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A Direct Approach for Managing Inventory with Lost Sales, Intermittent Demand, and Resource Constraints

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by

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STATEMENT OF CONTRIBUTION

A Direct Approach for Managing Inventory with Lost Sales, Intermittent Demand, and Resource Constraints

John Semple, Brian Downs

This paper develops a nonparametric method for managing inventory in situations where demand is intermittent, erratic, or not easily described by traditional parametric forms. Our method is based on the direct approach, a nonparametric technique proposed by Iyer and Schrage (1992). The Iyer/Schrage technique requires full backlogging of unsatisfied demand, and the authors observe that their theoretical results do not generalize to the case of lost sales. In this paper, we develop an alternative theoretical framework for using the direct approach when there are multiple items, lost sales, delivery lags and resource constraints. We show that, under very mild conditions, our direct approach produces closed-form cost expressions whose theoretical properties help build tractable models for complicated real-world inventory problems. Our results help bridge the gap between what is mathematically possible and computationally feasible in environments where stochastic models would be difficult or even impossible to implement. We demonstrate the practical relevance of our approach through application to a real-world distribution problem characterized by lost sales, delivery lags, irregular demand, multiple items, and multiple resource constraints.

A Direct Approach for Managing Inventory with Lost Sales, Intermittent Demand, and Resource Constraints

John Semple†

Brian Downs†

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Abstract. This paper develops a general class of periodic review order-up-to $S$ inventory models that are especially appropriate for situations where demand is intermittent, erratic, or not accurately described by parametric methods. Our model is designed to include multiple items, resource constraints, lags in delivery, and lost sales without sacrificing computational simplicity. Mild conditions are shown to ensure that the expected average holding cost and the expected average shortage cost are separable convex functions of the order-up-to levels $S_i$ for item $i$. This result holds even when demands are correlated across products and (or) time. We show how to use nonparametric techniques to produce convex closed-form estimates of the various cost functions. The practical usefulness of this model is illustrated through application to a real-world consignment inventory problem involving multiple resource constraints. The problem of determining optimal order-up-to levels is shown to be equivalent to a linear program.


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§1 Introduction and Literature Review

The vast majority of research in stochastic inventory theory begins with the assumption that the functional form of demand is completely known, even though information about the form is often limited in practice. As Iyer and Schrage (1992, pg. 1300) have observed, in applications

"...one cannot hope to make a perfectly correct specification, e.g., real distributions will not be exactly Normal or Poisson, etc."

When demand is irregular (for example, maintenance, repair and operating (MRO) purchases), the situation is even more difficult since smooth distributional forms such as the Poisson or Normal are inappropriate (Silver (1981)). Unfortunately, no amount of mathematical exactness can repair the errors introduced by an inexact approximation of the demand distribution.

To illustrate the complexities associated with irregular demand, we outline below the inventory management problem facing a distributor of industrial electric motors. The product line consists of hundreds of motors with a wide range of sizes and per unit costs. Since many of the distributor’s sales are to replace motors that have malfunctioned, demand for these items is intermittent, and their distributions are not easily modeled by convenient functional forms.

The distributor’s problem is complicated by item-specific delivery lead times that vary from one week to several months. The manufacturer pays for the costs of ordering and delivery, and shipments are made weekly to control these costs. The distributor adopts a one-week periodic review model to accommodate the manufacturer’s shipping schedule. Because the distributor has no ordering cost, an order-up-to $S$ (base-stock) policy is appropriate (see Love (1979)).

A final complication is the distributor’s restricted warehouse capacity, which is divided into categories having specific weight and storage limitations. This creates multiple resource
constraints that force the distributor to consider the tradeoffs between the stock levels of competing items. In this situation, importance must be placed on the aggregate performance of the stocking decision and not the mathematical optimality of individual items. As noted by Silver (1981), this is an important issue encountered in practice, and one that is often neglected. Moreover, given the scarcity and high cost of warehouse space in the distributor’s metropolitan area, a related issue is the appropriate amount of overall warehouse space. Answering these questions requires the development of closed-form expressions for the various inventory costs.

To address this problem, we introduce a general class of periodic review, order-up-to $S$ inventory models. These nonparametric models are distinguished from the majority of classical inventory models in a number of ways:

- No assumptions (including independence) about the functional forms of demand are made.
- There are item-specific lags in delivery that occur in conjunction with lost sales.
- The expected average holding cost and the expected average lost sales cost are shown to be separable convex functions of the order-up-to level $S_i$ for item $i$.
- Multiple items constrained by resources are handled easily.
- The solution procedure (linear programming) is computationally straightforward.

Our method is based on the direct approach, a nonparametric technique proposed by Iyer and Schrage (1992). In that paper, the authors assume full backlogging of unsatisfied demand, fixed setup costs, and fixed lead times. They develop theoretical results for computing $(s, S)$ control variables that are optimal with respect to the historical demand stream. The demand distribution is thus described directly from sample information instead of relying on an intermediate functional form. Simulation evidence presented by the authors suggests the direct approach works well when the distribution is correctly specified (e.g., Poisson, sample sizes $n \geq 40$, lead
times \( k \leq 6 \) and outperforms the classical stochastic approach when the distribution is incorrectly specified (e.g., Normal instead of true Poisson, \( n \geq 40, k \leq 5 \)). The authors observe that their theoretical results do not apply to the case of lost sales. In this paper, we present an alternative theoretical framework for using the direct approach when there are lost sales, multiple items, and resource constraints. We will assume zero setup cost (easily relaxed to linear setup costs) and item-specific lead times (fixed per product, but varying across products). We show that the direct approach produces closed-form cost approximations whose theoretical properties help build tractable models for complicated real-world inventory problems.

Existing techniques for avoiding demand specifications include the distribution-free method of Scarf (1958), who assumes that the first two moments of the demand distribution are known and then finds the order quantity that maximizes the expected profit for the worst-case distribution having the given mean and variance. Moon and Choi (1995, 1997), Moon and Gallego (1994) and Gallego (1992) have recently extended Scarf's results. Silver (1981, p. 640) has cited the lack of — and need for — computationally tractable methods for calculating control parameters (specifically order-up-to levels) in the case where demand is not easily described.

Of the previous convexity results appearing in the literature, most focus on showing cost convexity as a function of either the period beginning inventory position or the amount ordered (e.g., Arrow, Karlin and Scarf (1958), Vendemia, Patuwo and Hung (1995)). For general \((s, S)\) reorder-point/order-up-to systems arising in renewal theory, Sahin (1990) has shown that the cost rate function is convex in the variable \(s\) provided the difference \(\Delta = S - s \geq 0\) is held constant and there is full backlogging of unsatisfied demand. In the \((r,q)\) reorder-point/order-quantity system (where \(r\) is the reorder point and \(q\) is the fixed order quantity), Zipkin (1986) has shown that the average number of stockouts per unit time is a convex function of \((r,q)\) provided the demand density satisfies certain distributional assumptions. There do not appear to be any
convexity results for the shortage and holding costs (as functions of \( S \)) for general order-up-to \( S \) inventory systems with both lost sales and delivery lags.

With regard to previous multi-item inventory models, few include (or are easily adapted to include) resource restrictions. Hadley and Whitin (1963) consider the case where there is a single linear constraint on maximum inventory levels. Evans (1967) considers the case where there is a single constraint on production capacity and an initial stock level. He shows that in this case a base stock policy is optimal. There does not appear to be any work on the case of multiple items constrained by multiple resources.

With regard to delivery lags, the lost sales case (termed a model of Type I by Arrow, Karlin and Scarf (1958)) has received far less attention than the case of full backlogging (termed a model of Type II). This disparity is partially explained by the fact that the dynamic problem with full backlogging can be reformulated into a single period problem. As Porteus notes (1990, p. 636), "The reformulation into an equivalent single period problem depends critically on the assumption that shortages are backlogged...." While some results exist for lost sales in the single period case, the overall complexity of the problem has lead to a dearth of results for the dynamic case. Veinott (1965) described a variety of conditions under which a myopic base stock policy is optimal given the assumption of full backlogging of unsatisfied demand. Venoitt (1965, p. 217) assumes all products have equal delivery lags, noting that "A more realistic model would allow the delivery lag to vary with the product. But this seems to complicate the model considerably." Periodic review models that address both lost sales and delivery lags simultaneously have been analyzed in Arrow, Karlin and Scarf (1958), Gaver (1959), Nahmias (1979), Bartmann and Beckmann (1992), and Vendemia, Patuwo and Huang (1995). The last three works show how to compute optimal policies for individual items without closed form cost expressions. While this can be considered an advantage in those cases, closed-form cost
expressions are needed for computing the optimal joint policy in the resource-constrained multi-item case. A comprehensive review of the available theory can be found in Porteus (1990).

The analysis developed in the following sections is distinct from that found in the mainstream inventory literature. Indeed, to shoehorn this problem into an existing parametric multi-item inventory model is problematic from both a mathematical and a computational point of view. By contrast, our nonparametric model can be reduced to a piecewise linear convex programming problem — hence equivalent to a linear program — for minimizing the combined expected lost sales and holding costs. While developed and illustrated in the context of the motor distribution problem described earlier, we note that our results are quite general and adapt easily to other lost-sales environments.

The next section of the paper develops convexity results for the cost of lost sales as a function of the order-up-to levels $S_i$. The third section develops similar results for holding costs. The fourth section presents the separable convex programming formulation of our model in the context of a numerical example obtained from an electric motor distributor. The final section presents conclusions and some important issues for further research.

§2 A Direct Approach for Estimating Shortages with Lost Sales and Delivery Lags

Gaver (1958) and Arrow, Karlin and Scarf (1958) have demonstrated that the case of lost sales and delivery lags is fundamentally different from the case where lags occur with full backlogging of unsatisfied demand. Additionally, when an order-up-to $S$ policy is used, both have shown that the single period shortfall has the form

$$\int_S^\infty (y - S)f(S,y)dy$$

where $f(S,y)$ is the density for the period ending “deficit below base-stock level” (i.e., shortfall) distribution, and $y - S$ represents lost sales. Unlike the case where lead times are negligible or
unfilled demands are backlogged, the density $f(S,y)$ depends on the replenishment level $S$. The computation of this density is a difficult exercise; for even assuming that the period demands are i.i.d. random variables from a known distribution, $f(S,y)$ involves calculating the steady state distribution of period-beginning inventory levels followed by an integral convolution.

Similar complexities arise in the finite period models more commonly used in practice. Although shortages represent the primary cost incurred by most distributors, the following example illustrates that shortages (per period or on average) may lack any simplifying structure, including convexity.

**Example 1.** Suppose demand is one unit per period (with probability 1) over a four period horizon. Inventory is controlled using an order-up-to $S$ policy; specifically, at the start of each period an order is placed to bring the inventory position up to $S$ units. There is a one period lag in delivery, thus units ordered at the start of a period become part of the initial on hand inventory at the start of the next period. The initial on hand inventory at the start of the first period is one unit. The situation is summarized for three possible order-up-to levels in Table 1 below.

The single period shortages in periods 2 and 4 as well as the total shortages over the horizon are highlighted in bold. If a piecewise linear function is interpolated through these values, then the following conclusions are immediate: (i) the function describing shortages in a single period of the horizon and (ii) the function describing the average number of shortages per period are not, in general, convex functions of $S$. Additional examples suggest that lumpier demand patterns exacerbate these nonconvexities.
If the performances of different values of $S$ are to be tested, then it is reasonable to assume that the on hand inventory in the first period is equal to $S$. This ensures that all policies are judged relative to their best initial inventory position. Iyer and Schrage (1992) use this assumption in their analysis of the direct approach for the case of full backlogging. To see what effect, if any, this has on the lost sales case in example 1, consider the results presented in Table 2.

<< Table 2 Goes Approximately Here >>

While the shortages incurred in periods 2 and 4 are once again nonconvex (using interpolation), the average number of shortages per period over the horizon — the more important measure from the distributor’s perspective — is convex. This illustrates one of the main results of this section, namely, that the expected average shortage cost is a (piecewise linear) convex function of $S$ provided the on hand inventory in the first period is equal to $S$.

For ease of exposition, consider the case of a single item. The following notation will be helpful:

- $S = \text{Replenishment level.}$
- $A_t^S = \text{On-hand inventory at the beginning of period } t \text{ assuming a replenishment level } S.$
- $D_t = \text{Demand during period } t$
- $T = \text{Number of time periods (the planning horizon)}$
One period lags

Assuming for now that the delivery lag is only one period, the on hand inventory at the start of period $t$ can be expressed by the recursive formula

$$A_t^S = (S - A_{t-1}^S) + \max\{A_{t-1}^S - D_{t-1}, 0\}, \quad A_1^S = S$$

where $(S - A_{t-1}^S)$ is the order that arrives at the beginning of period $t$ and $\max\{A_{t-1}^S - D_{t-1}, 0\}$ is the carryover. The shortfall in period $t$ is given by

$$\max\{D_t - A_t^S, 0\},$$

and as shown in example 1, this function is not typically convex. Let $d = (d_1, d_2, \ldots, d_T)$ represent a single realization of the random demand vector $D = (D_1, D_2, \ldots, D_T)$ over periods $t = 1, 2, \ldots, T$. Letting $SC$ denote the fixed cost per unit shortfall, the average conditional shortage cost for replenishment level $S$ given demand $D = d$ is defined as

$$L_T(S|D = d) = SC \cdot \frac{\sum_{t=1}^{T} \max\{d_t - A_t^S, 0\}}{T}.$$

We will show that $L_T(S|D = d)$ is a convex function of $S$, the proof of which is broken into three parts: Lemma 2.1, Theorem 2.2 and Theorem 2.3. These three results provide insight into the relationship between the shortage cost and the replenishment level and additionally suggest a direct nonparametric estimation procedure. The first lemma is simple but included for completeness.

**Lemma 2.1.** Assume deliveries arrive following a one period lag. Given $d = (d_1, d_2, \ldots, d_T)$ and $A_1^S = S$, then $A_t^S \leq A_{t+1}^S \leq A_t^S + 1$ for all $t \leq T$.

**Proof.** The proof proceeds by induction on the number of time periods. It is trivially true for $t=1$. Suppose that it is true for time period $T-1$, i.e., $A_{T-1}^S \leq A_{T-1}^{S+1} \leq A_{T-1}^S + 1$. For time period $T$
There are two cases to consider. Case (i) \( A_{t-1}^S - d_{t-1} > 0 \). Then by the induction hypothesis we must have \( A_{t-1}^S - d_{t-1} \geq 0 \), hence \( A_t^S = S - d_{t-1} \) and \( A_t^{S+1} = S + 1 - d_{t-1} \), i.e., \( A_t^{S+1} = A_t^S + 1 \).

Case (ii) \( A_{t-1}^{S+1} - d_{t-1} \leq 0 \). Then also \( A_{t-1}^S - d_{t-1} \leq 0 \) by the induction hypothesis, thus \( A_t^S = (S - A_{t-1}^S) \) and \( A_t^{S+1} = (S + 1 - A_{t-1}^{S+1}) \). The induction hypothesis and the latter two equations require \( A_t^S \leq A_t^{S+1} \leq A_t^S + 1 \) and the lemma is proved.

The next theorem lays the groundwork for one of our main convexity results.

**Theorem 2.2.** Let \( d = (d_1, d_2, \ldots, d_T) \), \( A_1^S = S \), and \( T \geq 2 \). Then for the single period lag problem

1. If \( A_t^{S+1} = A_t^S + 1 \), then stockouts were not reduced in period \( T-1 \) by using replenishment level \( S+1 \) instead of \( S \).
2. If \( A_t^{S+1} = A_t^S \), then stockouts were reduced by one unit in time period \( T-1 \) using replenishment level \( S+1 \) instead of \( S \).

**Proof.** For period \( T \) we consider three mutually exclusive and collectively exhaustive scenarios.

The first two scenarios relate to part (i) of the theorem, the last scenario relates to part (ii).

**Scenario 1.** \( A_{t-1}^{S+1} - d_{t-1} > 0 \), hence \( A_{t-1}^S - d_{t-1} \geq 0 \) (by lemma 2.1). In this case no shortages are experienced in period \( T-1 \), thus \( A_t^{S+1} = S + 1 - d_{t-1} \) and \( A_t^S = S - d_{t-1} \). Consequently, \( A_t^{S+1} = A_t^S + 1 \) and shortages are not reduced in period \( T-1 \).

**Scenario 2.** \( A_{t-1}^{S+1} - d_{t-1} \leq 0 \) and \( A_{t-1}^S = A_{t-1}^{S+1} \), hence \( A_t^{S+1} = A_t^S + 1 \) since there is no carryover for either replenishment level, but the equal on hand inventory in the previous period means that one additional unit is contained in the order \( (S + 1 - A_{t-1}^{S+1}) \) placed under the \( S+1 \) replenishment level.
policy. The assumption \( A_{r-1}^{S+1} = A_{r-1}^S \) implies that shortages in the previous period are not improved by the \( S+1 \) replenishment policy.

**Scenario 3.** \( A_{r-1}^{S+1} - d_{r-1} \leq 0 \) and \( A_{r-1}^{S+1} = A_{r-1}^S + 1 \). These conditions imply that shortages in period \( T-1 \) are reduced by exactly one additional unit under the \( S+1 \) policy. Moreover, under this scenario, \( A_{r}^{S+1} = A_{r}^S \). This follows from the fact that there is no carryover under either the \( S+1 \) or \( S \) policy, and the orders placed in period \( T-1 \) are of equal size, i.e., \( S+1 - A_{r-1}^{S+1} = S - A_{r-1}^S \).

We are now in a position to prove convexity of \( L_T(S|D) \) as a function of \( S \).

**Theorem 2.3.** Let \( d = (d_1, d_2, \ldots, d_T) \) and \( A_i^S = S \). Then the average conditional shortage cost

\[
L_T(S|D = d) = SC \cdot \frac{\sum_{t=1}^{T} \max\{d_t - A_t^S, 0\}}{T}
\]

assuming a one period lag is a monotone decreasing convex function of the replenishment level \( S \).

**Proof.** Monotonicity follows directly from Lemma 2.1. Convexity will be shown using induction on the number of time periods. We may assume \( SC = 1 \), in which case the terms “shortage cost” and “shortage” become synonymous. For \( t=1 \), it is easily seen that \( L_1(S|D = d) \) has the convex graph (shown in bold) in Figure 1 below.

<<Figure 1 Goes Approximately Here>>

Assume the result is true for all \( t=1,2,\ldots,T-1 \). The induction hypothesis can be stated equivalently as
\[
I \cdot \left\{ L_t(S|D = d) - L_{t+1}(S+1|D = d) \right\} \geq t \cdot \left\{ L_t(S+1|D = d) - L_{t+1}(S+2|D = d) \right\}
\]
\[
\text{for } t=1,2,\ldots,T-1.
\]
Note that the vector \(d\) is truncated in (1) as needed, i.e., for period \(t\), \(d = (d_1,d_2,\ldots,d_t)\). To prove the theorem, it suffices to prove the inequality
\[
T \cdot \left\{ L_T(S|D = d) - L_{T-1}(S+1|D = d) \right\} \geq T \cdot \left\{ L_T(S+1|D = d) - L_{T-1}(S+2|D = d) \right\}
\]
We first break the problem into four mutually exclusive and collectively exhaustive cases as summarized below:

<table>
<thead>
<tr>
<th>Case I</th>
<th>Case III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A^{S+1}_T = A^S_T) and (A^{S+2}_T = A^{S+1}_T)</td>
<td>(A^{S+1}_T = A^S_T + 1) and (A^{S+2}_T = A^{S+1}_T)</td>
</tr>
<tr>
<td>Case II</td>
<td>Case IV</td>
</tr>
<tr>
<td>(A^{S+1}_T = A^S_T) and (A^{S+2}_T = A^{S+1}_T + 1)</td>
<td>(A^{S+1}_T = A^S_T + 1) and (A^{S+2}_T = A^{S+1}_T + 1)</td>
</tr>
</tbody>
</table>

Observe that for any value of \(S\), \(T \cdot L_T(S|D = d)\) can be expressed as
\[
T \cdot L_T(S|D = d) = (T - 1) \cdot L_{T-1}(S|D = d) + \max\{d_t - A^S_t, 0\}. \tag{3}
\]
In a similar fashion, \(T \cdot L_T(S|D = d)\) can also be expressed as
\[
T \cdot L_T(S|D = d) = (T - 2) \cdot L_{T-2}(S|D = d) + \max\{d_{t-1} - A^{S}_{t-1}, 0\} + \max\{d_t - A^S_t, 0\}. \tag{4}
\]
**Case I** \(A^{S+2}_T = A^{S+1}_T = A^S_T\), thus the shortages incurred in period \(T\) are the same for stocking policies \(S\), \(S+1\), and \(S+2\). Applying (3) with \(S, S+1, S+2\) and taking successive differences yields
\[
T \cdot \left\{ L_T(S|D = d) - L_T(S+1|D = d) \right\} = (T - 1) \cdot \left\{ L_{T-1}(S|D = d) - L_{T-1}(S+1|D = d) \right\} \tag{5}
\]
and

\[ T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\} = (T-1) \cdot \left\{ L_{T-1}(S+1|D=d) - L_{T-1}(S+2|D=d) \right\}. \]  

(6)

It follows from (5), (6) and the induction hypothesis that

\[ T \cdot \left\{ L_T(S|D=d) - L_T(S+1|D=d) \right\} \geq T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\}, \]

which proves Case I.

**Case II** \( A_{T+1}^S = A_T^S \) and \( A_{T+1}^{S+2} = A_T^{S+1} + 1 \). By Theorem 2.2, \( A_T^{S+1} = A_T^S \) implies that shortages are reduced in period \( T-1 \) by one unit using policy \( S+1 \) instead of policy \( S \). Shortages in period \( T \) are unchanged. Applying (4) with \( S, S+1 \) and taking successive differences yields

\[ T \cdot \left\{ (L_T(S|D=d) - L_T(S+1|D=d) \right\} = (T-2) \left\{ L_{T-2}(S|D=d) - L_{T-2}(S+1|D=d) \right\} + 1 \]  

(7)

Also by Theorem 2.2, \( A_T^{S+2} = A_T^{S+1} + 1 \) implies that shortages are not reduced in period \( T-1 \) using \( S+2 \) instead of \( S+1 \). Shortages in the final period may be reduced by at most one unit when adopting an \( S+2 \) level instead of \( S+1 \). Applying (4) with \( S+1, S+2 \) and taking differences

\[ T \cdot \left\{ (L_T(S+1|D=d) - L_T(S+2|D=d) \right\} = (T-2) \left\{ L_{T-2}(S+1|D=d) - L_{T-2}(S+2|D=d) \right\} + \delta \]  

(8)

where \( \delta = 1 \) if there are shortages in period \( T \) that are improved using an \( S+2 \) replenishment level instead of \( S+1 \), and \( \delta = 0 \) otherwise. In either case (\( \delta = 0 \) or 1), it is clear from (7), (8) and the induction hypothesis for \( t=T-2 \) that

\[ T \cdot \left\{ L_T(S|D=d) - L_T(S+1|D=d) \right\} \geq T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\}, \]

which proves Case II.

**Case III** \( A_{T+1}^{S+1} = A_T^S + 1 \) and \( A_{T+1}^{S+2} = A_T^{S+1} \). The condition \( A_T^{S+2} = A_T^{S+1} \) requires

\[ T \cdot \left\{ L_T(S+1|D=d) - L_T(S+2|D=d) \right\} = (T-1) \cdot \left\{ L_{T-1}(S+1|D=d) - L_{T-1}(S+2|D=d) \right\} \]  

(9)

The condition \( A_T^{S+1} = A_T^S + 1 \) requires
\[ T \cdot \left\{ L_T(S \mid D = d) - L_T(S + 1 \mid D = d) \right\} \geq (T - 1) \cdot \left\{ L_{T-1}(S \mid D = d) - L_{T-1}(S + 1 \mid D = d) \right\} \]  

Equations (9), (10) and the induction hypothesis for \( t = T - 1 \) imply

\[ T \cdot \left\{ L_T(S \mid D = d) - L_T(S + 1 \mid D = d) \right\} \geq T \cdot \left\{ L_T(S + 1 \mid D = d) - L_T(S + 2 \mid D = d) \right\}, \]

which proves Case III.

**Case IV** \( A^S_{T+1} = A^S_T + 1 \) and \( A^{S+2}_T = A^{S+1}_T + 1 \). Applying (3) with \( S, S+1 \) and taking successive differences yields

\[ T \cdot \left\{ L_T(S \mid D = d) - L_T(S + 1 \mid D = d) \right\} = (T - 1) \cdot \left\{ L_{T-1}(S \mid D = d) - L_{T-1}(S + 1 \mid D = d) \right\} + \delta \]  

where \( \delta = 1 \) if there is a shortage in period \( T \) that is improved using an \( S+1 \) replenishment level instead of \( S \), \( \delta = 0 \) otherwise. In a similar fashion,

\[ T \cdot \left\{ L_T(S + 1 \mid D = d) - L_T(S + 2 \mid D = d) \right\} = (T - 1) \cdot \left\{ L_{T-1}(S + 1 \mid D = d) - L_{T-1}(S + 2 \mid D = d) \right\} + \gamma \]  

where \( \gamma = 1 \) if there is a shortage in period \( T \) that is improved using \( S+2 \) instead of \( S+1 \), \( \gamma = 0 \) otherwise. Observe that \( \gamma = 1 \) implies \( \delta = 1 \). Equations (11), (12) and the induction hypothesis once again require

\[ T \cdot \left\{ L_T(S \mid D = d) - L_T(S + 1 \mid D = d) \right\} \geq T \cdot \left\{ L_T(S + 1 \mid D = d) - L_T(S + 2 \mid D = d) \right\}, \]

which completes the proof of Case IV and Theorem 2.3.

One can now extend \( L_T(S \mid D = d) \) to a function for the \( T \)-period expected average shortage

\[ L_T(S) = SC \cdot \sum_d L_T(S \mid D = d) \cdot P(D = d). \]  

Observe that \( L_T(S) \) does not require independence of the random variables \( D_t \ (t=1,2,\ldots,T) \); the \( T \)-period expected average shortage is a finite nonnegative combination of convex functions and therefore convex.
Multi-period lags in delivery

The situation where orders placed in period \( t \) do not arrive until period \( t+k \) can be handled in a manner similar to that presented for one period lags. The complicating feature is the introduction of an additional variable to explicitly keep track of the outstanding orders. Assuming a single item for ease of exposition, let

\[
O_t^S = \text{The number of units ordered at the start of period } t \text{ to arrive for use at the start of period } t+k.
\]

With this additional variable we can prove the following analog to Lemma 2.1.

**Lemma 2.4.** Suppose \( d = (d_1, d_2, \ldots, d_T) \) and \( A_t^S = S \) with no outstanding orders. Then

\[
A_t^S \leq A_{t+1}^S \leq A_t^S + 1 \text{ and } O_t^S \leq O_{t+1}^S \leq O_t^S + 1 \text{ for all } t \leq T.
\]

**proof:** See Appendix.

The next theorem generalizes the result of Theorem 2.2.

**Theorem 2.5.** Let \( d = (d_1, d_2, \ldots, d_T) \), and \( T \geq 2 \). Then for the \( k \)-period (\( k \geq 2 \)) lag problem

(i) If \( A_{t+1}^S = A_t^S + 1 \), then total stockouts over the preceding \( k \) time periods were not reduced using replenishment level \( S+1 \) instead of \( S \).

(ii) If \( A_{t+1}^S = A_t^S \), then total stockouts over the preceding \( k \) time periods were reduced by one unit using replenishment level \( S+1 \) instead of \( S \).

**proof.** See Appendix.

The next theorem is the analog to Theorem 2.3.

**Theorem 2.6.** Let \( d = (d_1, d_2, \ldots, d_T) \), \( A_t^S = S \) with no outstanding orders. Then the average conditional loss function

\[
L_r(S|D = d) = SC \cdot \frac{\sum_{t=1}^{T} \max\{d_t - A_t^S, 0\}}{T}
\]
for an item with $k$-period delivery lag is a monotone decreasing convex function of the replenishment level $S$.

**proof.** See Appendix.

The shortage cost can be further extended to include multiple products. In this case we define the conditional shortage cost for item $i$ with replenishment level $S_i$ given demand vectors $d_1, d_2, \ldots, d_n$ is denoted by

$$L_{i,T}(S_i | D_1 = d_1, D_2 = d_2, \ldots, D_n = d_n)$$

where now $D_i = (D_{i,1}, D_{i,2}, \ldots, D_{i,T})$ and $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,T})$ for items $i = 1, 2, \ldots, n$. For the calculations in Lemma 2.1 through Theorem 2.6, only the single demand vector $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,T})$ is needed for the analysis of item $i$. However, all possible demand vectors are needed to form the expected average shortage cost over all $n$ items and $T$ time periods:

$$L_T(S_1, \ldots, S_n) = \sum_{d_1, d_2, \ldots, d_n} \sum_{i=1}^n SC_i \cdot L_{i,T}(S_i | D_1 = d_1, \ldots, D_n = d_n) \cdot P(D_1 = d_1, \ldots, D_n = d_n). \quad (14)$$

Observe that convexity is preserved when demands are correlated across products and time since (14) still represents the sum of convex functions. This result does not require the demand distribution to be stationary.

Calculating the expected average shortage cost using (13) or (14) presents a formidable combinatorial challenge when the demand distribution is completely known. When the demand distribution is not fully understood (e.g., demand is intermittent or erratic), a direct approach is more appropriate. One such approach is obtained by replacing the $d_i$ in Theorem 2.3 (respectively Theorem 2.6) with observed demands for item $i$. This creates a direct sample estimate of $L_{i,T}(S_i)$, denoted by $\hat{L}_{i,T}(S_i)$, which does not depend on an explicit parametric
structure for the demand distribution. Furthermore, since this estimate provides the exact loss that would have been incurred for each policy \( S_i \), \( \hat{L}_{i,r}(S_i) \) is the best fit for the historical data. Using sample data in this manner forms the basis of the direct approach advocated by Iyer and Schrage (1992).

Our direct estimate is illustrated in Table 3 using historical data on a 20hp motor over a 12-period \((T=12)\) horizon. The delivery lag for this item is one period, the shortage cost is \( SC = $100/\text{unit} \), and the carrying cost is \$5/\text{unit} \). Various replenishment levels \( S_i = 0, 1, 2, ..., 14 \) are tested to see how they would have performed had they been used over the 12 periods. The on hand inventory for the first period is assumed to equal the order-up-to level being tested (i.e., \( A_i^{S_i} = S_i \)).

As demonstrated above, the shortage cost incurred in a single period of the horizon is not necessarily convex (for example, shortages in periods 3, 9, 11 and 12 are nonconvex). However, Theorem 2.3 and Theorem 2.6 guarantee that \( L_{i,r}(S_i) \) is a convex function of \( S_i \) provided \( A_i^{S_i} = S_i \). The graph of \( L_{i,r}(S_i) \) for the 20hp motor used in Table 3 is shown in figure 2 below.

§3 A Direct Approach for Estimating Holding Costs with Lost Sales and Delivery Lags

In parallel to the case for shortage costs, a direct approach can be used to estimate the expected average holding cost. It is assumed that holding costs are assessed on all unsold units at the end of a period, and we will assume a single item for the sake of expositional clarity. Let
\( h_i^S = \max \{A_i^S - d_i, 0\} \) to denote the number of units carried from period \( t \) to period \( t+1 \) using the order-up-to policy \( S \). Like the case for shortages, the per-period holding costs are not necessarily convex in \( S \) (witness periods 4 and 10 in Table 3). However, letting \( HC \) denote the holding cost (per unit per period), the average conditional holding cost for replenishment level \( S \) given demand \( D = d \), defined as

\[
H_T(S | D = d) = HC \cdot \frac{\sum_{t=1}^{T} \max \{A_t^S - d_t, 0\}}{T},
\]

has the following convexity property.

**Theorem 3.1.** Assume \( A_t^S = S \) and a delivery lag of \( k (k \geq 1) \) periods. Then

\[
H_T(S | D = d) = HC \cdot \frac{\sum_{t=1}^{T} \max \{A_t^S - d_t, 0\}}{T}
\]

is a monotone increasing convex function of \( S \).

**Proof.** We will show equivalently that the total number of units carried over any \( T \)-period horizon satisfies

\[
T \{ H_T(S + 2 | D = d) - H_T(S + 1 | D = d) \} \geq T \{ H_T(S + 1 | D = d) - H_T(S | D = d) \}
\]

To do so, we first partition the \( T \)-period horizon into disjoint segments defined by the following rule: the condition \( A_t^S + 1 = A_t^{S+1}, A_t^{S+1} + 1 = A_t^{S+2} \) always defines the start of a new segment. The horizon must consist of one or more consecutive segments. We will show that each of these segments satisfies the equation

\[
\sum_{i=t_0}^{t} (h_i^{S+2} - h_i^{S+1}) \geq \sum_{i=t_0}^{t} (h_i^{S+1} - h_i^S),
\]
where \( t_0 \) is the first period of the segment, and \((t \geq t_0)\) is any subsequent period that is part of the current segment. If equation (16) holds for each segment, then it must hold for the entire horizon and the theorem will be proved.

Every segment falls into one of three mutually exclusive and collectively exhaustive cases, each defined by the demand incurred in its initial period.

**Case 1.** Initial demand satisfies \( d_{t_0} \leq A_{t_0}^S \). In this case, the segment is one period long, and both summands in (16) consist of a single 0.

**Case 2.** Initial demand satisfies \( d_{t_0} \geq A_{t_0}^{S+2} \). In this case, all three policies experience a stockout in period \( t_0 \), and the orders placed at the start of period \( t_0 + 1 \) satisfy \( O_{t_0+1}^{S+2} = O_{t_0+1}^{S+1} + 1, O_{t_0+1}^{S+2} = O_{t_0+1}^{S} + 1 \). The remaining (outstanding) orders for each order-up-to level must satisfy \( O_{t}^{S+2} = O_{t}^{S+1} = O_{i}^{S} (t_0 - k + 2 \leq i \leq t_0) \) due to lemma 2.4 and the general relationship \( A_{t}^{S} + O_{t}^{S} + O_{t-1}^{S} + \cdots + O_{t-k+1}^{S} = S \) for all \( t, S \). This means that the period ending inventory levels for periods \( t_0, t_0 + 1, \ldots, t_0 + k \) are identical, thus the holding costs charged at the end of each period in the segment are identical using \( S, S+1, \) and \( S+2 \). The segment concludes at the end of period \( t_0 + k \), and the summands in (16) consist entirely of 0's. This demonstrates the truth of (16) for case 2.

**Case 3.** Initial demand satisfies \( d_{t_0} = A_{t_0}^{S+1} \), and the length of the segment is indeterminate (except it must consist of at least \( k+1 \) periods). In this case, a stockout in period \( t_0 \) occurs for policies \( S \) and \( S+1 \), but not for policy \( S+2 \). The segment in this case takes on the appearance of a "linked chain" like that shown in Figure 3 below:

**Figure 3.** A Segment ("Linked Chain") with Four Links.

<< Figure 3 Goes Approximately Here >>
The start of an odd link in period \( t \) is determined by \( O_{t+1}^{S+1} = O_t^S + 1 \) and \( O_{t+2}^{S+2} = O_t^{S+1} + 1 \); the start of an even link is determined by \( O_{t+1}^{S+1} = O_t^S \) and \( O_{t+2}^{S+2} = O_t^{S+1} + 1 \). All links are \( k \) periods long, and all periods except for \( t_0 \) are part of a link. A "+" sign indicates a period where \( h_t^{S+2} = h_t^{S+1} + 1 \) and \( h_t^{S+1} = h_t^S \); a "-" sign indicates a period where \( h_t^{S+2} = h_t^{S+1} \) and \( h_t^{S+1} = h_t^S + 1 \); an "=" sign indicates a period where either \( h_t^{S+2} = h_t^{S+1} = h_t^S \) or \( h_{t-1}^{S+2} = h_{t-1}^{S+1} + 1 \) and \( h_{t-1}^{S+1} = h_{t-1}^S + 1 \). For example, figure 3 documents the following history: at time period \( t_0 = a_1 - 1 \), \( A_{t+1}^{S+1} = A_{t+1}^S + 1 \), \( A_{t+1}^{S+2} = A_{t+1}^S + 1 \), all outstanding orders are equal, and \( d_{t+1} = A_{t+1}^{S+1} \). At time period \( t = a_1 \), the stockouts incurred under policies \( S \) and \( S+1 \) (but not \( S+2 \)) in period \( t_0 = a_1 - 1 \) prompt orders \( O_{t+1}^{S+2} = O_{t+1}^{S+1} \), \( O_{t+1}^{S+1} = O_{t+1}^S + 1 \) which will not arrive until the start of period \( t = a_1 + k \). A subsequent stockout in period \( t = a_2 - 1 \) \( (a_2 - 1 \leq a_1 + k - 1) \) caused by demand \( d_{a_1 - 1} \geq A_{a_1 - 1}^{S+2} \) means no units are carried over from this period for any of the three policies \( S, S+1, S+2 \). This is what perpetuates the chain; if a subsequent stockout does not occur in (or prior to) period \( a_1 + k - 1 \), the chain (and hence the segment) would end at period \( a_1 + k - 1 \) (since \( A_{a_1 + k}^{S+2} = A_{a_1 + k}^{S+1} + 1 \), \( A_{a_1 + k}^{S+1} = A_{a_1 + k}^S + 1 \), and a new segment would begin at \( a_1 + k \)). The chain continues until either the end of the horizon is reached or demand satisfies \( d_t \leq A_t^S \) for \( t \in [a_j + k, a_{j+1} + k - 1] \). The latter occurrence means \( A_{a_1 + k}^{S+2} = A_{a_1 + k}^{S+1} + 1 \) and \( A_{a_1 + k}^{S+1} = A_{a_1 + k}^S + 1 \), thus defining the start of a new segment at period \( a_{j+1} + k \). In either case, to show that equation (16) is true for this segment, it suffices to show that there are more +'s than -'s over each subinterval whose left hand endpoint is \( t = a_1 - 1 \). Moreover, it suffices to examine only subintervals whose right hand endpoint is the last period in a subchain with an even number
of links; these are the subintervals of the segment which minimize the disparity between total +'s and total -'s. The bookkeeping for a subchain with an even number of links is given below:

<table>
<thead>
<tr>
<th></th>
<th>+'s</th>
<th>-'s</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a_2-a_1)</td>
<td>(a_3-a_1-k-1)</td>
</tr>
<tr>
<td></td>
<td>(a_4-a_2-k-1)</td>
<td>(a_5-a_3-k-1)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(a_{2j}-a_{2j-2}-k-1)</td>
<td>(a_{2j+1}-a_{2j-1}-k)</td>
</tr>
<tr>
<td>TOTALS</td>
<td>(a_{2j}-a_1-(j-1)(k+1))</td>
<td>(a_{2j+1}-a_1-(j-1)(k+1)-k)</td>
</tr>
</tbody>
</table>

We need only show that \(a_{2j}-a_1-(j-1)(k+1) \geq a_{2j+1}-a_1-(j-1)(k+1)-k\), or equivalently, \(a_{2j+1} \leq a_{2j}+k\). But the latter must be true or else the chain is broken (i.e., our current segment has ended). This proves the validity of equation (16) for case 3.

The conditional average holding cost used above can be extended to the \(T\)-period expected average holding cost by analogy with (13), and the expected average holding cost for multiple products by analogy with (14). Therefore, the theoretical function describing expected average holding costs is a convex function of the \(S_i\) provided all items are initially stocked at their order-up-to levels. Replacing the \(d_i\)'s in Theorem 3.1 with observed demands over a \(T\)-period horizon produces a nonparametric estimate which is likewise convex and consistent with the observed data. Like the case for shortage costs, these results hold even when demands are correlated or the underlying distribution is nonstationary.
§4 An Illustration: Consignment Systems for Intermittent Demand

In this section we show how the nonparametric methods developed in the previous two sections can be used to compute order-up-to levels for the problem outlined in the introduction.

A direct consequence of the intermittent and erratic demand is that the distributor holds inventory on consignment, with the manufacturer (consignor) paying all incremental holding and ordering costs. As Culgin (1996) points out, this arrangement is gaining popularity as manufacturers try to encourage distributors to carry slow-moving, capital-intensive inventory items. The distributor receives a fixed percentage commission for each item sold (a sales fee), and an additional fixed percentage if the order can be filled from the distributor's stock (a warehousing fee). If the item is not on hand, the distributor will forgo the warehousing fee and the order will be shipped from the factory or another distributor. Thus the cost of a lost sale is easy to estimate, and the distributor has a significant interest in providing a high level of customer service.

In return for consignment, the distributor provides storage capacity and must carry only the manufacturer's product lines. The distributor incurs a fixed charge for acquiring adequate capacity (usually in the form of a leased warehouse) to meet peak storage requirements. It is the peak storage capacity (or its cost) that constrains the distributor's stocking decisions.

Although the incremental costs of acquiring additional capacity are easily incorporated into the model below, for simplicity we will assume that the distributor's storage capacity is fixed (a sunk cost). Therefore, the distributor's sole objective is to minimize the cost of shortages. For further expositional clarity, we will consider an abbreviated list of ten motors ($i=1, 2, ..., 10$) carried by the distributor during 1995-1996. These ten motors were among the top selling motors over that interval, and in the context of this short-list the distributor's objective can be restated as follows: determine order-up-to levels ($S_1, S_2, ..., S_{10}$) for items
that minimize the average cost of shortages subject to available capacity. Using the results of section 2, the objective function is approximated by the direct estimate

\[
\sum_{i=1}^{10} \hat{L}_{i,T}(S_i), \tag{OF}
\]

which is separable, piecewise linear and convex. Therefore, each \( \hat{L}_{i,T}(S_i) \) can be replaced by its separable programming equivalent (Charnes and Lemke (1954))

\[
\hat{L}_{i,T}(S_i) = \hat{L}_{i,T}(0) + \sum_{j=1}^{M_i} (\hat{L}_{i,T}(j) - \hat{L}_{i,T}(j-1)) S_y,
\]

where \( S_i = \sum_{j=1}^{M} S_y, \quad 0 \leq S_y \leq 1. \)

\( S_y \) measures the marginal change in \( \hat{L}_{i,T}(S_i) \) when raising \( S_i \) from \( j-1 \) to \( j \). The constant \( M_i \) is the smallest integer such that losses are zero for \( S_i \geq M_i \). For example, the motor summarized in Table 3 would use \( M_i = 14. \)

The first 26 weeks of sales from 1996 (\( T=26 \)) are used to construct the direct estimate \( \hat{L}_{i,T}(S_i) \) for \( i=1,\ldots,10 \). Sales figures include drop shipments made to customers when a motor was not in stock (hence resulting in a lost warehousing fee). Motors 2 through 10 experience a one week lag in delivery, motor 1 experiences a two week lag. The data are shown in Table 4 below.

<<Table 4 Goes Approximately Here>>

Multiple periods of low aggregate demand allow the on-hand inventory to approach the order-up-to levels. Consequently, our first constraint requires the total inventory on-hand and on-order to be no greater than the distributor's available storage capacity.
\[
\sum_{i=1}^{10} C_i^{(k)} y_i^{(k)} \leq C^{(k)} \quad \text{for } k = 1,2.
\]

Here, \( k \) indexes the different types of capacity available, floor (\( k=1 \)) and shelves (\( k=2 \)); \( C^{(k)} \) is the amount of type \( k \) capacity available; \( C_i^{(k)} \) is the amount of type \( k \) capacity required for one unit of item \( i \) (\( i=1,\ldots,10 \)); and \( y_i^{(k)} \geq 0 \) are allocation variables that record how many units of item \( i \) are allocated to type \( k \) capacity. Note that the \( y_i^{(k)} \) must therefore satisfy the linear equation

\[
S_i - \sum_{k=1,2} y_i^{(k)} = 0.
\]

If warehouse sizing and selection is an appropriate objective, then \( C^{(k)} \) is treated as a decision variable, and the objective function in (OF) is modified accordingly.

Motors come in one of ten standardized sizes. The smaller motors (less than 225 lbs.) come in boxes that can be shelved two or three deep and stacked three high. The larger motors (225 lbs. and up) come bolted to pallets, most of which can be shelved (one deep, no stacking). The largest motors (1000 lbs. and up) cannot be shelved and must be stored on the floor. Each motor's capacity requirement is given in terms of linear space and summarized in Table 5 below (NA means the motor must go on the floor).

<<Table 5 Goes Approximately Here>>

We assumed the distributor had one section of floor space (\( C^{(1)} = 120 \) in.) and two sections of shelving (\( C^{(2)} = 240 \) in.) to suit the abbreviated list of items used in our illustration. The linear programming formulation for the fixed-capacity stocking problem
Min \[ \sum_{i=1}^{10} \left[ L_i(0) + \sum_{j=1}^{M_i} (L_i(j) - L_i(j-1)S_y) \right] \]

s.t.

\[ \sum_{i=1}^{10} C_i^{(1)}y_i^{(1)} \leq C^{(1)} \] (floor space)

\[ \sum_{i=2}^{10} C_i^{(2)}y_i^{(2)} \leq C^{(2)} \] (shelf space)

\[ \sum_{j=1}^{M_i} S_y = y_i^{(1)} \quad i = 1 \]

\[ \sum_{j=1}^{M_i} S_y = y_i^{(1)} + y_i^{(2)} \quad i = 2, 3, \ldots, 10 \]

\[ 0 \leq S_y \leq 1, \quad y_i^{(1)}, y_i^{(2)} \geq 0. \]

The optimal stocking levels \( S_y^* \) \((i=1,\ldots,10)\), complete with their allocations to the different storage types \((y_i^{(k)} \quad k=1,2)\), are summarized in Table 6 below.

<< Table 6 Goes Approximately Here >>

With the exception of the large palletized motor (motor #1), all models are stocked. This is consistent with practical advice given to the distributor from other consignees in the motor distribution business. Although these stock levels would be used for the forthcoming period(s), it is interesting to note that in this example the warehousing fees accumulated ($2913.58) represent a little over 60% of the warehousing fees potentially available ($4814.35) for the items and periods analyzed. Not surprisingly, the dual multipliers for both types of shelving are $4.51 per inch, and greater capacity appears to be warranted.

In practice, these stocking decisions would be updated periodically (perhaps monthly) using a rolling horizon. The length of the horizon and the frequency with which updates are needed would be determined from practical experience. Computational results from real
problems having approximately 200-300 motors (approximately 2000 variables) suggest the entire procedure (cost estimation, LP generation, and solution) can be solved in about 20 seconds on a personal computer.

§5 Conclusions

We have presented a nonparametric approach that is useful in situations involving lumpy demand. Assuming an order-up-to-$S$ control policy, lost sales and delivery lags, we have demonstrated that the expected average shortage cost function and the expected average holding cost are convex functions of $S$ provided a simple initial condition holds. We develop closed-form nonparametric estimates of these functions that are easily computed and structurally consistent with the theoretical forms they approximate. The practical importance of these results is illustrated for a motor distribution system with resource constraints.

A somewhat lengthy list of new issues remain to be explored. Foremost among these is the accuracy of our nonparametric estimates relative to their true forms. Some convergence results for large $T$ or some other type of sensitivity analysis would be beneficial, but this appears to be an extremely difficult open question even for the case of full backlogging (for a detailed discussion, see Iyer and Schrage (1992)). Other possible topics include: using the model in conjunction with forecasts of future demands; using the model to calculate appropriate storage capacities (i.e., warehouse sizing and selection); devising more sophisticated computing informatics for large scale inventory systems with resource constraints; investigating the model under more specialized distributional assumptions. These topics are currently being pursued by the authors.
References


**TABLE 1.** Shortages as a function of $S$.

<table>
<thead>
<tr>
<th>Order-up-to Level</th>
<th>Per. 1</th>
<th>Per. 2</th>
<th>Per. 3</th>
<th>Per. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S=0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial On Hand Inventory</td>
<td>1 0 0 0</td>
<td>0 0 0 0</td>
<td></td>
<td>Avg.</td>
</tr>
<tr>
<td>Orders</td>
<td>0 0 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shortages</td>
<td>0 1 1 1</td>
<td>1</td>
<td></td>
<td>.75</td>
</tr>
<tr>
<td>$S=1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial On Hand Inventory</td>
<td>1 0 1 0</td>
<td>0 1 0 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orders</td>
<td>0 1 0 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shortages</td>
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<td></td>
<td>.5</td>
</tr>
<tr>
<td>$S=2$</td>
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<td></td>
</tr>
<tr>
<td>Initial On Hand Inventory</td>
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<td>1 1 1 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Orders</td>
<td>1 1 1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shortages</td>
<td>0 0 0 0</td>
<td>0 0 0 0</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

**TABLE 2.** Shortages as a function of $S$ (initial on hand inventory at order-up-to level).

<table>
<thead>
<tr>
<th>Order-up-to Level</th>
<th>Per. 1</th>
<th>Per. 2</th>
<th>Per. 3</th>
<th>Per. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S=0$</td>
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<td></td>
</tr>
<tr>
<td>Initial On Hand Inventory</td>
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<td>0 0 0 0</td>
<td></td>
<td>Avg.</td>
</tr>
<tr>
<td>Orders</td>
<td>0 0 0 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shortages</td>
<td>1 1 1 1</td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$S=1$</td>
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</tr>
<tr>
<td>Initial On Hand Inventory</td>
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<td>0 1 0 1</td>
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<td>Orders</td>
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<td>Shortages</td>
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<td>.5</td>
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<td>$S=2$</td>
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<td></td>
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<tr>
<td>Shortages</td>
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\[\begin{array}{cccccccccccccc}
& d_1 & d_2 & d_3 & d_4 & d_5 & d_6 & d_7 & d_8 & d_9 & d_{10} & d_{11} & d_{12} & \text{Avg. Short. Cost} & \text{Avg. Hold Cost} \\
0 & 0 & 4 & 10 & 1 & 0 & 6 & 0 & 2 & 3 & 1 & 3 & 6 & 300 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0.416666667 & 0 \\
2 & 2 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 1 & 216.6666667 & 0.416666667 \\
3 & 3 & 0 & 3 & 2 & 3 & 0 & 3 & 1 & 2 & 2 & 1 & 2 & 183.3333333 & 2.083333333 \\
4 & 4 & 0 & 4 & 3 & 4 & 0 & 4 & 2 & 2 & 3 & 1 & 3 & 150 & 5.416666667 \\
5 & 5 & 1 & 4 & 4 & 5 & 0 & 5 & 3 & 2 & 4 & 2 & 2 & 116.6666667 & 7.5 \\
6 & 6 & 2 & 4 & 5 & 6 & 0 & 6 & 4 & 3 & 5 & 3 & 3 & 91.666666667 & 10.416666667 \\
7 & 7 & 3 & 4 & 6 & 7 & 1 & 7 & 5 & 4 & 6 & 4 & 4 & 75 & 14.166666667 \\
8 & 8 & 4 & 4 & 7 & 8 & 2 & 8 & 6 & 5 & 7 & 5 & 5 & 58.333333333 & 17.916666667 \\
9 & 9 & 5 & 4 & 8 & 9 & 3 & 9 & 7 & 6 & 8 & 6 & 6 & 41.666666667 & 21.666666667 \\
10 & 10 & 6 & 4 & 9 & 10 & 4 & 10 & 8 & 7 & 9 & 7 & 7 & 33.333333333 & 25.833333333 \\
12 & 12 & 8 & 4 & 11 & 12 & 6 & 12 & 10 & 9 & 11 & 9 & 9 & 16.666666667 & 34.166666667 \\
13 & 13 & 9 & 4 & 12 & 13 & 7 & 13 & 11 & 10 & 12 & 10 & 0 & 8.333333333 & 38.333333333 \\
14 & 14 & 10 & 4 & 13 & 14 & 8 & 14 & 12 & 11 & 13 & 11 & 1 & 0 & 42.5 \\
\end{array} \]
Figure 1. $L_1(S|D = d)$ as a function of $S$.

Figure 2. Estimated shortage costs.
### TABLE 4. 26 Weeks of Sales

| ITEM ↓ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  | 25  | 26  |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1      | 0   | 0   | 0   | 0   | 0   | 3   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 2      | 0   | 0   | 0   | 0   | 1   | 0   | 0   | 5   | 12  | 0   | 0   | 11  | 4   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 12  |
| 3      | 0   | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 40  | 0   | 40  | 0   | 0   | 0   | 0   | 16  | 2   | 0   | 4   | 0   | 0   | 0   | 0   |
| 4      | 0   | 0   | 4   | 0   | 2   | 7   | 1   | 0   | 0   | 1   | 0   | 14  | 0   | 0   | 1   | 0   | 1   | 0   | 0   | 0   | 2   | 7   | 1   | 2   |
| 5      | 0   | 0   | 4   | 10  | 1   | 0   | 6   | 0   | 2   | 3   | 1   | 3   | 6   | 0   | 3   | 3   | 3   | 5   | 1   | 0   | 2   | 0   | 3   | 5   |
| 6      | 0   | 1   | 4   | 2   | 20  | 2   | 1   | 3   | 1   | 3   | 1   | 4   | 1   | 2   | 2   | 2   | 1   | 3   | 0   | 2   | 0   | 1   | 3   | 7   | 3   | 5   |
| 7      | 0   | 4   | 4   | 0   | 10  | 4   | 0   | 9   | 5   | 2   | 3   | 6   | 6   | 7   | 8   | 2   | 6   | 1   | 6   | 1   | 1   | 3   | 4   | 2   | 6   | 4   |
| 8      | 25  | 4   | 0   | 0   | 1   | 13  | 0   | 29  | 0   | 1   | 0   | 20  | 0   | 1   | 1   | 0   | 13  | 0   | 0   | 0   | 2   | 7   | 1   | 0   | 4   |
| 9      | 0   | 0   | 4   | 2   | 0   | 0   | 0   | 16  | 0   | 2   | 1   | 2   | 4   | 1   | 0   | 0   | 4   | 0   | 0   | 0   | 1   | 2   | 0   | 5   | 0   |
| 10     | 0   | 0   | 0   | 2   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 0   | 2   | 0   | 2   | 0   | 2   | 0   | 4   | 9   | 0   | 0   |

### TABLE 5. Capacity requirements $C_i^{(k)}$ (in inches).

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<tr>
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<td>4.7</td>
<td>4.7</td>
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<td>2.4</td>
<td>2.1</td>
<td>2.1</td>
<td>1.4</td>
<td>1.4</td>
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<tr>
<td>Shelving</td>
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<td>4.7</td>
<td>4.7</td>
<td>2.4</td>
<td>2.4</td>
<td>2.1</td>
<td>2.1</td>
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<td>$433</td>
<td>$482</td>
<td>$349</td>
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<td>$189</td>
<td>$147</td>
<td>$129</td>
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<td>$258</td>
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### TABLE 6 Stocking results, $S_i^r$

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<td>Shelving</td>
<td>$y_i^{(2)r}$</td>
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<td>13.25</td>
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<td>8</td>
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<tr>
<td>Repl. Level</td>
<td>$S_i^r$</td>
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<td>32.52</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>14</td>
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Figure 3. A Segment ("Linked Chain") with Four Links

\[
\begin{align*}
O_{a_1}^{S+2} &= O_{a_1}^{S+1} \\
O_{a_1}^{S+1} &= O_{a_1}^S + 1 \\
O_{a_2}^{S+2} &= O_{a_2}^{S+1} + 1 \\
O_{a_2}^{S+1} &= O_{a_2}^S \\
O_{a_2}^{S+2} &= O_{a_2}^{S+1} + 1 \\
O_{a_2}^{S+1} &= O_{a_2}^S \\
O_{a_3}^{S+2} &= O_{a_3}^{S+1} \\
O_{a_3}^{S+1} &= O_{a_3}^S + 1 \\
O_{a_3}^{S+2} &= O_{a_3}^{S+1} + 1 \\
O_{a_3}^{S+1} &= O_{a_3}^S \\
O_{a_4}^{S+2} &= O_{a_4}^{S+1} + 1 \\
O_{a_4}^{S+1} &= O_{a_4}^S \\
O_{a_4}^{S+2} &= O_{a_4}^{S+1} + 1 \\
O_{a_4}^{S+1} &= O_{a_4}^S \\
\end{align*}
\]
Appendix

Proofs of Lemma 2.4, Theorem 2.5 and Theorem 2.6

Proof of Lemma 2.4. The proof is once again by induction on T. The case for T=1 is straightforward and therefore omitted. Assume the truth of the proposition for all t ≤ T−1 (T ≥ 2). Before proceeding with the general induction step, observe that for general t

\[ A_i^s = O_{i-k}^s + \text{Max}\{A_{i-1}^s - d_{i-1}, 0\} \quad \text{(A1)} \]

and

\[ O_i^s = S - [A_i^s + O_{i-1}^s + \cdots + O_{i-k+1}^s], \quad \text{(A2)} \]

where we define \( O_{i-k}^s = 0 \) if \( t - k < 0 \). Taking successive differences between \( S+1 \) and \( S \) in (A2) yields

\[ O_i^{s+1} - O_i^s = 1 + (A_i^s - A_{i-1}^{s+1}) + (O_{i-1}^s - O_{i-1}^{s+1}) + \cdots + (O_{i-k+1}^s - O_{i-k+1}^{s+1}). \quad \text{(A3)} \]

Applying (A3) in the particular instance \( t = T-k \) results in

\[ O_{T-k}^{s+1} - O_{T-k}^s = 1 + (A_{T-k}^s - A_{T-k-1}^{s+1}) + (O_{T-k-1}^s - O_{T-k-1}^{s+1}) + \cdots + (O_{T-2k+1}^s - O_{T-2k+1}^{s+1}). \quad \text{(A4)} \]

The proof is divided into two cases, one of which has two subcases.

Case 1. \( O_{T-k}^{s+1} = O_{T-k}^s + 1 \). By the induction hypothesis, \( A_{T-k}^s \leq A_{T-k}^{s+1} \leq A_{T-k}^s + 1 \) and \( O_t^s \leq O_t^{s+1} \leq O_t^s + 1 \) for all \( t \leq T-1 \). This ensures that each term enclosed by parentheses in (A4) is nonpositive. Consequently, \( O_{T-k}^{s+1} = O_{T-k}^s + 1 \) can occur if and only if the following system of equalities hold in (A4):

\[ A_{T-k}^s = A_{T-k}^{s+1}, \quad O_{T-k-1}^s = O_{T-k-1}^{s+1}, \ldots, \quad O_{T-2k+1}^s = O_{T-2k+1}^{s+1}. \quad \text{(A5)} \]

The recursion (A1) and equation (A5) then imply the following sequence of beginning period inventory levels:

\[ A_{T-k+1}^s = A_{T-k+1}^{s+1}, \ldots, \quad A_{T-1}^s = A_{T-1}^{s+1}. \quad \text{(A6)} \]

Consequently, \( A_{T-k+1}^s = O_{T-k+1}^{s+1} + \text{Max}\{A_{T-k+1}^{s+1} - d_{T-k+1}, 0\} = A_{T-k+1}^s \).

We now show that \( O_T^s = O_T^{s+1} \) as well. Consider (A3) for \( t = T-1, T-2, \ldots, T-k+1 \). The induction hypothesis ensures that each of the terms on the right hand side of equation (A3) is nonpositive. Moreover, \((O_{T-k}^s - O_{T-k}^{s+1}) = -1 \) by the assumption for Case 1, and this term appears on the right hand side of (A3) for each \( t = T-1, T-2, \ldots, T-k+1 \). This, combined with the conditions in (A6), forces the right hand side of (A3) to be less than or equal to zero. It cannot be
negative since \( O_t^S \leq O_{t+1}^S \leq O_t^S + 1 \) for all \( t \leq T-1 \) by the induction hypothesis, so the following equalities must occur

\[
O_{T-1}^S = O_{T-1}^{S+1}, \quad O_{T-2}^S = O_{T-2}^{S+1}, \ldots, \quad O_{T-k+1}^S = O_{T-k+1}^{S+1}.
\]

Finally,

\[
O_t^{S+1} - O_t^S = 1 + (A_t^S - A_t^{S+1}) + (O_{t-1}^S - O_{t-1}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}) = 1 - 1 = 0.
\]

Case 2. \( O_{T-k}^{S+1} = O_{T-k}^S \). Then it follows immediately from the induction hypothesis and (A1) that

\( A_t^S \leq A_{T-k}^S \leq A_t^S + 1 \). It remains to show that \( O_t^S \leq O_{T-k}^{S+1} \leq O_t^S + 1 \). The latter is done by dividing Case 2 into two subcases: (Subcase I) \( O_{T-k}^{S+1} = O_{T-k}^S \) and \( O_{T-k}^S = O_{T-k}^S \); (Subcase II) \( O_{T-k}^{S+1} = O_{T-k}^S \) and \( O_{T-k}^S = O_{T-k}^S + 1 \).

Subcase I. Apply (A3) for period \( t = T-1 \) to obtain

\[
O_{T-1}^{S+1} - O_{T-1}^S = 1 + (A_{T-1}^S - A_{T-1}^{S+1}) + (O_{T-2}^S - O_{T-2}^{S+1}) + \cdots + (O_{T-k+1}^S - O_{T-k+1}^{S+1}). \tag{A7}
\]

Since \( O_{T-k}^{S+1} = O_{T-k}^S \), (A7) is equal to 0, which creates one of two possibilities: (a) \( A_{T-1}^S = A_{T-1}^S + 1 \) and \( O_{T-j}^S = O_{T-j}^S \) for \( j = 2, \ldots, k \); or (b) \( A_{T-k}^S = A_{T-k}^S \) and \( O_{T-j}^S = O_{T-j}^S \) (\( 2 \leq j \leq T-k \)) except for one fixed index \( m (2 \leq m \leq T-k+1) \) where \( O_{T-m}^S = O_{T-m}^S + 1 \). For possibility (a) we have

\[
O_{T-k}^S - O_{T-k-1}^S = 1 + (A_{T-k-1}^S - A_{T-k-1}^{S+1}) + (O_{T-k-2}^S - O_{T-k-2}^{S+1}) + \cdots + (O_{T-1}^S - O_{T-1}^{S+1})
\]

\[
= 1 + (A_{T-k-1}^S - A_{T-k-1}^{S+1}).
\]

For possibility (b), \( A_{T-k}^S = A_{T-k}^S \) implies \( A_{T-k}^S = A_{T-k}^S \) since \( O_{T-k}^{S+1} = O_{T-k}^S \) by the assumption for Case 2. Then

\[
O_{T-k}^S - O_{T-k-1}^S = 1 + (A_{T-k-1}^S - A_{T-k-1}^{S+1}) + (O_{T-k-2}^S - O_{T-k-2}^{S+1}) + \cdots + (O_{T-1}^S - O_{T-1}^{S+1})
\]

\[
= 1 + (O_{T-m}^S - O_{T-m}^{S+1}) = 1 - 1 = 0.
\]

This completes the proof of Subcase I.

Subcase II. By the assumptions of this case, \( O_{T-k}^{S+1} = O_{T-k}^S \) and \( O_{T-1}^S = O_{T-1}^S + 1 \). In this case the left hand side of equation (A7) is equal to 1, which forces all of the (nonpositive) terms in parentheses on the right hand side to be 0. As in Subcase I, \( A_{T-k}^S = A_{T-k}^S \) implies \( A_{T-k}^S = A_{T-k}^S \) from the assumption \( O_{T-k}^{S+1} = O_{T-k}^S \) of Case 2, and it follows that

\[
O_{T-k}^S - O_{T-k-1}^S = 1 + (A_{T-k-1}^S - A_{T-k-1}^{S+1}) + (O_{T-k-2}^S - O_{T-k-2}^{S+1}) + \cdots + (O_{T-1}^S - O_{T-1}^{S+1})
\]

\[
= 1 + (O_{T-1}^S - O_{T-1}^{S+1}) = 1 - 1 = 0.
\]
This completes the proof of Subcase II, Case 2, and Lemma 2.5.

**Proof of Theorem 2.5.** In period $T-k$, the following two equations must hold:

$$A_{T-k}^S + O_{T-k}^S + O_{T-k-1}^S + \ldots + O_{T-2k+1}^S = S$$

(A8)

$$A_{T-k}^{S+1} + O_{T-k}^{S+1} + O_{T-k-1}^{S+1} + \ldots + O_{T-2k+1}^{S+1} = S + 1$$

(A9)

These equations simply state that the on-hand inventory plus all outstanding orders (including the one made at the start of a period) must sum up to the replenishment level.

To prove part (i), observe that $A_{T}^S = A_T^S + 1$ can occur in one of two ways: (Case 1) the period ending inventory levels at time $T-1$ are equal and $O_{T-k}^{S+1} = O_{T-k}^S + 1$; or (Case 2) the period ending inventory levels at time period $T-1$ are unequal and $O_{T-k}^{S+1} = O_{T-k}^S$.

**Case 1.** The total amount of on-hand inventory available over periods $T-k, T-k+1, \ldots, T-1$ under the order up to $S$ policy is

$$A_{T-k}^S + O_{T-k-1}^S + \ldots + O_{T-2k+1}^S = S - O_{T-k}^S.$$  

(A10)

Under the $S+1$ policy, the total amount of on-hand inventory available over periods $T-k, T-k+1, \ldots, T-1$ is

$$A_{T-k}^{S+1} + O_{T-k-1}^{S+1} + \ldots + O_{T-2k+1}^{S+1} = S + 1 - O_{T-k}^{S+1}.$$  

(A11)

The amounts expressed in (A10) and (A11) are identical since $O_{T-k}^{S+1} = O_{T-k}^S + 1$. Because the period ending inventory levels are the same under both policies, equal amounts of inventory were moved over the periods $T-k, T-k+1, \ldots, T-1$. This ensures that stockouts were not improved by the $S+1$ policy over the $k$ periods preceding period $T$.

**Case 2.** A comparison of equations (A10) and (A11) reveals that the total on-hand inventory available over periods $T-k, T-k+1, \ldots, T-1$ is one unit greater under the $S+1$ policy. However, the additional unit is unused since the period ending inventory levels for period $T-1$ are assumed to be unequal (i.e., the $S+1$ policy has an additional unit which it carries over to period $T$). This ensures that stockouts were not improved by the $S+1$ policy over the $k$ periods preceding period $T$. This completes the proof of part (i) of the theorem.

To prove part (ii), observe that the condition $A_{T}^{S+1} = A_T^S$ can occur in precisely one way: the period ending inventory levels are the same under both replenishment policies and $O_{T-k}^{S+1} = O_{T-k}^S$. Equations (A10) and (A11) still apply, and the amount expressed in (A10) is one unit less than that expressed in (A11). The equal period ending inventories for period $T-1$ means that one additional unit was moved during periods $T-k, T-k+1, \ldots, T-1$ under the $S+1$ policy.
Consequently, stockouts were improved by precisely one unit over the \( k \) periods immediately preceding period \( T \).

**Proof of Theorem 2.6.** The proof is by induction on \( T \) and parallels that of Theorem 2.3. The assumption \( WF = 1 \) is used as before without loss of generality.

The case \( T=1 \) is again trivial, and we assume the truth of the theorem for \( t=1, \ldots, T-1 \). The problem is divided into the same four cases used in the proof of Theorem 2.3. However, the proofs of Case I, Case III and Case IV do not need to be repeated since they depend solely on an analysis of period beginning inventory levels in period \( T \). Consequently, only Case II, which involves results on lagged delivery times, needs to be redone.

**Case II.** \( A_{T}^{S+1} = A_{T}^{S} \) and \( A_{T}^{S+2} = A_{T}^{S+1} + 1 \). By Theorem 2.5, \( A_{T}^{S+1} = A_{T}^{S} \) implies that shortages are reduced in the preceding \( k \) time periods by one unit using policy \( S+1 \) instead of policy \( S \). Shortages in period \( T \) are unchanged. The difference in total shortages over \( T \) periods can be expressed as

\[
T \cdot \{(L_{T}(S|D=d) - L_{T}(S + 1|D=d)) = (T-k-1)\{L_{T-k-1}(S|D=d) - L_{T-k-1}(S + 1|D=d)\} + 1
\]

Also by Theorem 2.5, \( A_{T}^{S+2} = A_{T}^{S+1} + 1 \) implies that shortages are not reduced in the preceding \( k \) periods using \( S+2 \) instead of \( S+1 \). Shortages in the final period can be reduced by at most one unit using an \( S+2 \) replenishment level instead of \( S+1 \). Consequently,

\[
T \cdot \{(L_{T}(S + 1|D=d) - L_{T}(S + 2|D=d)) = (T-k-1)\{L_{T-k-1}(S + 1|D=d) - L_{T-k-1}(S + 2|D=d)\} + \delta \cdot 1
\]

where \( \delta = 1 \) if there are shortages in period \( T \) that are improved using an \( S+2 \) replenishment level instead of \( S+1 \), and \( \delta = 0 \) otherwise. In either case (\( \delta = 0 \) or 1), it is clear from the induction hypothesis for \( t=T-k-1 \) that

\[
T \cdot \{L_{T}(S|D=d) - L_{T}(S + 1|D=d)\} \geq T \cdot \{L_{T}(S + 1|D=d) - L_{T}(S + 2|D=d)\},
\]

which proves Case II and Theorem 2.6.
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