A_t^{S+1}) + (O_{t-1}^S - O_{t-1}^{S+1}) + \cdots + (O_{t-k+1}^S - O_{t-k+1}^{S+1}). \tag{A3}$$

Applying (A3) in the particular instance t=T-k results in

$$O_{T-k}^{S+1} - O_{T-k}^S = 1 + (A_{T-k}^S - A_{T-k}^{S+1}) + (O_{T-k-1}^S - O_{T-k-1}^{S+1}) + \cdots + (O_{T-2k+1}^S - O_{T-2k+1}^{S+1}). \tag{A4}$$

The proof is divided into two cases, one of which has two subcases.

Case 1. $O_{T-k}^{S+1} = O_{T-k}^S + 1$. By the induction hypothesis, $A_{T-k}^S \le A_{T-k}^{S+1} \le A_{T-k}^S + 1$ and $O_t^S \le O_t^{S+1} \le O_t^S + 1$ for all $t \le T-1$. This ensures that each term enclosed by parentheses in (A4) is nonpositive. Consequently, $O_{T-k}^{S+1} = O_{T-k}^S + 1$ can occur if and only if the following system of equalities hold in (A4):

$$A_{T-k}^S = A_{T-k}^{S+1},\ O_{T-k-1}^S = O_{T-k-1}^{S+1},\ \ldots\ldots,\ O_{T-2k+1}^S = O_{T-2k+1}^{S+1}. \tag{A5}$$

The recursion (A1) and equation (A5) then imply the following sequence of beginning period inventory levels:

$$A_{T-k+1}^S = A_{T-k+1}^{S+1},\ \ldots\ldots\ ,\ A_{T-1}^S = A_{T-1}^{S+1}. \tag{A6}$$

Consequently, $A_T^{S+1} = O_{T-k}^{S+1} + Max\left\{A_{T-1}^{S+1} - d_{T-1},\ 0\right\} = A_T^S + 1$.

We now show that $O_T^S = O_T^{S+1}$ as well. Consider (A3) for $t = T-1, T-2, \ldots, T-k+1$. The induction hypothesis ensures that each of the terms on the right hand side of equation (A3) is nonpositive. Moreover, $(O_{T-k}^S - O_{T-k}^{S+1}) = -1$ by the assumption for Case 1, and this term appears on the right hand side of (A3) for each $t = T-1, T-2, \ldots, T-k+1$. This, combined with the conditions in (A6), forces the right hand side of (A3) to be less than or equal to zero. It cannot be negative since $O_t^S \le O_t^{S+1} \le O_t^S + 1$ for all $t \le T-1$ by the induction hypothesis, so the following equalities must occur

$$O_{T-1}^S = O_{T-1}^{S+1},\ O_{T-2}^S = O_{T-2}^{S+1}, \dots,\ O_{T-k+1}^S = O_{T-k+1}^{S+1}.$$

Finally,

$$O_T^{S+1} - O_T^S = 1 + (A_T^S - A_T^{S+1}) + (O_{T-1}^S - O_{T-1}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}) = 1 - 1 = 0.$$

Case 2. $O_{T-k}^{S+1} = O_{T-k}^S$. Then it follows immediately from the induction hypothesis and (A1) that $A_T^S \le A_T^{S+1} \le A_T^S + 1$. It remains to show that $O_T^S \le O_T^{S+1} \le O_T^S + 1$. The latter is done by dividing Case 2 into two subcases: (Subcase I) $O_{T-k}^{S+1} = O_{T-k}^S$ and $O_{T-1}^{S+1} = O_{T-1}^S$; (Subcase II) $O_{T-k}^{S+1} = O_{T-k}^S$ and $O_{T-1}^{S+1} = O_{T-1}^S + 1$.

<u>Subcase I</u>. Apply (A3) for period t=T-1 to obtain

$$O_{T-1}^{S+1} - O_{T-1}^S = 1 + (A_{T-1}^S - A_{T-1}^{S+1}) + (O_{T-2}^S - O_{T-2}^{S+1}) + \cdots + (O_{T-k}^S - O_{T-k}^{S+1}). \tag{A7}$$

Since $O_{T-1}^{S+1} = O_{T-1}^S$, (A7) is equal to 0, which creates one of two possibilities: (a) $A_{T-1}^{S+1} = A_{T-1}^S + 1$ and $O_{T-j}^S = O_{T-j}^{S+1}$ for j=2, ..., k; or (b) $A_{T-1}^{S+1} = A_{T-1}^S$ and $O_{T-j}^{S+1} = O_{T-j}^{S1}$ $(2 \le j \le T-k)$ except for one fixed index m $(2 \le m \le T-k+1)$ where $O_{T-m}^{S+1} = O_{T-m}^{S1} + 1$. For possibility (a) we have

$$\begin{aligned} O_T^{S+1} - O_T^S &= 1 + (A_T^S - A_T^{S+1}) + (O_{T-1}^S - O_{T-1}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}) \\ &= 1 + (A_T^S - A_T^{S+1}). \end{aligned}$$

For possibility (b), $A_{T-1}^{S+1} = A_{T-1}^S$ implies $A_T^{S+1} = A_T^S$ since $O_{T-k}^{S+1} = O_{T-k}^S$ by the assumption for Case 2. Then

$$\begin{aligned} O_T^{S+1} - O_T^S &= 1 + (A_T^S - A_T^{S+1}) + (O_{T-1}^S - O_{T-1}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}) \\ &= 1 + (O_{T-m}^S - O_{T-m}^{S+1}) = 1 - 1 = 0. \end{aligned}$$

This completes the proof of Subcase I.

<u>Subcase II</u>. By the assumptions of this case, $O_{T-k}^{S+1} = O_{T-k}^S$ and $O_{T-1}^{S+1} = O_{T-1}^S + 1$. In this case the left hand side of equation (A7) is equal to 1, which forces all of the (nonpositive) terms in parentheses on the right hand side to be 0. As in Subcase I, $A_{T-1}^{S+1} = A_{T-1}^S$ implies $A_T^{S+1} = A_T^S$ from the assumption $O_{T-k}^{S+1} = O_{T-k}^S$ of Case 2, and it follows that

$$\begin{aligned} O_T^{S+1} - O_T^S &= 1 + (A_T^S - A_T^{S+1}) + (O_{T-1}^S - O_{T-1}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}) \\ &= 1 + (O_{T-1}^S - O_{T-1}^{S+1}) = 1 - 1 = 0. \end{aligned}$$

This completes the proof of Subcase II, Case 2, and Lemma 4.1.

PROOF OF THEOREM 4.2. In period *T-k*, the following two equations must hold:

$$A_{T-k}^{S} + O_{T-k}^{S} + O_{T-k-1}^{S} + \cdots + O_{T-2k+1}^{S} = S \tag{A8}$$

$$A_{T-k}^{S+1} + O_{T-k}^{S+1} + O_{T-k-1}^{S+1} + \cdots + O_{T-2k+1}^{S+1} = S + 1 \tag{A9}$$

These equations simply state that the on-hand inventory plus all outstanding orders (including the one made at the start of a period) must sum up to the replenishment level.

To prove part (*i*), observe that $A_T^{S+1} = A_T^S + 1$ can occur in one of two ways: (Case 1) the period *ending* inventory levels at time *T*-1 are equal and $O_{T-k}^{S+1} = O_{T-k}^{S} + 1$; or (Case 2) the period ending inventory levels at time period *T*-1 are unequal and $O_{T-k}^{S+1} = O_{T-k}^{S}$.

Case 1. The total amount of on-hand inventory available over periods *T-k*, *T-k*+1,....., *T*-1 under the order up to *S* policy is

$$A_{T-k}^{S} + O_{T-k-1}^{S} + \cdots + O_{T-2k+1}^{S} = S - O_{T-k}^{S}. \tag{A10}$$

Under the *S*+1 policy, the total amount of on-hand inventory available over periods *T-k*, *T-k*+1,....., *T*-1 is

$$A_{T-k}^{S+1} + O_{T-k-1}^{S+1} + \cdots + O_{T-2k+1}^{S+1} = S + 1 - O_{T-k}^{S+1}. \tag{A11}$$

The amounts expressed in (A10) and (A11) are identical since $O_{T-k}^{S+1} = O_{T-k}^{S} + 1$. Because the period ending inventory levels are the same under both policies, equal amounts of inventory were moved over the periods *T-k*, *T-k*+1,....., *T*-1. This ensures that stockouts were not improved by the *S*+1 policy over the *k* periods preceding period *T*.

Case 2. A comparison of equations (A10) and (A11) reveals that the total on-hand inventory available over periods *T-k*, *T-k*+1,....., *T*-1 is one unit greater under the *S*+1 policy. However, the additional unit is unused since the period ending inventory levels for period *T*-1 are assumed to be unequal (i.e., the *S*+1 policy has an additional unit which it carries over to period *T*). This ensures that stockouts were not improved by the *S*+1 policy over the *k* periods preceding period *T*. This completes the proof of part (*i*) of the theorem.

To prove part (*ii*), observe that the condition $A_T^{S+1} = A_T^S$ can occur in precisely one way: the period ending inventory levels are the same under both replenishment policies and $O_{T-k}^{S+1} = O_{T-k}^{S}$. Equations (A10) and (A11) still apply, and the amount expressed in (A10) is one unit less than that expressed in (A11). The equal period ending inventories for period *T*-1 means that one additional unit was moved during periods *T-k*, *T-k*+1,....., *T*-1 under the *S*+1 policy.

Consequently, stockouts were improved by precisely one unit over the k periods immediately preceding period T.

PROOF OF THEOREM 4.3. The proof is by induction on T and parallels that of Theorem 3.3. The assumption $WF = 1$ is used as before without loss of generality.

The case T=1 is again trivial, and we assume the truth of the theorem for t=1,.....,T-1. The problem is divided into the same four cases used in the proof of Theorem 3.3. However, the proofs of Case I, Case III and Case IV do not need to be repeated since they depend solely on an analysis of period ending inventory levels. Consequently, only Case II, which involves results on lagged delivery times, needs to be redone.

Case II. $A_T^{S+1} = A_T^S$ and $A_T^{S+2} = A_T^{S+1} + 1$. By Theorem 4.2, $A_T^{S+1} = A_T^S$ implies that shortages are reduced in the preceding k time periods by one unit using policy S+1 instead of policy S. Shortages in period T are unchanged. The difference in total shortages over T periods can be expressed as

$$T \cdot \left\{(L_T(S|D=d) - L_T(S+1|D=d)\right\} = (T-k-1)\left\{L_{T-k-1}(S|D=d) - L_{T-k-1}(S+1|D=d)\right\} + 1$$

Also by Theorem 4.2, $A_T^{S+2} = A_T^{S+1} + 1$ implies that shortages are not reduced in the preceding k periods using S+2 instead of S+1. Shortages in the final period can be reduced by at most one unit using an S+2 replenishment level instead of S+1. Consequently,

$$T \cdot \left\{(L_T(S+1|D=d) - L_T(S+2|D=d)\right\} =$$
$$= (T-k-1)\left\{L_{T-k-1}(S+1|D=d) - L_{T-k-1}(S+2|D=d)\right\} + \delta \cdot 1$$

where $\delta = 1$ if there are shortages in period T that are improved using an S+2 replenishment level instead of S+1, and $\delta = 0$ otherwise. In either case ($\delta = 0$ or 1), it is clear from the induction hypothesis for t=T-k-1 that

$$T \cdot \left\{L_T(S|D=d) - L_T(S+1|D=d)\right\} \geq T \cdot \left\{L_T(S+1|D=d) - L_T(S+2|D=d)\right\},$$

which proves Case II and Theorem 4.3.

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